Rooks, Recurrences and Residues

Shaoshi Chen

Department of Mathematics
North Carolina State University, Raleigh

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Outline

- **Motivation**: enumerating 3D Walks.

- **Integrability problems**:

  Given \( f \in K(y, z) \), decide whether

  \[
  f = D_y(g) + D_z(h) \quad \text{for some } g, h \in K(y, z).
  \]

- **Telescoping problems**:

  Given \( f \in k(x, y, z) \), find \( L \in k(x)\langle D_x \rangle \) such that

  \[
  L(x, D_x)(f) = D_y(g) + D_z(h) \quad \text{for some } g, h \in k(x, y, z).
  \]
Enumerating 3D Rook Walks

The Rook moves in a straight line as below in first quadrant of the 3D space.

\[ R_n: \] The number of different Rook walks from \((0, 0, 0)\) to \((n, n, n)\).
2D-diagonals

\( f(m, n) \): the number of different Rook walks from \((0, 0)\) to \((m, n)\).

\[
F(x, y) = \sum_{m, n \geq 0} f(m, n) x^m y^n = \frac{1}{1 - \frac{x}{1-x} - \frac{y}{1-y}}.
\]

The diagonal of \(F(x, y)\) is

\[
\text{diag}(F) := \sum_{n \geq 0} f(n, n) x^n.
\]

Notation: \( \mathbb{F} \) an algebraically closed field of char zero \((= \bar{\mathbb{Q}}, \mathbb{C}, \ldots )\).

Lemma: Let \( G := y^{-1} \cdot F(y, x/y) \) and \( L(x, D_x) \) be a linear differential operator with coefficients in \( \mathbb{F}(x) \). Then

\[
\underbrace{L(x, D_x)(G)}_{\text{Telescoper}} = D_y(H) \quad \text{with} \quad H \in \mathbb{F}(x, y) \quad \Rightarrow \quad L(\text{diag}(F)) = 0
\]
Residues

Assume that $K$ be a field of characteristic zero.

**Definition.** Let $f \in K(y)$. The residue of $f$ at $\beta_i \in \overline{K}$ w.r.t. $z$, denoted by $\text{res}_y(f, \beta_i)$, is the coefficient $\alpha_{i,1}$ in

$$f = p + \sum_{i=1}^{n} \sum_{j=1}^{m_i} \frac{\alpha_{i,j}}{(y - \beta_i)^j},$$

where $p \in K[y]$, $\alpha_{i,j}, \beta_i \in \overline{K}$.

**Lemma.** $f = D_y(g)$ with $g \in K(y) \iff$ All residues of $f$ w.r.t. $y$ are zero.

**Remark.** This lemma is not true for algebraic functions!!!

**Hermite Reduction.**

$$f = D_y(g) + \frac{A}{B},$$

where $\deg_y(A) < \deg_y(B)$ and $B$ squarefree.

**Poisson formula.**

$$\text{res}_y(f, \beta_i) = \frac{A(\beta_i)}{D_y(B)(\beta_i)}.$$
Let $\mathbb{F}(x)\langle D_x \rangle$ be the ring of linear differential operators in $x$ with coefficients in $\mathbb{F}(x)$.

**Problem.** For $f \in \mathbb{F}(x, y)$, find $L \in \mathbb{F}(x)\langle D_x \rangle$ such that

$$L(x, D_x)(f) = D_y(g)$$

for some $g \in \mathbb{F}(x, y)$. (Telescoper)

**Simpler Problem.** For $h \in \mathbb{F}(x, y)$, decide whether

$$h = D_y(g)$$

for some $g \in \mathbb{F}(x, y)$

**Answer.** $h = D_y(g)$ iff $\text{res}_y(h, \beta) = 0$ for any root $\beta$ of the $\text{den}(h)$.

**Idea.** To find $L \in \mathbb{F}(x)\langle D_x \rangle$ such that $h = L(f)$ has only zero residues.
Telescoping via Residues: The **Bivariate** Rational Case

**Hermite Reduction.**

\[ f = D_y(g_1) + \frac{A}{B}, \quad \text{where } \deg_y(A) < \deg_y(B) \text{ and } B \text{ squarefree.} \]

**Rothstein-Trager Resultant.** \( R(x, z) := \text{resultant}_y(B, A - zD_y(B)) \).

\[ R(x, \text{res}_y(A/B, \beta)) = 0 \quad \text{for any root } \beta \text{ of } B \text{ in } \overline{\mathbb{F}(x)}. \]

**Theorem (Abel 1827).** There exists \( L \in \mathbb{F}(x)[D_x] \) s.t. \( L(\gamma) = 0 \) for any root \( \gamma \in \overline{\mathbb{F}(x)} \) of \( R(x, z) \).

\[ L(\text{res}_y(f, \beta)) = \text{res}_y(L(f), \beta) = 0 \quad (\forall \beta) \quad \Rightarrow \quad L(f) = D_y(g). \]
Telescopers for 2D Rook Walks

For the 2D Rook walks, the rational function is

\[ f := \frac{(-1 + y)(-y + x)}{y(y - 2x - 2y^2 + 3xy)} \]

**Resultant:** The Rothstein-Trager Resultant is

\[ R(x, z) := (-x + 2zx)(40z^2x^2 + x - 2x^2 + x^3 - 4z^2x - 36z^2x^3) \]

So the residues of \( f \) w.r.t. \( y \) are respectively

\[ r_1 = \frac{1}{2}, \quad r_2 = \frac{\sqrt{(9x - 1)(x - 1)}}{18x - 2}, \quad r_3 = -\frac{\sqrt{(9x - 1)(x - 1)}}{18x - 2} \]

**Annihilators for residues:** \( L_1 = D_x \) and

\[ L_2 = L_3 = (9x^2 - 10x + 1)D_x + (18x - 14) \]

Finally, the telescoper for \( f \) is

\[ L := (9x^2 - 10x + 1)D_x^2 + (18x - 14)D_x. \]
Recurrences

\( R(n) \): the number of different Rook walks from \((0, 0)\) to \((n, n)\).

Let \( S_n \) be the shift operator defined by \( S_n(R(n)) = R(n + 1) \).

\[
L(x, D_x) \left( \sum_{n \geq 0} R(n)x^n \right) = 0 \quad \Rightarrow \quad P(n, S_n)(R(n)) = 0.
\]

For the 2D Rook walks, we get the linear recurrence:

\[
R(n + 2) = \frac{(-10n - 14)R(n + 1) + 9nR(n)}{n + 2} \quad (R(1) = 2, \ R(2) = 14).
\]

Running the recurrence, \( R(n) \) is as follows.

2, 14, 106, 838, 6802, 56190, 470010, 3968310, ... \[ \text{OEIS: A051708} \]
Enumerating 3D Walks

The Rook moves in 3-dimensional space.

**Question:** How many different Rook walks from \((0, 0, 0)\) to \((n, n, n)\)?
3D-diagonals

\( f(m, n, k) \): the number of different Rook walks from \((0, 0, 0)\) to \((m, n, k)\).

\[
F(x, y, z) = \sum_{m,n \geq 0} f(m, n, k) x^m y^n z^k = \frac{1}{1 - \frac{x}{1-x} - \frac{y}{1-y} - \frac{z}{1-z}}.
\]

The diagonal of \( F(x, y, z) \) is

\[
\text{diag}(F) := \sum_{n \geq 0} f(n, n, n) x^n.
\]

**Lemma:** Let \( \tilde{F} := (yz)^{-1} \cdot F(y, z/y, x/z) \) and \( L(x, D_x) \in \mathbb{F}(x)\langle D_x \rangle \). Then

\[
L(x, D_x)(\tilde{F}) = D_y(G) + D_z(H)
\]
with \( G, H \in \mathbb{F}(x, y, z) \Rightarrow L(\text{diag}(F)) = 0. \)
Telescoping Problems

Telescopers for trivariate rational functions:
Given $f \in \mathbb{F}(x, y, z)$, find $L \in \mathbb{F}(x)\langle D_x \rangle$ such that

$$L(x, D_x)(f) = D_y(g) + D_z(h) \quad \text{for some } g, h \in \mathbb{F}(x, y, z).$$

Telescopers for bivariate algebraic functions:
Given $\alpha(x, y)$ algebraic over $\mathbb{F}(x, y)$, find $L \in \mathbb{F}(x)\langle D_x \rangle$ such that

$$L(x, D_x)(\alpha) = D_y(\beta) \quad \text{for some algebraic } \beta(x, y) \text{ over } \mathbb{F}(x, y).$$

Goal: The two telescoping problems above are equivalent!
Integrability Problems

Rational Integrability:

Given \( f(y, z) \in \mathbb{E}(y, z) \), decide

\[
f = D_y(g) + D_z(h) \quad \text{for some } g, h \in \mathbb{E}(y, z).
\]

If such \( g, h \) exist, we say that \( f \) is rational integrable w.r.t. \( y \) and \( z \).

Algebraic Integrability:

Given \( \alpha(y) \) algebraic over \( \mathbb{E}(y) \), decide

\[
\alpha = D_y(\beta) \quad \text{for some algebraic } \beta \text{ over } \mathbb{E}(y).
\]

If such \( \beta \) exists, we say that \( \alpha \) is algebraic integrable w.r.t. \( y \).

Goal: The two integrable problems above are equivalent!
Residues

**Definition.** Let \( f \in \mathbb{E}(y)(z) \). The *residue* of \( f \) at \( \beta_i \) w.r.t. \( z \), denoted by \( \text{res}_z(f, \beta_i) \), is the coefficient \( \alpha_{i,1} \) in

\[
f = p + \sum_{i=1}^{n} \sum_{j=1}^{m_i} \frac{\alpha_{i,j}}{(z - \beta_i)^j},
\]

where \( p \in \mathbb{E}(y)[z] \), \( \alpha_{i,j}, \beta_i \in \mathbb{E}(y) \).

**Lemma.** Let \( f \in \mathbb{E}(y)(z) \) and \( \beta \in \mathbb{E}(y) \).

- \( \partial(\text{res}_z(f, \beta)) = \text{res}_z(\partial(f), \beta) \) with \( \partial \in \{D_x, D_y\} \).
- \( f = D_z(g) \iff \text{All residues of } f \text{ w.r.t. } z \text{ are zero.} \)

**Remark.** The second assertion is not true for algebraic functions!!!
Equivalence between Two Integrability Problems

**Theorem** (Picard1902). Let \( f = A/B \in \mathbb{F}(y, z) \). Then

\[
f = D_y(g) + D_z(h) \iff \text{res}_z(f, \beta) = D_y(\gamma\beta) \text{ for all } \beta \text{ s.t. } B(\beta) = 0.
\]

**Example 1.** Let \( f = (y + z)^{-1} \). Since \( \text{res}_z(f, -y) = 1 = D_y(y) \), \( f \) is rational Integrable w.r.t. \( y \) and \( z \). In fact,

\[
f = D_y \left( \frac{y}{y + z} \right) + D_z \left( -\frac{y}{y + z} \right).
\]

**Example 2.** Let \( f = (yz)^{-1} \). Since \( \text{res}_z(f, 0) = (y)^{-1} \) is not algebraic integrable, \( f \) is not rational integrable w.r.t. \( y \) and \( z \).
Equivalence between Two Telescoping Problems

Assume that $\mathbb{E} = \mathbb{F}(x)$.

**Theorem** (Telescoping). Let $f \in \mathbb{F}(x, y, z)$ and $L \in \mathbb{F}(x)(D_x)$. Then

$L(x, D_x)$ is a telescoper for $f$ w.r.t. $y$ and $z$

$L(x, D_x)$ is a telescoper for every residue of $f$ w.r.t. $z$

**Remark.**

$L_i(x, D_x)(\alpha_i) = D_y(\beta_i)$, $1 \leq i \leq n$

$L = \text{LCLM}(L_1, L_2, \ldots, L_n)$ is a telescoper for all $\alpha_i$. 
Differentials and Residues

Let $K = \mathbb{F}(x, y)(\alpha)$ where $\alpha$ is an algebraic function over $\mathbb{F}(x, y)$. Think of $\alpha(x, y)$ as a parameterized family of algebraic functions of $y$ (with parameter $x$).

**Differentials.**

$$\Omega_{K/\mathbb{F}(x)} := \{ \beta \, dy \mid \beta \in K \}.$$

- $df = 0$ for all $f \in \mathbb{F}(x)$ and $D_x(\beta dy) = D_x(\beta)dy$.

**Residues.** Let $\mathcal{P}$ be a place of $K$ (with no ramification). Then any $\beta \in K$ has a $\mathcal{P}$-adic expansion

$$\beta = \sum_{i \geq \rho} a_i t^i, \quad \text{where } \rho \in \mathbb{Z}, \ a_i \in \mathbb{F}(x) \text{ and } t \in K.$$

The residues of $\beta$ at $\mathcal{P}$ is $a_{-1}$, denoted by $\text{res } \mathcal{P}(\beta)$.

- $\text{res } \mathcal{P}(D_x(\beta)) = D_x(\text{res } \mathcal{P}(\beta))$. 

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Differential Equations for Residues

Let $K = \mathbb{F}(x, y)(\alpha)$ and $\beta = A/B$ with $A \in \mathbb{F}(x)[y, \alpha]$ and $B \in \mathbb{F}(x)[y]$. Let $B^*$ be the squarefree part of $B$ w.r.t. $y$.

**Theorem.** There exists $L \in \mathbb{F}(x)\langle D_x \rangle$ such that all residues of $L(\alpha)$ are zero and

$$\deg_{D_x}(L) \leq [K : \mathbb{F}(x, y)] \cdot \deg_y(B^*).$$

**Definition.** A differential $\omega \in \Omega_{K/\mathbb{F}(x)}$ is of second kind if all residues of $\omega$ are zero.

**Lemma.**

- If $\omega$ is exact i.e. $\omega = d(\beta)$, then $\omega$ is of second kind.
- Let $\Phi_{K/\mathbb{F}(x)} := \{\text{differentials of second kind}\}/\{\text{exact differentials}\}$. Then

$$\dim_{\mathbb{F}(x)}(\Phi_{K/\mathbb{F}(x)}) = 2 \cdot \text{genus}(K).$$
Telescopers for Bivariate Algebraic Functions

Algorithm. Given $\alpha(x, y)$ algebraic over $\mathbb{F}(x, y)$, do

1. Compute $L_1 \in \mathbb{F}(x)\langle D_x \rangle$ such that $\omega = L_1(\alpha) \, dy$ is of second kind.

2. Find $a_0, \ldots, a_{2g} \in \mathbb{F}(x)$ with $g := \text{genus}(K)$ with $K = \mathbb{F}(x, y)(\alpha)$, not all zero, such that

   $$a_{2g} D_x^{2g}(\omega) + \cdots + a_0 \omega = d(\beta) \quad \text{for some } \beta \in K.$$

Remark.

- If $\alpha \in \mathbb{F}(x, y)$, Step 2 is not needed since $g = 0$.
- If $\omega$ is of second kind, so is $D_x^i(\omega)$ for all $i \in \mathbb{N}$. 
Manin’s example

The elliptic integral

\[ I(x) := \int_{\Gamma} f(x, y) \, dy, \quad \text{where} \quad f = \frac{1}{\sqrt{y(y - 1)(y - x)}}. \]

Telescopener for \( f \)

\[ L(x, D_x)(f) = D_y \left( \frac{2y(y - 1)}{(-y + x)\sqrt{-y(y - 1)(-y + x)}} \right), \]

where

\[ L = (4x^2 - 4x)Dx^2 + (8x - 4)Dx + 1. \]

Then \( I(x) \) satisfies the Picard-Fuchs equation

\[ D_x^2(I(x)) + \frac{2x - 1}{x(x - 1)} D_x(I(x)) + \frac{1}{4x(x - 1)} I(x) = 0. \]
Telescopers for 3D Rook Walks

Transformation. \( F = P/Q := (yz)^{-1}f(y, z/y, x/z) \).

\[
P = \frac{(-1 + y)(y - z)(-z + x)}{zy(zy - 2 yx - 2 z^2 + 3 xz - 2 y^2 z + 3 y^2 x + 3 z^2 y - 4 zyx)}
\]

Residues. Roots of \( R(x, y, u) := \text{Resultant}_z(Q, P - u \cdot D_z(Q)) \) are

\[
r_1 = \frac{y - 1}{y(3y - 2)}, \quad r_2 = -r_3 = \frac{(y - 1)^2}{y(3y - 2) \sqrt{-4y^3 + 16xy^2 + 4y^2 - y - 24xy + 9x}}.
\]

Telescopers. \( L_1 = D_x \) and \( L_2 = L_3 \) with

\[
L_2 = D_x^3 + \frac{4608 x^4 - 6372 x^3 + 813 x^2 + 514 x - 4}{x (-2 + 121 x + 475 x^2 - 1746 x^3 + 1152 x^4)} D_x^2
\]

\[
+ \frac{4 (576 x^3 - 801 x^2 - 108 x + 74)}{x (-2 + 121 x + 475 x^2 - 1746 x^3 + 1152 x^4)} D_x
\]
Recurrences for 3D Rook Walks

\[ L = \text{LCLM}(L_1, L_2, L_3) \] is a telescopers for \( \mathbb{F}(x, y, z) \).

\[ \downarrow \]

\[ L(x, D_x) \left( \sum_n f(n, n, n)x^n \right) = 0 \]

Recurrence. Let \( r(n) := f(n, n, n) \). From \( L(x, D_x) \) via \text{gfun}, we get

\[
(1152n^2 + 1152n^3)r(n) + (-7830n - 3204 - 6372n^2 - 1746n^3)r(n + 1) + (2957n + 762 + 2238n^2 + 475n^3)r(n + 2) + (4197n + 4698 + 1240n^2 + 121n^3)r(n + 3) + (-22n^2 - 80n - 96 - 2n^3)r(n + 4) = 0.
\]

With initial values \( r(0) = 1, r(1) = 6, r(2) = 222, r(3) = 9918 \), we get

\[ 1, 6, 222, 9918, 486924, 25267236, 1359631776, 75059524392, \ldots \]
Equivalence.

\[ L(x, D_x)(f) = D_y(g) + D_z(h), \quad f, g, h \in \mathbb{F}(x, y, z) \]

\[ \Leftrightarrow \]

\[ L(x, D_x)(\alpha) = D_y(\beta) \quad \text{for any residue } \alpha \text{ of } f \text{ w.r.t. } z. \]

Note. One can also reduce rational \( m \) vars to algebraic \( m - 1 \) vars.

Order Bound. Let \( K = \mathbb{F}(x, y)(\alpha) \) and \( n \) be the number of poles of \( \alpha \).

\[ L(x, D_x)(\alpha) = D_y(\beta) \quad \Rightarrow \quad \text{ord}(L) \leq [K : \mathbb{F}(x, y)] \cdot n + 2 \cdot \text{genus}(K). \]

Future Work. Walks in higher dimension (4D, 5D, ...).