The Galois Group of a Picard-Vessiot Extension
Part I
The Zariski Topology on Affine n-Space
and
The Definition of Linear Algebraic Group.

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1 Linear Algebraic Groups.

1.1 The Zariski topology on affine n-space.

$C :=$ algebraically closed field, characteristic 0
$X := (X_1, \ldots, X_n), \quad X_1, \ldots, X_n$ indeterminates
$\mathbb{A}^n := C^n, \quad$ affine n-space
$x := (x_1, \ldots, x_n), \quad$ point in $\mathbb{A}^n$
$C[X]$:=polynomial algebra over C
Since C is infinite, we may identify $C[X]$ with an algebra of functions on X.

We define the Zariski topology on $\mathbb{A}^n$.

$V \subseteq \mathbb{A}^n$ is *closed* (is an *affine variety*) if there exists a finite set

$$F_1, \ldots, F_r$$

of polynomials in $C[X]$ such that

$$V = \{x \in \mathbb{A}^n : F_1(x) = \cdots = F_r(x) = 0\}.$$

V is closed $\Longleftrightarrow$ there exists an ideal

$$\mathfrak{a} = (F_1, \ldots, F_r)$$

with a finite basis in $C[X]$ such that

$$V = \{x \in \mathbb{A}^n : F(x) = 0\} \quad \forall F \in \mathfrak{a}.$$

Theorem 1 *(Hilbert Basis Theorem) Every ideal in $C[X]$ has a finite basis.*

Corollary 2 *($C[X]$ is Noetherian) Every ascending sequence of ideals is finite.*

$V(\mathfrak{a}) :=$ the closed set of zeros of the ideal $\mathfrak{a}$ of $C[X]$.

Let $\mathfrak{a}$ be an ideal in $C[X]$.

$$V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}}).$$

Let V be a subset of $\mathbb{A}^n$. Let

$$\mathfrak{a} = \{F \in C[X] : F(x) = 0 \quad \forall x \in V\}.$$

$\mathfrak{a}$ is a radical ideal in $C[X]$.

$I(V) := \mathfrak{a}$.

$\overline{V} :=$ the closure of V.

Clearly, $I(V) = I(\overline{V})$.

V and I are order reversing.

$$\begin{aligned} \mathfrak{a} &\subseteq \mathfrak{b} \Longrightarrow V(\mathfrak{b}) \subseteq V(\mathfrak{a}) \\ V &\subseteq W \Longrightarrow I(W) \subseteq I(V) \end{aligned}$$

Also,

$$\begin{aligned} V((1)) &= \emptyset \\ V((0)) &= \mathbb{A}^n \\ V(\mathfrak{a} \cap \mathfrak{b}) &= V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b}) \\ V\left(\sum_i \mathfrak{a}_i\right) &= \bigcap_i V(\mathfrak{a}_i) \end{aligned}$$

Since C is algebraically closed, we have a theorem that reduces the Zariski topology to a theory of radical ideals of a Noetherian C-algebra.

Theorem 3 *(Hilbert Nullstellensatz) The mapping*

$$\mathfrak{a} \longmapsto V(\mathfrak{a})$$

from the set of radical ideals in $C[X]$ to the set of closed sets in $\mathbb{A}^n$, and the mapping

$$V \longmapsto I(V)$$

from the set of closed sets in $\mathbb{A}^n$ to the set radical ideals in $C[X]$ are inclusion reversing, bijective, and inverse to each other.

Therefore,

$$\begin{aligned} I(\emptyset) &= (1) \\ I(\mathbb{A}^n) &= (0) \\ I(V \cup W) &= I(V) \cap I(W) \\ I\left(\bigcap_i V_i\right) &= \sum_i I(V_i) \end{aligned}$$

A topological space is *Noetherian* if every descending sequence of closed sets is finite.

Corollary 4 *The topological space $\mathbb{A}^n$ is Noetherian.*

Since every subspace of a Noetherian space is Noetherian, every closed subset of $\mathbb{A}^n$ is Noetherian.

Example 5 *Let $\mathfrak{a} = (XY^2+2Y^2, X^4-2X^2+1, 1-Z(Y-X^2+1)) \subseteq C[X,Y,Z]$.*
Let (x,y,z) be in $V(\mathfrak{a})$. Then, $\left(x^2-1\right)^2 = 0$. Therefore, $x = \pm 1$.
So, $y = 0$. It follows that $0 = 1$. So, $V(\mathfrak{a}) = \emptyset$, and $\mathfrak{a} = (1)$

A topological space is *reducible* if it is the union of two proper closed subsets. Otherwise, it is *irreducible.*

Exercise 6 *A Hausdorff space is irreducible if and only if it is reduced to a point.*

Lemma 7 *(Springer, Linear Algebraic Groups) Let V be a topological space.*

1. *$S \subseteq V$ is irreducible if and only it its closure is irreducible.*
2. *Let $\varphi : V \to W$ be a continuous map to a topological space W. If V is irreducible, so is its image $\varphi(V)$.*
3. *If V is irreducible, every nonempty open subset of V is dense in V.*

Since a point in a topological space V is irreducible, every point is contained in a maximal irreducible subspace V' of V (Zorn's Lemma). V' is called an *irreducible component* of V. So, V is the union of its irreducible components. An irreducible component is closed.

Proposition 8 *Let V be a nonempty closed subset of $\mathbb{A}^n$. V is irreducible if and only if $I(V)$ is prime.*
Proof. *Suppose $V = U \cup W$, U, W proper closed subsets of V.*
Then, U, W are closed in $\mathbb{A}^n$.

$$\begin{aligned} I(V) &= I(U) \cap I(W) \\ I(U) &\nsubseteq I(W) \qquad I(W) \nsubseteq I(U) \end{aligned}$$

Choose $F \in I(U) \setminus I(W)$, and $G \in I(W) \setminus I(U)$.
$FG \in I(V)$. $F \notin I(V)$ and $G \notin I(V)$. Thus, $I(V)$ is not prime.
We leave the converse as an exercise.
Since (0) is a prime ideal in $C[X]$, it follows that $\mathbb{A}^n$ is irreducible. ■

Let $V = V(X_1X_2)$. Then,

$$I(V) = (X_1) \cap (X_2).$$

$I(V)$ is not prime. V is the union of the coordinate axes of $\mathbb{A}^2$.

The point 0 in affine 1-space (the affine line) can be defined as the irreducible closed set $V(X^2)$. The ideal $\mathfrak{a} = (X^2)$ is not prime, but its radical $\sqrt{\mathfrak{a}} = (X) = I(0)$ is prime.

Remark 9 *The bijective correspondence* $\mathfrak{a} \longmapsto V(\mathfrak{a})$ *between radical ideals of* $C[X]$ *and closed subsets of* $\mathbb{A}^n$
restricts to a bijective correspondence $\mathfrak{p} \longmapsto V(\mathfrak{p})$ *between prime ideals of* $C[X]$ *and irreducible closed subsets of* $\mathbb{A}^n$*.*

Example 10 *Show that the ideal* $\mathfrak{p} = (Y - X^2, Z - XY)$ *in* $\mathbb{C}[X, Y, Z]$ *is prime.*

Suppose $FG \in \mathfrak{p}$*.* $V(\mathfrak{p}) = \left\{ \left(t, t^2, t^3\right) : t \in \mathbb{C} \right\}$ *– the so-called "twisted cubic." Therefore,* $F\left(t, t^2, t^3\right) G\left(t, t^2, t^3\right) = 0$ *for all* $t \in \mathbb{C}$*. Therefore, since polynomials in* 1 *variable have only a finite number of roots, one of the polynomials vanishes identically in* t*. Therefore, either* $F(X, Y, Z)$ *or* $G(X, Y, Z)$ *is in* $\mathfrak{p}$*.*

Together, the Hilbert Basis Theorem (the Noetherianity of $C[X]$),
and the Hilbert Nullstellensatz (closed sets are determined by radical polynomial ideals)
prove that every closed set is a finite union of distinct maximal irreducible closed subsets:

Corollary 11 *Let* $\mathfrak{a}$ *be a proper radical ideal in* $C[X]$*. Then,*

$$\mathfrak{a} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r,$$

where $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ *are the distinct minimal prime ideals containing* $\mathfrak{a}$*.*

$\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are called the *prime components* of $\mathfrak{a}$.

Corollary 12 *Let* V *be a non-empty closed set in* $\mathbb{A}^n$*. Then,*

$$V = V_1 \cup \cdots \cup V_r,$$

where $V_1, \ldots, V_r$ *are the distinct maximal irreducible closed subsets of* V*.*

Exercise 13 *Let* V *be a closed subset of* $\mathbb{A}^n$*.*

1. *An open subset* U *of* V *is dense in* V *if and only if* $U \cap V_i \neq \emptyset$, $i = 1, \ldots, m$.
2. *The intersection of two dense open subsets of* V *is nonempty.*

A topological space is *connected* if it is not the union of two disjoint proper closed subsets.
Every irreducible space is connected. Not every connected space is irreducible.

The union of the coordinate axes in $\mathbb{A}^2$ is connected, but is reducible, as we just saw.

By Zorn's Lemma, every point in a topological space V is contained in a maximal connected subspace V' of V. V' is called a *connected component* of V. V is the union of its connected components. The following exercise shows that a connected component of V is closed.

Exercise 14
1. *The closure of a connected subspace of a topological space is connected.*

2. *No proper nonempty subset of a connected space can be both open and closed.*

3. *If $\varphi : V \to W$ is a continuous map from the connected space V into the topological space W, then $\varphi(V)$ is connected.*

4. *Let V be a Noetherian topological space. V is the disjoint union of finitely many connected closed subsets, its connected components. The connected components of V are both open and closed in V. A connected subset of V is contained in a connected component of V. If $\varphi : V \to V$ is a continuous automorphism, φ permutes the connected components of V.*

5. *A closed subset V of $\mathbb{A}^n$ fails to be connected if and only if there exist two ideals $\mathfrak{a}$, $\mathfrak{b}$ in $C[X]$ such that $\mathfrak{a} + \mathfrak{b} = C[X]$, and $\mathfrak{a} \cap \mathfrak{b} = I(V)$.*

For us, the most important ring associated with a closed set V is the *ring of polynomial functions* on V. This ring is called the *coordinate ring* of V.

The coordinate ring of V is $C[V] = C[X]/\mathfrak{a}$, where $\mathfrak{a} = I(V)$.

If $\gamma = X \mod \mathfrak{a}$, then the coordinate ring of V equals $C[\gamma]$.

An element a in a ring R is *nilpotent*, if there is a positive integer k such that $a^k = 0$.

R is *reduced* if 0 is the only nilpotent element in R. Since $\mathfrak{a}$ is a radical ideal, $C[V]$ is reduced.

Let Fun(V) be the C-algebra of C- valued functions on V.

We define a C-algebra homomorphism φ: $C[V] \to$Fun(V):

$$\varphi(f)(x) = F(x), \quad x \in V, F \in C[X], \quad f \text{ the residue class of } F \bmod \mathfrak{a}.$$

Clearly, the kernel of φ is (0).So, we may identify $C[V]$ with a subalgebra of Fun(V).

It is called the *ring of polynomial functions on V.*

Proposition 15 *Let V be a closed set in $\mathbb{A}^n$.*

The bijective correspondence established by the Hilbert Nullstellensatz induces a bijective correspondence between the closed subsets of V and the radical ideals of $C[V]$.

$$W \leftrightarrow \mathfrak{a},$$

where $a = I(W) \bmod I(V)$. In this correspondence, irreducible closed subsets correspond to prime ideals.

Lemma 16 *Let $\mathfrak{a}$ be a radical ideal in $C[X]$. There is a bijective mapping*

$$\varphi : V(\mathfrak{a}) \to Hom_C(C[V], C).$$

Proof. *Write $C[V] = C[\gamma]$. Let $x \in V(\mathfrak{a})$. Define*

$$\chi_x : C[V] \to C$$

by

$$\chi_x(f) = f(x).$$

Clearly, $\chi_x \in Hom_C(C[V], C)$ *Note that* $\chi_x(\gamma) = x$. *Conversely, let* $\chi \in Hom_C(C[V], C)$. *Set*

$$x = \chi(\gamma).$$

Then,

$$0 = F(\gamma), \qquad F \in \mathfrak{a}.$$

Thus,

$$0 = \chi(F(\gamma)) = F(\chi(\gamma)).$$

Therefore,

$$x = \chi(\gamma) \in V(\mathfrak{a}).$$

Since

$$\chi_x(\gamma) = \gamma(x) = x,$$

it follows that

$$\chi = \chi_x.$$

So, the mapping from $V(\mathfrak{a})$ *to* $Hom_C(C[V], C)$ *is surjective. It is injective, since*

$$\chi_x = \chi_y \Longrightarrow x = \chi_x(\gamma) = \chi_y(\gamma) = y.$$

■

We call χ_x the *evaluation homomorphism* ($\gamma \longmapsto x$) defined by x.

Let V be a closed subset of $\mathbb{A}^n$.
A map $\varphi : V \to W$, W a closed subset of $\mathbb{A}^m$ is a *morphism* (of algebraic varieties) if for $x \in V$,

$$\varphi(x) = (f_1(x), \dots, f_m(x)), \qquad , f_1, \dots, f_m \in C[V].$$

φ defines a C-algebra homomorphism

$$\varphi^* : C[W] \to C[V],$$

defined by

$$\varphi^*(g)(x) = g(\varphi(x)).$$

If $C[V] = C[\gamma]$, and $C[W] = C[\varsigma]$, then
$\varphi^*(\varsigma) = (f_1(\gamma), \dots, f_m(\gamma)) \in C[V]^m$.
Call $f_1, \dots, f_m$ the *coordinate functions of* φ.

Lemma 17 *A morphism $\varphi : V \to W$ of affine varieties is continuous.*
Proof. *let T be a closed subset of W. Let $\mathfrak{a}$ be an ideal in $C[W]$ such that $T = V(\mathfrak{a})$. Then,*

$$\begin{aligned} x &\in \varphi^{-1}(W) \iff \varphi(x) \in W \\ &\iff f(\varphi(x)) = 0 \ \ \forall f \in \mathfrak{a} \\ &\iff \varphi^*(f)(x) = 0 \ \ \forall f \in \mathfrak{a}. \end{aligned}$$

So, $\varphi^{-1}(W) = V((\varphi^*(\mathfrak{a}))$. ∎

If T is a closed subset of $\mathbb{A}^p$ and

$$V \overset{\varphi}{\to} W \overset{\psi}{\to} T,$$

then

$$(\psi \circ \varphi)^* = \varphi^* \circ \psi^*.$$

Therefore, φ is an isomorphism of affine varieties if and only if φ^* is an isomorphism of C-algebras.

Exercise 18 *Let $\varphi : V \to W$ be a morphism of affine varieties.*

1. *If V is irreducible (resp. connected), then the closure $\overline{\varphi(V)}$ is irreducible (resp. connected).*

2. *If φ is an automorphism of V, then φ permutes the irreducible (and connected) components of V.*

3. *If φ is an automorphism of V, and V' is a closed subset of V, then $\varphi(V)$ is a closed subset of V.*

4. *If $W = \overline{(\varphi(V))}$, then φ^* is injective.*

5. *If φ^* is surjective, then $\varphi(V)$ is a closed subset of W.*

6. *If φ^* is injective, then $\varphi(V)$ is dense in W.*

For proofs of the next theorem and corollary, see Springer, *Linear Algebraic Groups, Chapter 1.*

Theorem 19 *(Chevalley) Let B be a domain finitely generated over a subring A, and let C be an algebraically closed field.*
For every $b \neq 0$ in B, there exists $a \neq 0$ in A
such that every homomorphism $\alpha : A \to C$ such that $\alpha(a) \neq 0$
extends to a homomorphism $\beta : B \to C$.

Corollary 20 *(Images of constructible sets are constructible)*
Let $\varphi : V \to W$ be a morphism of affine varieties. Then, $\varphi(V)$ contains an open dense subset of $\overline{\varphi(V)}$.

Let $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$ be affine varieties.

The Cartesian product $V \times W = \{(x,y) : x \in V, y \in W\}$ is an affine variety:

Let $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_m)$ be families of indeterminates.

$V \times W \subseteq \mathbb{A}^{n+m}$. If $\mathfrak{a} = I(V) \subseteq C[X]$, and $\mathfrak{b} = I(W) \subseteq C[Y]$, then $I(V \times W) \subseteq C[X,Y]$.

$I(V \times W) = \mathfrak{a} \cdot C[X,Y] + \mathfrak{b} \cdot C[X,Y]$, the ideal in $C[X,Y]$ generated by $\mathfrak{a} \cup \mathfrak{b}$.

$C[X]$ and $C[Y]$ are linearly disjoint over C.

The polynomial algebra $C[X,Y]$ is canonically C-isomorphic to $C[X] \otimes_C C[Y]$.

$(C[X] \otimes_C C[Y]) \backslash I(V \times W)$ is canonically C-isomorphic to $C[X] \backslash \mathfrak{a} \otimes_C C[Y] \backslash \mathfrak{b} = C[V] \otimes_C C[W]$.(Zariski-Samuel, *Commutative Algebra*, I, Theorem 35, p. 184).

So, the coordinate ring of $V \times W$ is $C[V] \otimes_C C[W]$.

If $f \in C[V], g \in C[W]$, then, for $(x,y) \in V \times W$, $(f \otimes g)(x,y) = f(x)g(y)$.

Spose V and W are irreducible. Then, $C[V]$ and $C[W]$ are integral domains. Since C is algebraically closed, $C[V] \otimes_C C[W]$ is an integral domain. Therefore, $V \times W$ is irreducible.

Example 21

$$V = V(X_1^2 + X_2^2 - 1) \subseteq \mathbb{A}^2$$

$$W = V(Y_1) \subseteq \mathbb{A}^2$$

$$V \times W = V\left(X_1^2 + X_2^2 - 1, Y_1\right) \subseteq \mathbb{A}^4$$

$$C[V] = C[\gamma_1, \gamma_2], \gamma_1^2 + \gamma_2^2 = 1$$

$$C[W] = C[\varsigma_2]$$

$$\begin{aligned} C[V \times W] &= C[\gamma_1, \gamma_2, \varsigma_2], \gamma_1^2 + \gamma_2^2 = 1 \\ &= C[\gamma_1, \gamma_2] \otimes_C C[\varsigma_2]. \end{aligned}$$

1.2 The closed subgroups of $GL(n)$.

The set $M(n)$ of $n \times n$ matrices with entries in C = affine n^2-space.

$c = (c_{ij})$ = the point $(a_{11}, \ldots, a_{1n}, \ldots, a_{n1}, \ldots, a_{nn})$.

$$GL(n) = \{c \in M(n) : \det c \neq 0\}.$$

$GL(n)$ is a dense open subset of $M(n)$.

Identify it with the closed subset V of $\mathbb{A}^{n^2+1}$ defined by the equation

$$X_{n^2+1} \det X = 1.$$

With this identification, the coordinate ring of $GL(n)$ is $C\left[X, \frac{1}{\det X}\right]$.

$C\left[X, \frac{1}{\det X}\right]$ is the localization of $C[X]$ by the multiplicative set M of non-negative powers of $\det X$.

$C\left[X, \frac{1}{\det X}\right]$is an integral domain. $C[X]$ is a subring of $C\left[X, \frac{1}{\det X}\right]$.

The closed subsets of $GL(n)$ are in bijective correspondence with the radical ideals of $C\left[X, \frac{1}{\det X}\right]$. Irreducible closed subsets correspond to prime ideals.

Identify $\left(c, \frac{1}{\det c}\right)$ with the matrix c. The following proposition tells us which radical ideals of $C[X]$ are the defining ideals of closed subsets of $GL(n)$.

Proposition 22 *A radical ideal $\mathfrak{a}$ is the defining ideal of a closed subset of $GL(n)$ if and only if no prime component of $\mathfrak{a}$ contains $\det X$.*

The proof is in the appendix to this section.

Example 23 $\mathfrak{a} = \left(X_{11}^2 X_{22} - X_{11}, X_{11}X_{12}X_{22} - X_{12}, X_{11}X_{12}, X_{11}X_{21}\right)$
is not the defining ideal of a closed subset of $GL(n)$.

$V(\mathfrak{a})$ has two components V_1, V_2, where $V_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix}, a \neq 0 \right\}$, $V_2 = \left\{ \begin{pmatrix} 0 & 0 \\ b & c \end{pmatrix}, b, c \in C \right\}$.

$\mathfrak{a} = (X_{11}X_{22} - 1, X_{12}, X_{21}) \cap (X_{11}, X_{12})$. *The second prime component contains* $\det X$.

Let V be a closed subset of $GL(n)$, and let $C\left[\gamma, \frac{1}{\det \gamma}\right]$ be its coordinate ring.

We have the short exact sequence:

$$0 \longrightarrow \mathfrak{a} \to C\left[X, \frac{1}{\det X}\right] \overset{\varphi}{\to} C\left[\gamma, \frac{1}{\det \gamma}\right] \to 0, \quad , \varphi(X) = \gamma, \quad , \varphi\left(\frac{1}{\det X}\right) = \frac{1}{\det \gamma}.$$

A subgroup G of $GL(n)$ that is also a closed subset is called a *closed subgroup* (*linear algebraic group*).

$GL(1)$ is denoted by $\mathbb{G}_m$

A subgroup G of $\mathbb{G}_m$ is closed if and only if there is a polynomial F in $C[X]$, X an indeterminate, such that $G = V(F)$.

G is the finite set of roots of a polynomial in 1 indeterminate.

The proper subgroups of $\mathbb{G}_m$ are finite groups: the groups of m^{th} roots of unity.

Other examples:

1. The *special linear group* $SL(n) = \{c \in GL(n) : \det c = 1\}$.

2. The *upper triangular group* $T(n) = \{c \in GL(n) : c_{ij} = 0, \quad i > j\}$.
3. The *upper triangular unipotent group* $U(n) = \{c \in T(n) : c_{ii} = 1, i = 1, \ldots, n\}$.
4. The *diagonal group* $D(n) = \{c \in GL(n) : c_{ij} = 0, i \neq j\}$.
5. The *orthogonal group* $O(n) = \{c \in GL(n) : c^t c = 1_n\}$, 1_n the $n \times n$ identity matrix.
6. The *special orthogonal group* $SO(n) = O(n) \cap SL(n)$.
7. A closed subgroup of $GL(2n)$ is the *symplectic group* $SP(2n) = \{c \in GL(2n) : c^t jc = 1_{2n}\}$, where

$$j = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}.$$

These are some of the so-called *classical groups*. A more prosaic closed subgroup of $GL(n)$ is

$$G = \{c \in GL(n) : \exists p \in \mathbb{N} : (\det c)^p = 1\}.$$

2 Appendix: The defining ideal in $C[X]$ of a closed subset of $GL(n)$.

We want to prove the following proposition:

Proposition 24 *A radical ideal $\mathfrak{a}$ in $C[X]$ is the defining ideal of a closed subset of $GL(n)$ if and only if no prime component of $\mathfrak{a}$ contains $\det X$.*

An ideal $\mathfrak{a}$ of $C[X]$ is *contracted* if there is an ideal $\mathfrak{b}$ of $C\left[X, \frac{1}{\det X}\right]$ with $\mathfrak{a} = \mathfrak{b} \cap C[X]$.
If $\mathfrak{a}$ is an ideal of $C[X]$, $\mathfrak{a}^e = C\left[X, \frac{1}{\det X}\right] \cdot \mathfrak{a}$ is called the *extension* of $\mathfrak{a}$.
Let $\mathfrak{a}$ be an ideal and G be an element of $C[X]$. G is prime to $\mathfrak{a}$ if for any $F \in C[X]$,

$$FG \in \mathfrak{a} \Longrightarrow F \in \mathfrak{a}.$$

Note that G is prime to $\mathfrak{a}$ if and only if every positive power of G is prime to $\mathfrak{a}$.

Lemma 25 *$\mathfrak{a}$ is contracted if and only if $\det X$ is prime to $\mathfrak{a}$.*
Proof. *If $\mathfrak{a} = \mathfrak{b} \cap C[X]$, $\mathfrak{b}$ an ideal of $C\left[X, \frac{1}{\det X}\right]$, then*

$$\det X \cdot F \in \mathfrak{a}, \quad F \in C[X] \Longrightarrow F \in \mathfrak{b} \cap C[X] = \mathfrak{a}.$$

Spose $\det X$ *is prime to $\mathfrak{a}$. Let $F \in \mathfrak{a}^e \cap C[X]$. Then,*

$$F = \sum_{j=1}^{p} H_i F_i, \quad H_i \in C\left[X, \frac{1}{\det X}\right], \quad F_i \in \mathfrak{a}, \quad i = 1, \ldots, p.$$

There exists a nonnegative integer k such that $G_i = \det X^k \cdot H_i \in C[X]$, $i = 1, \ldots, p$. Thus,

$$\det X^k F = \sum_{j=1}^{p} G_i F_i \in \mathfrak{a}.$$

It follows that $F \in \mathfrak{a}$.

$$\det X \prod_{\mathfrak{q} \neq \mathfrak{p}} F_q \in \mathfrak{a}.$$

Spose $\prod_{\mathfrak{q} \neq \mathfrak{p}} F_q$ is in $\mathfrak{a}$. Then, $\prod_{\mathfrak{q} \neq \mathfrak{p}} F_q \in \mathfrak{p}$. Therefore, for some $\mathfrak{q} \neq \mathfrak{p}$, $F_{\mathfrak{q}} \in \mathfrak{p}$. Thus, $\mathfrak{a}$ is not contracted. ■

Proposition 26 *(Zariski-Samuel, Commutative Algebra, Vol. I, Theorem 15, p. 223)The maps $\mathfrak{b} \longmapsto \mathfrak{b} \cap C[X]$ from the set of all ideals in $C\left[X, \frac{1}{\det X}\right]$ to the set of contracted ideals in $C[X]$,*

and

$$\mathfrak{a} \longmapsto \mathfrak{a}^e = C\left[X, \frac{1}{\det X}\right] \cdot \mathfrak{a}$$

from the set of contracted ideals in $C[X]$ to the set of ideals in $C\left[X, \frac{1}{\det X}\right]$, are inclusion preserving, bijective and inverse to one another,

and preserve the ideal-theoretic operations of forming intersections and radicals.