# A Set-Theoretical Approach to Model Theory

R.C. Churchill

Hunter College and the Graduate Center of CUNY, and the University of Calgary

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**Contents**

$\mathbb{N}$ *always denotes the set* $\{0, 1, 2, 3, \dots\}$ *of natural numbers, and*

$\mathbb{Z}^+$ *always denotes the set* $\{1, 2, 3, \dots\} = \mathbb{N} \setminus \{0\}$ *of positive integers.*

*Rings are always assumed to admit unities, and ring homomorphisms are assumed to preserve unities.*

*Groups are written multiplicatively, with identity denoted* $e$*, unless specifically stated to the contrary.*

# 1. Introduction

> Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk.
>
> [God made the whole numbers, all the rest is the work of man.]
>
> Leopold Kronecker

Kronecker died in 1891. When axiomatic set theory was developed in the early 1920s, closely following the ratification of the Nineteenth Amendment of the United States Constitution, an appropriate revision might have been:

> God made sets, all the rest is a human construct.

When category theory entered the picture in mid-century a further revision seemed necessary:

> God made classes, all the rest is a human construct.

This last formulation is a good summary of how this author views mathematics, but I need to be more specific: for me the "mathematical world" must be treated as being distinct from the "real world." In this matter I seem to be in good company.

> ... I ... use the word 'reality' ... with two different connotations. ... By physical reality I mean the material world, the world of day and night, earthquakes and eclipses, the world which physical science tries to describe.
>
> [But] ... for me, and I suppose for most mathematicians, there is another reality, which I will call 'mathematical reality'.
>
> G.H. Hardy

The quote is from [Hardy, §22], and continues with additional comments which are quite relevant to these notes.

> ... there is no sort of agreement about the nature of mathematical reality among either mathematicians or philosophers. Some hold that it is 'mental' and that in some sense we construct it, others that it is outside and independent of us. ...
>
> I should not wish to argue any of these questions here even if I were competent to do so, but I will state my own position dogmatically in order to avoid any misapprehensions. I believe that mathematical reality lies outside us, that our function is to discover or *observe* it, and that the theorems which we prove, and which we describe grandiloquently as our 'creations', are simply our notes of our observations. This view has been held, in one form or another, by many philosophers of high reputation from Plato onwards, and I shall use the language which is natural to a man who holds it. A reader who does not like the philosophy can alter the language: it will make very little difference to my conclusions.

I share a good deal of Hardy's philosophy[1], but I again need to be more specific. For me the mathematical world contains nothing but classes (which include sets[2], and therefore functions), whereas the real world, which encompasses death and taxes, *also encompasses the notations and formulas, the set-theoretical axioms, and the logical constructs, such as sentential inference, which serve as indispensable guides for understanding the mathematical world.*

Such a viewpoint comes from experience. I have never held a natural number in my hands, let alone seen one, but I am quite confident when writing down an expression such as[3] $3 + 5 = 8$ that my symbols actually represent something "out there." A physicist reading this might be reminded of a "parallel universe." That concept is similar to my mathematical world in the sense of being separate and physically inaccessible, while simultaneously being amenable to description by means of formulas (which for me include equations).

For someone of my philosophical persuasion a definition such as "a polynomial (in a single variable $x$) is an expression of the form $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$" makes no sense: it confuses notation with what that notation is intended to represent,

[1]But not, unfortunately, his mathematical abilities!

[2]We will eventually explain our view of the distinction between sets and classes, which is *not* that a set is a class which is an element of a set.

[3]I checked this particular sum on several calculators, so am fairly confident that it is correct, or at least in the right ballpark.

and as a consequence offers not even a hint as to what mathematical world entity is being described[4]. Moreover, it is not clear to me that I could easily use such a definition to prove anything about polynomials in a manner which would be regarded as *currently acceptable mathematical rigor*[5]. When training to become a professional mathematician those in my generation would eventually encounter a definition of a polynomial based ultimately on set theory. For example, if one requires coefficients in a commutative ring $R$ one could define a polynomial[6] to be a function from a collection[7] $\{(x,n)\}_{n\in\mathbb{N}}$ into $R$ which has value 0 for almost all[8] $(x,n)$, wherein $x$ can denote any set disjoint from $R$. From this perspective, and assuming $R=\mathbb{Z}$, the notation $3x^{12}-5x^3+7$ would be regarded as a "short-hand" representation of the function

$$\begin{array}{lcrl} (x,12) & \mapsto & 3, & \\ (x,3) & \mapsto & -5, & \\ (x,0) & \mapsto & 7, & \text{and} \\ (x,n) & \mapsto & 0 & \text{if } n\notin\{0,3,12\}. \end{array}$$

Alternatively, a "free object" approach to polynomials might be preferable to those who favor categorical formulations, and is more in keeping with the idea of "substituting for variables." In either case one generally reverts to the classical notation once the definition has been thoroughly dissected at the formal level and absorbed into the unconscious, the advantage being that one is now able proceed with confidence that something in the mathematical world is actually being discussed[9].

My experiences listening to talks on model theory often remind me of my long-ago struggles to understand the definition of a polynomial. Model theorists refer

[4]On the other hand, such a definition is probably sufficient for the majority of non-mathematicians who use polynomials in their work, and they no doubt outnumber professional mathematicians. I am not advocating changing the way polynomials are introduced in secondary schools.

[5]I doubt if a nineteenth century mathematician would have been bothered by the definition I have given in quotes. But my experience has been that, *even if they communicated their definition in that form*, they understood what they were doing at a far deeper level, and that their arguments, once understood, are not generally difficult to formulate in contemporary terms. Indeed, a good deal of "modern mathematical constructs" (sheaf cohomology, algebraic function fields, etc.) could be viewed as aids for understanding what the masters already knew. Example: in [Gunn, §4, Theorem 6, p. 51], Gunning attributes a theorem involving an exact cohomology sequence to Weierstrass.

[6]A slight variation of the definition I give is found in [Lang, Chapter II, §3, p. 97].

[7]Think of $(x,n)$ as $x^n$.

[8]I.e. all but at most finitely many.

[9]Mathematical "formalists" (and probably many others!) would regard all this as nonsense. For the pure formalist mathematics amounts to a collection of real-world rules governing meaningless marks on paper.

constantly to "sets of formulas," which for me makes no sense, since these two entities inhabit completely different worlds[10]. For example, in [Marker, Chapter 4, §1, Definition 4.1.1, p. 115] one finds

> **Definition 4.4.1 :** Let $p$ be the set of $\mathcal{L}_A$-formulas in free variables $v_1, \ldots, v_n$. We call $p$ an *n-type* if ....

Model Theory is a very active field, and has made enormous contributions to mathematics, e.g. E. Hrushovski's proof of the geometric Mordell-Lang conjecture in algebraic function fields[11]. It is particularly useful in differential algebra, which is how Model Theory caught my attention and why I wanted to learn about it. Unfortunately, my philosophical approach to mathematics seemed to prevent me from benefiting fully from the many contacts I have with model theorists at the CUNY Graduate Center[12].

Fortunately, I eventually realized[13] that if one views model theory from the standpoint of universal algebras the concept of a "set of formulas" can be developed in such a way as to be as mathematically straightforward as the concepts of a group and of a topological space. These notes indicate, probably in far more detail than anyone would ever care to see, how this can be done. Specifically, I begin with an entity, which I call a "UA type" (for "universal algebra type"), having a simple set-theoretic definition, associate a category with each such entity, prove that this category admits free objects, and indicate how and why it seems quite natural to refer the elements of the free objects as "formulas," or at least as "propositions," even though (in this author's opinion) that is not what they "really" are[14].

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[10]Formulas are human constructs; sets are not. Indeed, when one lists the axioms of set theory [as will be done later in these notes] one of the first axioms that appears is that of "specification." This (roughly) states that if $X$ is a set and $p(x)$ is a "formula," then there is a subset $Y \subset X$ consisting precisely of those $x \in X$ for which $p(x)$ is true. As one reads further down such a list if is often the case that one will eventually encounter a statement such as "We have yet to assume the existence of any set." Given the statement of the Axiom of Specification, and the fact that, if supplied with real-world paper and pencil I can create formulas at will, it seems implicit from such a presentation that *formulas cannot be sets.*

[11]Reference [Bou] seems to cover all the background, both in model theory and Abelian varieties that allow one to understand both the statement and proof. (A fair knowledge of algebraic geometry is assumed.)

[12]To be honest I have to admit that much of my frustration is of my own making: I have never attended a full course in Model Theory, despite many opportunities to do so, and it has been years since I took courses in logic. I wanted to quickly learn what was going on in that field, primarily because of the applications to differential algebra, but I did not go about doing that in a very intelligent way.

[13]Which should not be mistaken for an assertion that my current understanding is actually correct!

[14]To graduate from propositions to more general formulas one must move up from sentential

This approach is reminiscent of the way Kolchin formulated differential algebra in [Kol] (and elsewhere), although he did not use categorical terminology. There my UA type is replaced with a pair $(\Delta, R)$ consisting of a set $\Delta$ (the *derivation operators*) and a commutative ring $R$, and each such pair is associated with a category having as objects $R$-algebras on which the elements $\delta \in \Delta$ act as commuting derivations[15], and having as morphisms $R$-algebra homomorphisms which commute with the action of each $\delta$. He proves the category admits a free object on any set $X = \{x_\alpha\}$, the elements of which are called *differential polynomials* in the *differential indeterminates* $x_\alpha$.

Contemporary texts on Model Theory begin with a slightly more ambitious program than what I have outlined at the end the penultimate paragraph. They open with a generalization of a UA type which in [Marker, Chapter 1, §1. Definition 1.1.1, p. 8] is called a *language*, but[16] in [Roth, Chapter 1, §1.1, p. 3] is called a *signature* ("languages" do not appear in [Roth] until page 11, and "signatures" never appear [at least in the index] in [Marker]). The analogue of what I have labeled a $T$-algebra is called a *structure*, and, as one would expect, these references establish (generally not using categorical language) that for any fixed language/signature all such entities and their associated homomorphisms form a category.

My introduction to the universal algebra approach[17] was a casual reading of portions of [B-M], which I subsequently began to study in earnest. Unfortunately, almost immediately[18] I ran into set-theoretic difficulties with a proof of one of the basic results needed from universal algebra. My resolution of that difficulty consumes about 34 pages (§5-9) of these notes!

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inference ("Propositional Calculus") to "first order logic" ("Predicate Calculus"). These notes end with sentential inference, but once the philosophy of that construction is understood the form one expects the extension to take (if not the intermediate technical details) becomes predictable (see, e.g. [B-M, Chapters 3-5]).

[15]A *derivation* on an $R$-algebra $A$ is a mapping $\delta : a \in A \mapsto a' \in A$ such that for any $a, b \in A$ one has

- $(a+b)' = a' + b'$ and
- $(ab)' = ab' + a'b$.

Derivations $\delta_1, \delta_2$ on $A$ *commute* if $\delta_1(\delta_2(a)) = \delta_2(\delta_1(a))$ for all $a \in A$.

[16]Individuals interested in learning Model Theory need to know that no two workers in the field use the same terminology. Despite this, they all remain friends. This does, however, present problems for those not in the field when it comes to choosing which terminology to use, and I am friends with both David Marker and Philipp Rothmaler (or at least I think was before I started writing these notes)!

[17]Which I soon discovered could be found in many places; particularly in books on that subject.

[18]On pages 5 & 6 of that reference.

I have the same set-theoretic difficulties with the proof of the existence of an algebraic closure for a field given by Lang in [Lang, Chapter V, §2, pp. 231-2]. Since that result (if not the proof) is assumed familiar to readers, for the purposes of this introduction I will here describe my problem in that context. Given a field $K_0$ Lang proves that one can construct a field extension $K_1$ of $K_0$ which contains a root of every polynomial in[19] $K_0[x]$. (I am fine with this portion of the argument.) He then applies the analogous construction to $K_1$, obtaining a field $K_2$ which contains a root of every polynomial in $K_1[x]$ (again fine); then applies $\dots$, thereby obtaining a recursively defined sequence $K_0 \subset K_1 \subset K_2 \subset \cdots$ of fields. He completes the proof by verifying that the set-theoretic union $\cup_{n\in\mathbb{N}} K_n$ admits the structure of a field (this is easy), and that this union, with this field structure, is an algebraic closure of $K_0$ (again easy).

So what is the problem? For me the problem is that, from a set-theoretic viewpoint, *it is not clear that union* $\cup_{n\in\mathbb{N}} K_n$ *makes sense.* To invoke the Axiom of Unions (which will be stated in §6) one needs to know that the collection $\{K_n\}_{n\in\mathbb{N}}$ is a set, and since this collection is defined recursively, that would require realizing the underlying recursive function $n \mapsto K_n$ by means of the Recursion Theorem[20]. But *I could not see how this could be done.*

However, I did see how to make the argument rigorous by moving to the axioms Saunders Mac Lane has proposed as a (temporary) set-theoretic foundation for (a good deal of) category theory. These are explained in §6, and that is a major reason why the notes are so lengthy. (There are additional reasons for dealing with category theory: as already indicated, we make heavy use of free objects.) Once that work is out of the way we apply the results to Lang's argument before moving on the the real goal, which is the universal algebra construction which bothered me in the first place. Those who are not concerned about my set-theoretic and philosophical problems[21] should stop reading here[22] and start looking elsewhere. Reference [B-M] is recommended (it was my main reference, and I like the approach very much), but readers are warned that much of the terminology found there is not, in my experience, what is currently used in model theory. I have tried to employ contemporary terminology[23].

During the talk it was pointed out that one can make sense out of my neme-

[19]Throughout this paragraph $x$ denotes a single indeterminate.

[20]This theorem will be stated, proved, and illustrated in §7.

[21]Or would "hang-ups" be more appropriate?

[22]Assuming that has not already happened.

[23]Which brings me back to my point that no two model theorists seem agree on terminology. They do seem agree, however, that terminology used by any other model theorist, even if totally appropriate, should be avoided at all costs.

sis unions with appeals to the Axiom of Specification rather than to the Axiom of Unions[24]. In particular, for that purpose there was no need to introduce Mac Lane's axioms. However, because elsewhere I deal with categories, and because it clarified (for this author) the role of the Axiom of Replacement, I decided to leave things as I originally developed them[25].

A remark on terminology is in order. In §6 I go to great lengths to distinguish between "sets" and "classes," but in other places I cast off formality and write about "collections" without specifying the meaning. I do this in part, as Halmos writes in [Hal, §1, p. 1],

> ... to avoid terminological monotony ... ,

but also to indicate points within the exposition where I regard the level of the discussion as being informal. To illustrate, suppose we agree that the integers $\mathbb{Z}$ form a set and we want to assert the same about the even integers. Two possible choices for doing this, which for this author represent opposite extremes, are to state that

> The collection of even integers is a set.

and to state that

> By the Axiom of Specification those $n \in \mathbb{Z}$ satisfying $\exists m \in \mathbb{Z}$ such that $n = 2m$ constitute a subset of $\mathbb{Z}$, and therefore a set. This is the *set of even integers.*

Absent any need to stress some very technical point(s), and offered only these two choices, I would opt for the first without a second thought. Even though it is open to (or even "invites") misinterpretation, I am confident that my meaning would be clear to anyone at the level I would expect of readers. The problem, of course, is the implicit circularity: the statement could easily be interpreted as "the set of even integers is a set," which is not very informative.

[24] In [Hal, §19, pp. 74-5] Halmos uses that axiom to deal with an example involving a set-theoretic construction completely analogous to the two which had bothered me.

[25] In other words, I was getting a severe case of "burn out" from writing these notes, particularly the footnotes, and had no desire to essentially start all over.

## 2. UA Types

A *UA type* is[26] an ordered pair $(T, \mathrm{ar})$ consisting of a set $T$ and a function $\mathrm{ar} : T \to \mathbb{N}$; $T$ is the *underlying set* of the UA type; the elements of $T$ are the *operations* (*of the UA type*, as opposed to "on the UA type"); ar is the associated[27] *arity function* (and UA is an abbreviation for "universal algebra"). In practice one denotes the UA type by $T$ (in the same spirit as denoting a topological space $(X, \tau)$ by $X$). If we define

$$T_n := \mathrm{ar}^{-1}(\{n\}), \qquad n \in \mathbb{N}, \tag{2.1}$$

then one has

$$T = \textstyle\coprod_{n \geq 0} T_n \qquad \text{(disjoint union)}. \tag{2.2}$$

In these notes the sets $T_n$ within the decomposition (2.2) will generally be finite, and that suggests expressing $T$ in the form

$$T = \{t_{01}, t_{02}, \ldots, t_{0m_0}, t_{11}, \ldots, t_{1m_1}, t_{21}, \ldots\},$$

where

$$T_n = \{t_{n1}, t_{n2}, \ldots, t_{nm_n}\}, \qquad n = 0, 1, 2, \ldots .$$

However, in model theory, which is of primary importance for us, this is rarely (if ever) done. Instead the $t_{nj}$ are replaced by various symbols[28] which prove very convenient for applications, but which can initially lead one to believe (incorrectly) that one is not actually dealing with (elements of) sets[29]. For example, instead of $T = \{t_{01}, t_{11}, t_{21}, t_{22}\}$ one is (far) more likely to encounter notation along the lines of

$$T = \{0, -, +, \cdot\}, \tag{2.3}$$

[26] In [B-M] what we call a UA type is simply called a "type." We have avoided that terminology since in model theory that word has a completely different meaning. Two common synonyms for "UA type" are: *similarity type* and *type of universal algebra* (which strikes one more as an example of a definition than an actual definition). I settled on "UA type" to keep universal algebras in mind, but readers are warned that this terminology is not used by other authors.

[27] "arity" rhymes with "clarity."

[28] Such conventions are also used, but far more sparingly, in other areas of mathematics. For example, a relation on a set $X$, which by definition is a subset $R \subset X \times X$, is often denoted by a symbol such as $\sim$ or $<$, and rather than adhere to the set-theoretic custom of writing $(x_1, x_2) \in R$ one writes something like $x_1 R x_2$ or, in these two particular cases, $x_1 \sim x_2$ and $x_1 < x_2$.

[29] Elements of sets are always sets. In particular, when one writes $t_{nj} \in T$ both $t_{nj}$ and $T$ represent sets.

(read 0, $-$, $+$, and $\cdot$ as "zero, inversion, plus" and "times" (or "dot") respectively), in which case writing $+ \in T$ would make perfect sense, and rather than $T_0 = \{t_{01}\}$, $T_1 = \{t_{11}\}$ and $T_2 = \{t_{21}, t_{22}\}$ one will probably see

$$T_0 = \{0\}, \qquad T_1 = \{-\}, \qquad \text{and} \qquad T_2 = \{+, \cdot\}. \tag{2.4}$$

The reason for such notation will quickly become evident.

# 3. $n$-ary Operations

*Here $A$ denotes a set.*

Define

$$(3.1) \qquad \begin{cases} A^0 & := \{\emptyset\}, \\ A^1 & := A; \quad \text{and} \\ A^{n+1} & := A^n \times A \ \text{ if } n \geq 1 \text{ and } A^n \text{ has been defined.} \end{cases}$$

A function $f : A^n \to A$ is called an[30] *$n$-ary operation* (on $A$), although when $n$ is small alternate terminology is generally employed: when $n = 0, 1$ or $2$ one speaks of a[31] *nullary, unary* or *binary operation* respectively. When one is considering a collection of such operations on $A$ with varying $n$ one simply refers to *arity operations.*

**Examples 3.2 :**

(a) By definition all nullary operations on $A$ are of the form $\emptyset \in A^0 \mapsto a \in A$, and there is (obviously) a unique such mapping for each $a \in A$. The nullary operations on $A$ can thereby be identified with the points of $A$, and we will adhere to this convention. Put another way: *each nullary operation on $A$ amounts to "distinguishing"* (or "focusing on", or "picking out") *a specific point of $A$.*

(b) In any ring $R$ the *additive inverse mapping* $r \in R \mapsto -r \in R$ is a unary operation.

(c) In any group $G$ the *multiplicative inverse mapping* $g \in G \mapsto g^{-1} \in G$ sending an element to its multiplicative[32] inverse is a unary operation.

(d) Any derivation $\delta : R \to R$ on a ring $R$ is a unary operation.

(e) Addition and multiplication are binary operations on any ring.

[30] "n-ary" rhymes with "plenary," although for some it rhymes with "hen dairy" (emphasis on "hen"). The controversy was supposedly resolved by the Treaty of Btfsplk, but this seems not to be the case.

[31] "nullary" rhymes with "scullery," and "unary" with "spoon-ah-ree," although for some it rhymes with "you carry" (emphasis on "you"). This matter was also on the agenda at Btfsplk.

[32] If the group were written additively the terminology would need to be changed accordingly, since (b) would then be a special case.

(f) For any ring $R$ and any $n \geq 1$ the mapping $(r_1, r_2, \ldots, r_n) \in R^n \mapsto \sum_{j=1}^{n} r_j \in R$ is an $n$-ary opertion on $R$.

(g) (Boolean operations) Let $X$ be a non-empty set and let $A := \mathcal{P}(X)$ denote the power set of $X$, i.e. the set of all subsets of $X$. Then the *complementation mapping* $S \in A \mapsto (X \setminus A) \in A$ is a unary operation on $A$, and the union and intersection mappings $(S,T) \in A \times A \mapsto S \cup T$ and $(S,T) \in A \times A \mapsto S \cap T$ are binary operations on $A$.

Any collection of arity operations on $A$ gives rise to further arity operations by means of Cartesian products of these operations, restrictions (including constant mappings), various diagonal mappings $\mathrm{diag}_n(A) : a \in A \mapsto (a, a, \ldots, a) \in A^n$ with $n \geq 1$ (note these include the identity mapping $\mathrm{id}_A : A \to A$), and finite compositions thereof. Any such operations is said to be *induced* by the given operations[33].

**Examples 3.3 :**

(a) (Subtraction) Let $R$ be a ring and let $\sigma : R \times R \to R$ and $\rho : R \to R$ denote the corresponding addition and the additive inverse mappings. Then the induced operation $\sigma \circ (\mathrm{id}_A \times \rho) : (r,s) \in R^2 \mapsto \sigma(r, \rho(s)) = r - s \in R$ is a binary operation.

(b) Let $G$ be a group and let $\mu : G \times G \to G$ and $\rho : G \to G$ denote the corresponding multiplication and multiplicative inverse mappings. Then the induced operation $\mu \circ (\rho \times \mathrm{id}_A) : (g,h) \in G^2 \mapsto g^{-1}h \in G$ is a binary operation.

(c) (Fixing variables) Let $f : A \times A \to A$ be a binary operation on a set $A$ and let $b$ be an element of $A$. Then the mappings $a \in A \mapsto f(b,a) \in A$ and $a \in A \mapsto f(a,b) \in A$, which amount to restricting $f$ to $\{b\} \times A$ and $A \times \{b\}$ respectively, are unary operations on $A$. One might describe these induced mappings as resulting by "fixing the value of first variable at (i.e. to be) $b$" and "fixing the value of the second variable at $b$" respectively. Similarly, if $f : A^n \to A$ and $1 \leq m \leq n$, then by fixing (the values of) $m$ variables one obtains an $(n-m)$-ary operation on $A$.

[33]The definition is, admittedly, somewhat vague. It would seem preferable to impose some sort of "algebraic" structure on the collection of all arity operations on a set, and when this set is a $T$-algebra define the operations "induced" the collection $\{t_A\}$ to be the operations within the intersection of all those "algebras" containing the given collection. However, since the several conditions we have stated explicitly are easily verified in all the examples we will encounter, the added generality did not seem worth the effort.

# 4. Algebras of UA Type T

*In this section* $T = (T, \text{ar})$ *denotes a UA type.*

A set $A$ is an[34] *algebra of UA type* $T$, which for brevity we refer to as a[35] $T$-*algebra*, if for each $n \in \mathbb{N}$ with $T_n \neq \emptyset$ there is a mapping from $T_n$ into the $n$-ary operations on $A$. The $n$-ary function $A^n \to A$ associated with an element $t \in T_n$ is denoted[36] $t_A$ when $n > 0$, but when $n = 0$ the convention introduced in Example 3.2(a) is adopted: in that instance $t_A$ denotes the value at $\emptyset$ of the nullary function $\{\emptyset\} \to A$ associated with $t$. Thus

$$
t \in T_n \quad \Rightarrow \quad \begin{cases} t_A \in A & \text{if } n = 0, \quad \text{whereas} \\ t_A : A^n \to A & \text{if } n > 0. \end{cases} \tag{4.1}
$$

In particular: *to define a* $T$*-algebra structure on a set* $A$ *it suffices to associate a point of* $A$ *with each* $t \in T_0$ (the points need not be distinct) *and, for each* $n > 0$*, an* $n$*-ary operation* $t_A : A^n \to A$ *with each* $t \in T_n$ (the $n$-ary operations associated with distinct points of a fixed $T_n$ need not be distinct).

Any $T$-algebra is assumed to possess all those arity operations induced by the various $t_A$. In other words, when working with a $T$-algebra one has all these induced operations at one's disposal.

In analogy with the terminology used with group actions, when $A$ is a $T$-algebra one says that the "operations of $T$ act on $A$," or simply that[37] $T$ "acts on $A$." When there is a need to distinguish the set $A$ from the UA type $T$, i.e. to regard $A$ independently of the operations associated with $T$, we refer to $A$ as the *underlying set* of the $T$-algebra[38].

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[34]One could just as easily introduce the notation of a *coalgebra of UC type*: one would replace $n$-ary operations $A^n \to A$ with $n$-airy *co-operations* $A \to \coprod_{j=1}^{n} A$, wherein the coproduct symbol $\coprod$ indicates the disjoint union. These have no doubt been studied, but most likely using different terminology.

[35]The terminology "$T$-algebra" is fairly common, but does not always have the meaning we have assigned. We are following [B-M, Chapter I, §1, Definition 1.5, p. 2]. In [Roth, Chapter 2, §2.2, p. 12], what we call a $T$-algebra is called a *term algebra*.

[36]Notation such as $t_{A^n}$ might seem preferable, but becomes a bit cumbersome.

[37]The "$T$ acts on $A$" terminology is not standard, but I find it convenient.

[38]Again there is a simple analogy with conventions used in topology: a topological space $X$ is, if one wants to be reasonably precise, an ordered pair $(X, \tau)$ consisting of a set $X$ and a collection $\tau$ of subsets thereof satisfying certain properties (which are assumed familiar). However, in practice one condenses the notation $(X, \tau)$ to $X$, whereupon "$X$" suddenly represents two distinct entities: the pair $(X, \tau)$, and the set "underlying" the topological space.

**Examples 4.2 :**

(a) Let $T = \{t_{21}, t_{22}\} = T_2$, let $R$ be a ring, and let $\alpha : R \times R \to R$ and $\mu : R \times R \to R$ be the respective addition and multiplication functions on $R$. Then the mapping

$$\text{(i)} \qquad \begin{cases} t_{21} & \mapsto & \alpha \\ t_{22} & \mapsto & \mu \end{cases}$$

gives $R$ the structure of a $T$-algebra.

In this description we are regarding the ring $R$ as (at least) a 3-tuple $(R, \alpha, \mu)$ of sets, with the functions $\alpha$ and $\mu$ being subsets of $(R \times R) \times R$. In particular, the symbols $t_{21}$, $t_{22}$, $\alpha$ and $\beta$ in (i) are to be regarded as representing sets.

(b) Example (a) was presented in a manner which (hopefully) can easily be understood by any mathematician. But it texts on Universal Algebra and/or Model Theory it would be presented more along the following lines[39].

Let $T = \{+, \cdot\}$. (It is implicit from the use of the symbols $+$ and $\cdot$ that the two operations are assumed binary.) Then any ring $R$ is a $T$-algebra: map $+$ to the addition function $(r_1, r_2) \in R \times R \mapsto r_1 + r_2 \in R$ and $\cdot$ to the multiplication function $(r_1, r_2) \in R \times R \mapsto r_1 r_2 \in R$. (We now begin to see the reason for the $+, \cdot$ notation.)

As the ring example ending the previous paragraph suggests, when $A$ is a $T$-algebra and symbolic notation (such as $+$ or $\cdot$) is used to represent a binary operation $t \in T$ the same notation is generally used in connection with the corresponding function $t_A : A^2 \to A$. For example, when $T = \{+, \cdot\}$ and $a_1, a_2 \in A$ one would write the function value $+_A(a_1, a_2)$ as $a_1 + a_2$ and the function value $\cdot_A(a_1, a_2)$ as $a_1 \cdot a_2$. Similar abbreviations are also used for unary operations: if $T = T_1 = \{-\}$ and $a \in A$ the corresponding function value $-_A(a)$ would be written as $-a$ if, say, $A$ were a ring and one had additive inverses in mind. On the other hand, when a multiplicative group $G$ is regarded as a $\{-\}$-algebra the function value $-_G(g)$ at a point $g \in G$ would be denoted $g^{-1}$.

Such abbreviations are also used with nullary operations, but in that context one must not forget that the actions on a $T$-algebra $A$ are identified with points

[39] In fact such a presentation would probably be little (or no) more than: Any ring is a $\{+, \cdot\}$-algebra.

of $A$. For example, if $T = \{0\}$ the (unique) value $0_A(\emptyset)$ of the corresponding function $0_A : \{\emptyset\} \to A$ would be written as $0$. (Think of it this way: the function $0_A$ "picks out" the element of $A$ which one wishes to designate by means of the symbol $0$. This adds a bit of formalism to such statements as "the additive identity of any ring will be denoted by $0$.")

(c) In Example (b) one could assume $R$ is the trivial ring $0$, in which case the addition and multiplication functions are identical. Viewing this ring as a $T$-algebra creates a distinction between these two operations which is otherwise absent.

(d) Any ring is also an $S$-algebra for $S = \{+\}$ or $\{\cdot\}$. In particular, there is no claim that both binary operations of the ring are under consideration. Similarly, there is no claim that rings are the only $\{+, \cdot\}$-algebras.

A mapping $f : A \to B$ between $T$-algebras is a *homomorphism* (*of $T$-algebras*) if

$$f(t_A(a_1, a_2, \ldots, a_n)) = t_B(f(a_1), f(a_2), \ldots, f(a_n)) \qquad \text{for all} \qquad t \in T, \tag{4.3}$$

which for $n = 0$ we take to mean

$$f(t_A) = t_B \tag{4.4}$$

(recall (4.1)). Example: Any ring homomorphism is a homomorphism of $\{+, \cdot\}$-algebras.

Conditions (4.3) and (4.4) are often summarized by the statement: *all operations of $T$ are preserved.* One might also[40] describe these conditions by: $f$ is "equivariant" w.r.t. the actions of the operations of $T$ on $A$ and $B$.

It should be evident that we are dealing with a category: the objects are $T$-algebras; the morphisms are homomorphisms between such entities. This is the *category of $T$-algebras*, which we denote by $\mathcal{A}_T$. When we refer to an *isomorphism* of $T$-algebras, this context is implicit. To keep the category in mind we will often refer to $T$-algebra homomorphisms as morphisms.

Suppose $A$ is a $T$-algebra and $B$ is a subset of $A$. We say that $B$ is a $T$-*subalgebra* of $A$ if $B$ is "closed under every $t_A$," i.e. if

$$t_A(b_1, b_2, \ldots, b_n) \in B \qquad \text{whenever} \qquad b_1, b_2, \ldots, b_n \in B \quad \text{and} \quad t \in T_n. \tag{4.5}$$

[40]But one does not.

Example: any subring of a ring $R$ is a $\{+,\cdot\}$-subalgebra of the $\{+,\cdot\}$-algebra $R$. If $A$ and $B$ are $T$-algebras satisfying $B \subset A$ and $t_B = t_A|_{B^{\mathrm{ar}(t)}}$ for all $t \in T$ then $A$ is an *extension* of $B$, and when that is the case $B$ is clearly a $T$-subalgebra of $A$.

Our immediate goal is to prove the existence of free objects in $\mathcal{A}_T$. For reasons which will become clear in §8, this will require a substantial amount of preliminary work.

# 5. The Recursion Theorem

The Recursion Theorem is the theoretical justification for allowing recursive definitions in mathematics, i.e. "definitions by induction" as exemplified by (3.1). We present this classical result as a "warm up" for our discussion of set theory; we will eventually require a generalization.

**Theorem 5.1 (The Recursion Theorem) :** *Let $X$ be a non-empty set, let $x_0 \in X$, and let $f : X \to X$ be any function. Then there is a unique function $g : \mathbb{N} \to X$ such that*

(i) $$g(0) = x_0$$

*and is such that the diagram*

(ii) $$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle g} \\ \mathbb{N} & \xrightarrow{n \mapsto n+1} & \mathbb{N} \end{array}$$

*commutes. Equivalently: there is a unique function $g : \mathbb{N} \to X$ such that* (i) *holds and*

(iii) $$g(n+1) = f(g(n)) \qquad \textit{for all} \qquad n \geq 0.$$

The theorem is attributed to Dedekind (1888) in [Potter$_2$, Chapter II, §5.3, (5.3.1), p. 93].

**Proof :** In the proof[41] we view functions set-theoretically as in Example 4.2(a), i.e. we view a function $p : Y \to Z$ between sets $Y$ and $Z$ as a subset $p \subset Y \times Z$ such that for each $y \in Y$ there is precisely one $z \in Z$ such that $(y, z) \in p$.

Let $C \subset \mathcal{P}(\mathbb{N} \times X)$ denote all those subsets $A \subset \mathbb{N} \times X$ which contain $(0, x_0)$ and satisfy $(n+1, f(x)) \in A$ whenever $(n, x) \in A$. The collection obviously contains $\mathbb{N} \times X$, and is therefore non-empty.

We will show that the intersection $g := \cap_{A \in C} A$ is a function from $\mathbb{Z}$ into $X$ satisfying properties (i)-(iii). To establish this it suffices to prove that

(iv) $$W = \mathbb{N},$$

[41] Which is adapted from [Hal, Section 12, pp. 48-9].

where $W \subset \mathbb{Z}$ denotes the collection of those $n \in \mathbb{Z}$ such that $(n,x) \in g$ for precisely one $x \in X$. We argue by induction on $n$.

By construction we have $(0, x_0) \in g$. If $(0,y) \in g$ also holds and $y \neq x_0$ then $\widehat{A} := g \setminus \{(0,y)\} \in C$. Since $\widehat{A}$ is a proper subset of $g$, this would contradict the fact that $g \subset A$ for all $A \in C$. The case $n = 0$ is thereby established.

Now suppose $0 \leq n \in \mathbb{N}$ and $n \in W$, hence that there is a unique $x \in X$ such that $(n,x) \in g$. Then from the definition of $C$ we have $(n+1, f(x)) \in g$, and if $n+1 \notin g$ there must be a $y \in X$, $y \neq f(x)$, such that $(n+1, y) \in g$. In this instance one checks that[42] the set $\widetilde{A} := g \setminus \{(n+1,y)\}$ satisfies $\widetilde{A} \in C$, and we again contradict the fact that $g \subset A$ for all $A \in C$.

Turning to uniqueness, suppose $\widehat{g} : \mathbb{N} \to X$ also satisfies (i)-(iii). Then from (i) we have $\widehat{g}(0) = x_0 = g(0)$, and $\widehat{g}$ and $g$ therefore argree at 0. However, if $0 \leq n \in \mathbb{N}$ and $\widehat{g}$ and $g$ agree at $n$ then from (iii) we see that

$$\widehat{g}(n+1) = f(\widehat{g}(n)) = f(g(n)) = g(n+1).$$

The two functions $\widehat{g}$ and $g$ therefore agree at all $n \in \mathbb{N}$, and $\widehat{g} = g$ is thereby established. **q.e.d.**

It is generally easy to sense when the Recursion Theorem is being used implicitly, but even then the associated set $X$ and the function $f : X \to X$ may not be so easy to determine. The following formulation can sometimes be helpful in that regard.

**Corollary 5.2**[43]**:** *Let $X$ be a non-empty set, let $x_0 \in X$, and suppose $\{f_n : X \to X\}_{n \in \mathbb{N}}$ is any family of functions. Then there is a unique function $g : \mathbb{N} \to X$ such that*

(i) $$g(0) = x_0$$

*and such that for each $n \in \mathbb{N}$ the diagram*

(ii) $$\begin{array}{ccc} X & \xrightarrow{f_n} & X \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle g} \\ \mathbb{N} & \xrightarrow{m \mapsto m+1} & \mathbb{N} \end{array}$$

*commutes. Equivalently: there is a unique function $g : \mathbb{N} \to X$ such that* (i) *holds and*

(iii) $$g(n+1) = f_n(g(n)) \qquad \textit{for all} \qquad n \geq 0.$$

[42] For more detail see [Hal, Section 12, p. 49].

[43] See [Potter$_2$, Chapter II, §5.3, (5.3.3), p. 94].

**Proof :** Define $\widetilde{X} := \mathbb{N} \times X$ and define $\widetilde{f} : \widetilde{X} \to \widetilde{X}$ by

(iv) $$\widetilde{f} : (n, x) \mapsto (n + 1, f_n(x)).$$

Then by Theorem 5.1 there is a unique function $\widetilde{g} : \mathbb{N} \to \widetilde{X}$ such that

(v) $$\widetilde{g}(0) = (0, x_0)$$

and

(vi) $$\widetilde{g}(n + 1) = \widetilde{f}(\widetilde{g}(n)) \qquad \text{for all} \qquad n \in \mathbb{N}.$$

Let $\pi_1 : (n, x) \to \mathbb{N}$ and $\pi_2 : (n, x) \in \widetilde{X}$ denote the projections onto the first and second factors respectively, i.e. $\pi_1 : (n, x) \mapsto n$ and $\pi_2 : (n, x) \mapsto x$. Define $g := \pi_2 \circ \widetilde{g} : \mathbb{N} \to \mathbb{N}$ and write $\widetilde{g}$ accordingly, i.e. as

$$\widetilde{g}(n) = ((\pi_1 \circ \widetilde{g})(n), g(n)), \qquad n \in \mathbb{N}.$$

We claim that $\widetilde{g}(n)$ must have the form

(v) $$\widetilde{g}(n) = (n, g(n)) \qquad \text{for all} \qquad n \in \mathbb{N},$$

i.e, that $\pi_1 \circ \widetilde{g} = \mathrm{id}_{\mathbb{N}}$. For $n = 0$ this is obvious from (v). If it is true for $0 \le n \in \mathbb{N}$ then we see from the calculations

$$\begin{aligned} \widetilde{g}(n+1) &= \widetilde{f}(\widetilde{g}(n)) \\ &= f(n, g(n)) && \text{(by the induction hypothesis)} \\ &= (n+1, f_n(g(n))) && \text{(by (iv))} \\ &= (n+1, g(n+1)) && \text{(because } g(n+1) := \pi_2(\widetilde{g}(n+1))) \end{aligned}$$

that it also holds for $n + 1$, and the claim is thereby established.

Equality (iii) is apparent from the final two lines of this last calculation. **q.e.d.**

**Corollary 5.3 :** *There is precisely one function $h : \mathbb{N} \to \mathbb{N}$ such that*

$$h(n) = n! := n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1 \qquad \textit{for all} \qquad n \ge 0.$$

**Proof :** Take $X := \mathbb{N}$ and $f_n : m \in \mathbb{N} \mapsto nm \in \mathbb{N}$ in Corollary 5.2. (Note that the collection $\{f_n\}$ is not defined recursively.) **q.e.d.**

In practice this corollary would most likely be expressed in the form of a definition: the "factorial" $n!$ of any integer $n \in \mathbb{N}$ is defined by[44]

$$(5.4) \qquad n! := \begin{cases} 1 & \text{if } n = 0\,, \\ n \cdot (n-1)! & \text{if } n \geq 1. \end{cases}$$

Unfortunately, even after one works through the usual axioms of set theory[45] and the consequent principle of mathematical induction (which is assumed familiar), it is not at all clear mathematically, even though it seems quite clear intuitively, that (5.4) provides a sufficient amount of information to specify a unique function[46] $g : \mathbb{N} \to \mathbb{N}$. The Recursion Theorem, which stands imposingly behind Corollary 5.3, is what saves the day: the information is indeed sufficient. In general, *one should be able to back up any definition by recursion by appealing to the Recursion Theorem*[47].

[44]Of course there is also a non-recursive definition of $n!$ for $n \geq 0$, i.e.

$$n! := \Gamma(n+1),$$

where $\Gamma : x \in (0, \infty) \mapsto \int_0^\infty t^{x-1}e^{-t}\,dt$ is the classical Gamma function (see, e.g. [Rudin, Chapter 8, §8.17-22, pp. 192-195]). But for the proof of interest by Lang there is no evident alternate approach to defining the relevant sequence of fields.

[45]Those axioms will be reviewed in the following section.

[46]A function $f : X \to Y$ between sets $X$ and $Y$ is, by definition, a subset of $X \times Y$ satisfying certain properties, and the axioms of set theory are quite specific and fairly restrictive with regard to how subsets can be defined.

[47]On the other hand, if one did this with every recursive definition encountered one would never make any progress on the actual topic of interest.

## 6. Remarks on the Set-Theoretical Foundations of Category Theory

To continue with the study of $T$-algebras we need a somewhat lengthy digression on set theory for reasons described in the Introduction. Readers comfortable with set/class distinctions and category theory should consider skipping to §7 immediately.

When set theory was first introduced it appeared to unify a great deal of mathematics in a very efficient way, but it was quickly realized that one had to be careful about just what collections could be regarded as sets. For example, *the collection of all sets cannot qualify as a set*: for if the collection $X$ of all sets is a set we could use the Axiom of Specification (see (b) below) to construct the subset $Y$ consisting of those sets $x$ satisfying $x \notin x$. One now asks if $Y \in Y$. If so it must be the case that $Y \notin Y$, which is an obvious contradiction. On the other hand, the alternative $Y \notin Y$ fares no better; the contradiction[48] is now $Y \in Y$. This indirect proof, before being interpreted as such, was known as *Russell's paradox*.

The confusion engendered by such a simple argument leads to an obvious question: how can one be confident, in any particular mathematical discussion, that when one refers to certain collections as "sets" they are deserving of that designation? Thus far the general agreement has been to base set theory on a collection of axioms which seem intuitively plausible, and which collectively eliminate all known paradoxes, and to adhere to these axioms, at least implicitly, when doing mathematics. In such an approach "set" (some prefer "class") and "membership" are taken as primitive concepts, i.e. the terms are never defined, but at an intuitive level it can be useful to think of a set as a mound of pebbles, possibly consisting of just one pebble, or as a collection of mounds of pebbles, and of "membership" in one of these sets as being one of the pebbles within that mound, or the single pebble, or as one of the mounds within that collection of mounds. A real-world symbol $\in$, which we call the *membership symbol*, is introduced to help in describing sets and their members, and real-world rules, i.e. axioms, based on experience and needs, are listed to govern the ways in which one is allowed to deal with sets.

Preferably all the axioms should be "based on experience," i.e. they should be "intuitively plausible" in the spirit of the axioms of plane Euclidean geometry. In that subject "point" and "line" are primitives, but because of the mental images these terms elicit it does not take a great deal of effort for the mathematically inclined to accept an axiom such as "two distinct lines meet (i.e., intersect) in at most one

[48] From the point of view of pure logic the actual contradiction, in both instances, is the sentence $(Y \in Y) \wedge (Y \notin Y)$. (One reads the symbol $\wedge$ as "and.")

point." One might regard "the union of two sets is a set" as a comparable axiom in set theory[49].

Unfortunately, a few of the set-theoretic axioms we will list may not seem intuitively plausible, an example for this author[50] being the Axiom of Foundation. In that case the appearance is by necessity, i.e. so as to reduce the chances of uncovering another paradox similar to that of Russell.

The current leaders as far as formal axiomatic set-theoretic systems are concerned are the two[51] associated with the name pairs[52] Zermelo-Fraenkel (ZF) and Hilbert-Bernays (HB). However, for mathematicians who do not specialize in that area, or in foundations, a less formal ("naive") approach generally suffices in everyday practice[53]. That is the route we will take.

In what follows the symbol $\Rightarrow$ should be read as "implies" unless otherwise stated, and the symbol $\Leftrightarrow$ as "if and only if" or "is equivalent to."

Here is our list[54]:

(a) (The Axiom of Extension) Two sets $X$ are equal if and only if $x \in X \Leftrightarrow x \in Y$;

- This axiom governs how the equality sign "=" (which for this author belongs to real-world logic) is to be used in connection with the membership symbol $\in$.

- One reads $a \in B$ as "$a$ is in $B$," or "$a$ is a member of $B$," or "$a$ is an element of $B$." These last two choices leave one with the impression that "members of sets" and "elements of sets" are something other than sets. That is a false impression:

*the elements of a set can only be sets.*

In more formal presentations of set theory, e.g. the appendix to [Kelly], one generally sticks with small letters, e.g. one would write

[49]Readers are assumed familiar with unions, intersections and complements of sets, i.e. with a working knowledge of set theory as has been used in the previous sections of these notes. However, readers are not assumed familiar with formal presentations of the discipline.

[50]Intuition is, after all, subjective.

[51]For workers in set theory the Morse-Kelly (MK) axioms are also quite important. See the appendix on set theory in [Kelly].

[52]Technically, ZFC and HBC. The C indicates that the Axiom of Choice [in our list in the form or Zermelo's postulate] is included.

[53]The paradigm for such an approach is [Hal], but that reference does not encompass category theory.

[54]What we list are basically the ZF axioms. For more precise statements see e.g. [B-M, Chapter VI, §2, pp. 52-56].

$a \in b$ rather than $a \in B$, thereby avoiding being fooled by the notation. We will, nevertheless, adopt that misleading notation: it is used in practically all mathematics not concerned with formal set theory or foundations.

• If $A$ and $B$ are sets and $x \in A \;\Rightarrow\; x \in B$ we say that $A$ is a *subset* of $B$, or that $B$ *contains* $A$, and indicate this by writing $A \subset B$ or $B \supset A$ respectively. Note that $A = B \;\Rightarrow\; A \subset B$. $A$ is a *proper subset* of $B$, or $B$ *properly contains* $A$, if $A \subset B$ and $A \neq B$. From the Axiom of Extension we see that

$$X = Y \qquad \Leftrightarrow \qquad X \subset Y \text{ and } Y \subset X.$$

(b) (The Axiom of Specification[55]) If $X$ is a set, and if $p(t)$ is a statement[56] having the property that $p(x)$ can be determined either true or false for each $x \in X$, then there is a set $Y$ which consists precisely of those $x \in X$ for which $p(x)$ is true (in particular, $Y \subset X$);

• One generally specifies the set $Y$ by writing $\{\, x \in X \;:\; p(x)\,\}$. Including the "$\in X$" in this notation is crucial. For example, writing $\{\, x \;:\; x \notin x\,\}$ could quickly lead to the Russell paradox, whereas writing $\{\, x \in X \;:\; x \notin x\,\}$, where $X$ is the (assumed) set of all sets, could simply be the first step in a repetition of our proof by contradiction that there is no such set.

• If the existence of at least one set $X$ is assumed (which has yet to be the case), it follows by taking $p(x)$ to be the statement $x \neq x$ that there is a set $\emptyset$, called the *empty set* or *null set*, which contains no elements. Any other set is *non-empty*.

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[55] This is also called the *comprehension axiom*, or the *comprehension principle, for elements of* $X$.

[56] Halmos uses the word "condition" [Hal, Section 2, p. 6], and goes on to state (his $x$ is my $t$ and his $S(x)$ is my $p(t)$):

> A "condition" here is just a sentence. The symbolism $S(x)$ is intended to indicate that the letter $x$ is *free* in the sentence $S(x)$; that means that $x$ occurs in $S(x)$ at least once without being introduced by one of the phrases "for some $x$" or "for all $x$."

The terminologies "statement, condition" and "sentence" are admittedly vague. Readers who would like to see finer precision can find far more detail elsewhere, e.g. the first five chapters of [Eisbg]. (However, that particular reference uses the Hilbert-Bernays approach to set theory.)

- Suppose $X$ is a set containing a point $x_0$ one wishes to be removed. If this is done, and all other points remain, does the remaining collection constitute a set? The Axiom of Specification ensures this will be the case, since that result can be described by $\{\, x \in X \,:\, x \neq x_0 \,\}$.

- The *difference* $X \setminus Y$ of sets $X$ and $Y$ is defined by $\{x \in X : x \notin Y\,\}$. The Axiom of Specification guarantees this is a set. (This definition does *not* assume $Y \subset X$.)

- It will prove useful, for our purposes, to refer to the set $X$ in the statement of the Axiom of Specification as the *ambient set*. Readers are warned that this is not standard terminology.

- Let $X = \mathbb{N}$, and let $p(x)$ be the statement: $x \in \mathbb{N}$ is even, greater than $2$, and is the sum of two prime numbers. The Axiom of Specification ensures that $Y := \{x \in \mathbb{N} \,:\, p(x)\,\}$ is a set, since (in principle) one can determine if $p(x)$ is true for any given positive even integer $x$. Example: for $x := 23,212,867,370$ the statement is true since (as everybody knows) $x$ is the sum of the two (obviously[57]) prime integers $27,484,207$ and $23,185,383,163$, and we thereby conclude that $x \in Y$. The *Goldbach conjecture*, which has yet to be either established or refuted, is that $Y$ is the set of positive even integers greater than $2$.

(c) (The Axiom of Pairing) For any two sets $X$ and $Y$ there is a set $Z$ which admits both $X$ and $Y$ as elements;

- By means of the Axiom of Specification one can then guarantee the existence of a subset $\{X, Y\} \subset Z$ having only $X$ and $Y$ as elements, i.e.

$$\{X, Y\} := \{z \in Z \,:\, z = X \text{ or } z = Y\,\}.$$

$\{X, Y\}$ is the *pair* formed by $X$ and $Y$.

- When $Y = X$ one writes $\{X, Y\}$ as $\{X\}$ and refers to this last set as a “singleton” (set).

[57] Particularly when you have a good computer algebra package on your office computer!

• When $Y$ is replaced by $\{X, Y\}$ one writes the pair $\{X, \{X, Y\}\}$ as $(X, Y)$ are refers to such sets as "ordered pairs". The definition enables one to prove that $(X, Y) = (Z, W)$ if and only if $X = Z$ and $Y = W$.

(d) The Cartesian product of any two sets is a set;

• The definition of the Cartesian product of two sets is assumed familiar. However, since the concept will soon arise in greater generality, it seems worth repeating here. The Cartesian product $X \times Y$ of sets $X$ and $Y$ is the collection of all ordered pairs $(x, y)$ with $x \in X$ and $y \in Y$.

• A *function* between sets $X$ and $Y$ is defined to be a subset $f \subset X \times Y$ such that for each $x \in X$, $(x, y_1) \in f$ and $(x, y_2) \in f \Rightarrow y_1 = y_2$. One indicates such an $f$ by the notation $f : X \to Y$, and for any ordered pair $(x, y) \in f$ one refers to $y$ as the *value of $f$ at $x$* and indicates this by writing $y$ as $f(x)$.

• The *range* of a function $f : X \to Y$ between two sets is denoted $f(X)$ and consists of those $y \in Y$ such that $y = f(x)$ for some $x \in X$.

• A function $f : X \to Y$ between sets $X$ and $Y$ is: *injective*, or is *one*(*-to*)*-one*, or is an *injection*, if $x_1, x_2 \in X$ and $x_1 \neq x_2$ imply $f(x_1) \neq f(x_2)$; *surjective*, or is *onto*, or is a *surjection*, if $Y = f(X)$; *bijective*, or is *one*(*-to*)*-one onto*, or is a *bijection*, if it is both injective and surjective.

(e) (The Axiom of Powers) Given any set $X$ there is a set $Y$ with the property that every subset of $X$ is an element of $Y$.

• By means of the Axiom of Specification one can then guarantee the existence of a set consisting precisely of the subsets of $X$, i.e.

$$\mathcal{P}(X) := \{\, y \in Y \ : \ y \subset X \,\}.$$

$\mathcal{P}(X)$ is the *power set* of $X$.

• Note that the assignment $x \in X \mapsto \{x\} \in \mathcal{P}(X)$ defines an injection $\sigma_{\mathcal{P}(X)} : X \to \mathcal{P}(X)$, which at the intuitive level would indicate that $\mathcal{P}(X)$ is "at least as big" as $X$. One might also argue, again at the intuitive level, that $\mathcal{P}(X)$ is actually "bigger" than $X$ since there can be no surjection $\tau : X \to \mathcal{P}(X)$. Indeed, if such a $\tau$ exists the set $S = \{x \in X : x \notin \tau(x)\}$ would be in the range of $\tau$, and we could therefore pick a point $x_0 \in X$ such that $\tau(x_0) = S$. From the definition of $S$ we then see that

$$x_0 \in S \qquad \Leftrightarrow \qquad x_0 \notin \tau(x) \qquad \Leftrightarrow \qquad x_0 \notin S,$$

and we therefore have a contradiction[58].

(f) (The Axiom of Replacement[59]) Let $X$ be a set and let $p(s,t)$ be a statement having the the following two properties:

- $p(x,y)$ can be determined either true of false for each pair of sets $(x,y)$ with $x \in X$; and
- $p(s,t)$ is "functional" for such pairs in the sense that
  - for each $x \in X$ there is a set $y$ such that $p(x,y)$ is true, and
  - if $p(x,y)$ is true and $p(x,z)$ is true then $y = z$.

Then there is a set $Y$ consisting of those sets $y$ such that $p(x,y)$ is true for some $x \in X$.

• The simplest consequence of this axiom is that: *the range of any function is a set.*

**Proof :** Let $f : X \to W$ be a function between sets $X$ and $W$ and let $p(s,t)$ be the statement $t \in W$ and $(s,t) \in f$. Then the collection of sets $y$ such that $p(x,y)$ is true for some $x \in X$ is precisely the image $f(X)$ of $f$. **q.e.d.**

[58]The argument is curiously similar to that associated with Russell's paradox.

[59]The statement is adapted from [Mac$_2$, Chapter I, §6, p. 23]. The result is also known as the Axiom of Substitution.

• The range of a function $f : X \to Y$ between sets is also called the *image* of the function, or the *image of* $X$ (*under* $f$). Since for any subset $A \subset X$ the restriction[60] $f|_A : A \to Y$ is again a function, the *image* $f|_A(A)$ is again a set. This set is written $f(A)$ and is called the *image* (*of* $A$) (*under* $f$).

• The image of a function $f : X \to Y$ between sets is also called a *family of elements of* $Y$, or a *family of sets*, the sets being the elements of $f(X)$, and when this terminology is used the notation and terminology undergo radical changes: in place of $f : X \to Y$ one writes $\{y_x\}_{x \in X}$, without specific mention of $f$, but with the understanding that

$$y_x := f(x),$$

and one refers to $X$ as the *index*(*ing*) *set* (of the family). Specific reference to $Y$ is also quite often omitted.

• One of the major reasons for introducing the Axiom of Replacement is to enable one to deal with "ordinal numbers." Since we will have no occasion to do so in these notes, we refer interested readers elsewhere, e.g. to [Hal, §19] or to [Roth, Chapter 7, §5, Lemma 7.5.5, pp. 100-101].

(g) (The Axiom of Unions) For any set $X$ there is a set $\cup X$ consisting of all sets $z$ having the property that $z \in x$ for some $x \in X$. $\cup X$ is the *union* of the elements of $X$.

• This definition is often written

$$\cup X := \{\, z \,:\, z \in x \;\text{ for some }\; x \in X \,\},$$

but it is worth mentioning explicitly that the notation does *not* indicate an application of the Axiom of Specification: no ambient set has been given.

• When $X = \{A, B\}$ is a pair one writes $\cup X$ as $A \cup B$, the displayed definition in the previous item would generally be written

$$A \cup B := \{\, x \,:\, x \in A \;\text{ or }\; x \in B \,\},$$

[60] Readers are assumed familiar with this terminology.

and one would refer to $A \cup B$ as the *union of the sets* $A$ and $B$. This construction with sets is assumed thoroughly familiar to readers, but that might not be the case that $\cup X$ is familiar, hence relating them seems appropriate.

• Note that

$$\cup\emptyset = \emptyset.$$

Otherwise there is a set $z$ such that $z \in x$ for some $x \in \emptyset$, whereas $x \in \emptyset$ contradicts the fact that $\emptyset$ has no points.

• Intersections get little attention in axiomatic treatments of set theory since they are easily handled using (b) or variations thereof. The major thing one needs to keep in mind is that $\cap\emptyset$ is not defined (and the same therefore holds for empty families of sets, i.e. families indexed by the empty set): otherwise contradictions result (just as with attempts to define division within rings by $0$). See, for example, [Hal, Section 9, p. 35].

(h) (The Axiom of Foundation[61]) Every non-empty set $X$ contains an element $y_X$ such that if $z \in y_X$ then $z \notin X$.

• Given sets $A$ and $B$, think of $A \in B$ as $A$ being "below" $B$. The Axiom of Foundation asserts that nothing in $X$ is below $y_X$, hence that $y_X$ is at the "foundation" of $X$ (but need not be unique in that regard).

• A statement equivalent to the Axiom of Foundation is: each non-empty set $X$ contains an element $y_X$ such that $X \cap y_X = \emptyset$.

• When taken in combination with the Axiom of Specification this axiom provides further insurance against paradoxes of the type discovered by Russell. Indeed, an immediate consequence is: *no set can be an element of itself.*

**Proof :** This is obvious for the empty set, since that set has no elements. So assume $X$ is a non-empty set and $X \in X$. By the Axiom of Foundation there is an element $y_X \in X$ such that $y_X \notin X$, and we are thereby faced with the contradiction $y_X \in X$ and $y_X \notin X$. **q.e.d.**

[61]This is also called the Axiom of Regularity.

(i) (The Axiom of Infinity) The collection $\mathbb{N}$ of natural numbers is a set;

- This is the first axiom which asserts the existence of a set.

- The *successor* of a set $X$ is defined to be the set $X^+ := X \cup \{X\}$. The Axiom of Infinity is often stated in the more general form: *there is a set which contains the empty set and also contains the successor of each of its elements.* One can then define $0 := \emptyset,\ 1 := 0^+,\ 2 := 1^+$, etc. and apply the Axiom of Specification to this set to realize $\mathbb{N}$ as a subset (and therefore as a set). Moreover, one can subsequently prove that $\mathbb{N}$ satisfies the "Peano Axioms" and establish the principle of mathematical induction. From that point one can formulate rigorous definitions of addition an multiplication in $\mathbb{N}$ and prove all the familiar properties of, and the relationship between, these binary operations, e.g. associativity and the distributive law[62]. Working though such constructions would be too much of a diversion for our purposes. All that we require are the initial statement and the principle of mathematical induction[63].

and

(j) (Zermelo's Postulate) If $\{X_\alpha\}$ is any family of pairwise disjoint non-empty sets there is a set $Y$ such that $Y \cap X_\alpha$ is a singleton for each index $\alpha$.

In the everyday practice of mathematics one frequently encounters "families of subsets" of a given set $X$. (For example, a topology on a set $X$ is defined as a family of subsets satisfying various properties.) The terminology refers to the image of a function $f : I \to \mathcal{P}(X)$ and, so as to conform with the notation introduced in the final bulleted item of (f), would generally be written

$$\{X_\iota\}_{\iota \in I}, \qquad \text{wherein} \qquad X_\iota := f(\iota). \tag{6.1}$$

In this context the union of the elements of $f(X)$ is expressed in a somewhat different (and probably far more familiar) way. By definition we have (after adjusting notation to fit the present context)

$$\cup f(I) := \{\, x : x \in y \ \text{ for some } \ y \in f(I)\,\}.$$

[62]See, e.g. [Hal, §12, pp. 46-49].
[63]Which we have no intention of proving!

However, the equivalences

$$\begin{aligned} y \in f(I) \quad &\Leftrightarrow \quad y = f(\iota) \ \text{ for some } \ \iota \in I \\ &\Leftrightarrow \quad y = X_\iota \ \text{ for some } \ \iota \in I, \end{aligned}$$

imply that $\cup f(I)$ could also be written

$$\cup f(I) := \{\, x \,:\, x \in X_\iota \ \text{ for some } \iota \in I \,\}.$$

In fact (as every reader knows!) one writes the definition of $\cup f(I)$ in this case as

$$\cup_{\iota\in I} X_\iota := \{\, x \,:\, x \in X_\iota \ \text{ for some } \iota \in I \,\}, \tag{6.2}$$

and one generally refers to the "union of the family $\{X_\iota\}_{\iota\in I}$" without explicitly mentioning the function $f$. Moreover, when $I$ is understood, or naming this index set is not essential, one would abbreviate $\{X_\iota\}_{\iota\in I}$ to $\{X_\iota\}$, and $\cup_{\iota\in I}X_\iota$ to $\cup_\iota X_\iota$, or even to $\cup X_\iota$.

One goes a bit further when $I = \mathbb{N}$ (or subsets thereof, by means of simple modifications of what follows), in which case one can meaningfully speak of finite, countable, and at-most countable sets. In those instances examples of the notation used (assumed thoroughly familiar to readers) are: $\cup_{j=0}^n A_j$ (for finite unions) and $\cup_{j=0}^\infty A_j$ (for [countably] infinite unions), although in the second case one might just as likely use $\cup_{n\in\mathbb{N}} A_n$.

Zermelo's Postulate is equivalent to (but is, at least for this author, far easier to grasp intuitively than) the "Axiom of Choice". To detail that axiom and verify this assertion define the *Cartesian product* $\prod_{\alpha\in I} X_\alpha$ of any family of (sub)sets (of some set $X$) to be the collection of all functions $f : I \to \cup_{\alpha\in I} X_\alpha$ such that $f(\alpha) \in X_\alpha$ for all $\alpha \in I$. It is easily seen from the Axiom of Specialization that this collection $\prod_{\alpha\in I} X_\alpha$ of subsets of $I \times \cup_{\alpha\in I} X_\alpha$ must a set: whether or not this set is non-empty is another matter. Elements $f \in \prod_{\alpha\in I} X_\alpha$ are called *choice functions* for the family $\{X_\alpha\}$: each selects ("chooses") a single point, i.e. $f(\alpha)$, from each $X_\alpha$.

When $I = \{\alpha_1, \alpha_2\}$ has exactly two elements each such function $f : I \to \prod_\alpha X_\alpha$ is uniquely determined by the ordered pair $(f(\alpha_1), f(\alpha_2)) \in X_{\alpha_1} \times X_{\alpha_2}$ and, conversely, each ordered pair $(x_1, x_2) \in X_{\alpha_1} \times X_{\alpha_2}$ uniquely determines a function $f : I \to X_1 \cup X_2$, i.e., $\alpha_j \in I \mapsto x_j \in X_{\alpha_j}$. One easily concludes, in this restricted context, that there is is a bijection between the previously defined (using ordered pairs) Cartesian product $X_{\alpha_1} \times X_{\alpha_2}$ and our newly defined (using functions) Cartesian product $\prod_{\alpha\in I} X_\alpha$. The existence of this bijection is our justification[64] for

[64] And that of others. See, e.g. [Hal, §9, p. 36].

regarding the current (via functions) definition as a generalization of that assumed in (e).

**Theorem 6.3 :** *The following statements are equivalent:*

(a) *Zermelo's Postulate; and*

(b) (The Axiom of Choice) *the Cartesian product of any non-empty family of non-empty sets is non-empty.*

There are formulations of the Axiom of Choice which do not involve "families," e.g. see[65] [B-M, Chapter VI, §2, (ZF7), p. 55].

**Proof :**

(a) $\Rightarrow$ (b) : Given any non-empty family $\{Y_\alpha\}_{\alpha\in I}$ of non-empty subsets of a set $Y$ define a family of subsets $\{X_\alpha\}_\alpha$ of the set $X := Y\times I$ by $X_\alpha := Y_\alpha\times\{\alpha\}$, $\alpha\in I$. By construction the $X_\alpha$ are pairwise disjoint (this is the only reason for their construction from the given $Y_\alpha$), and we can therefore use Zermelo's Postulate to guarantee the existence of a set $Z$ such that $Z\cap X_\alpha$ consists of a single point $(y_\alpha,\alpha)$ for each $\alpha\in I$. The function $f:\alpha\in I\mapsto y_\alpha\in Y_\alpha$ is a choice function for the family $\{Y_\alpha\}$.

(b) $\Rightarrow$ (a) : Given any non-empty family $\{X_\alpha\}_{\alpha\in I}$ of non-empty pairwise disjoint sets let $f : I\to\cup_\alpha X_\alpha$ be a choice function. (Such $f$ exist since (b) is assumed true.) Then $Y := f(I)$ is a set by (f), and $Y\cap X_\alpha = \{f(\alpha)\}$ for each $\alpha\in I$. **q.e.d.**

Axiomatic formulations of set theory seemed perfectly acceptable[66] to the majority of working mathematicians until category theory arrived on the scene in the middle of the last century: suddenly the need to incorporate the collection of all sets into mainstream mathematics became blatantly apparent.

> *The development of category theory . . . posed problems for the set theoretic foundations of mathematics.*
>
> S. Mac Lane [$\text{Mac}_1$, p. 192].

[65] Although these authors use the "Axiom of Choice" terminology, the statement of the result is closer to the spirit of Zermelo's Postulate.

[66] Modulo a few odd consequences of the Axiom of Choice which have the status of skeletons in a family closet. The 1924 Banach-Tarski paradox is a good example, see e.g. [French]: that particular anomaly can be eliminated by replacing Zermelo's Postulate with the so-called "Axiom of Determinancy" (see, e.g. [$\text{Potter}_2$, §15.7, pp. 275-280]), but other counter-intuitive conclusions then arise.

Fortunately Mac Lane, one of the founding fathers of category theory, realized that in many contexts (which will include those of interest to us) one can circumvent the known difficulties quite easily, simply by adding one more axiom[67]. Although he used different terminology[68], what mathematicians now seem to accept, often at an unconscious level, is an axiomatic formulation of set theory along the following lines[69]:

- In the statements of all the axioms listed above[70] replace all occurrences of the word "set" by "class". Of course this change is only semantic: one now refers to the empty class, to the Cartesian product of two classes; to the power class of a class (rather than to the power set of a set); to the range of a function $f : X \to Y$ between two classes; families of (sub)classes; and to the union of the elements of a class. In particular, elements of classes are now classes[71].

- Add to these relabelled axioms the following *Axiom of the Existence of a Universe*: There is a class $U$, the elements of which are called[72] *sets*, such that:

  (k) elements of sets are sets[73]; and

  ($\ell$) axioms (a)-(i) and the corresponding bulleted items remain valid when applied to the elements of $U$ (but not to $U$ itself), and always result in elements of $U$. Specifically:

[67] An alternative "solution," adopted by many, is to be aware of the problem, but to devote minimal time worrying about it. Since the repair work can involve considerable effort, there is ample justification for assuming this position. For example, in [Eisbd, Appendix A5, §A5.1, pp. 698] one finds:

> [The fact that there there is no set of all sets leads to] . . . difficult foundational issues.
> . . . We shall take a naive approach, and simply ignore the problem.

[68] In fact in [M-B, Chapter XV, §1, pp. 506-8] Mac Lane and Birkhoff more-or-less employ our terminology, but in that reference the set theory is based on the Hilbert-Bernays (HB) axioms, which already distinguish between sets and classes. Because of this many seem to assume, when first learning about category theory, that the HB system lies at the foundations of the subject and justifies making the distinction. But the fact of the matter is, the ZF approach works just as well, as one can see from [$\text{Mac}_2$, Chapter I, §6, p. 23].

[69] Considerably more detail can be found in [$\text{Potter}_1$].

[70] This guarantees that we are still dealing with ZF theory.

[71] In HB theory "sets" refer to classes which are elements of classes. That is *not* our definition of a set.

[72] What we call classes and sets are called sets and small sets by Mac Lane. But in the experience of this author the terminology we use seems more common outside category theory.

[73] As opposed to being classes which may not be sets.

- The Axioms of Extension and Specification hold as initially stated in (a) and (b);
- the pair and ordered pair of any two sets is a set, as is the singleton set of any set;
- the Cartesian product of any two sets is a set;
- the the power set of any set is a set;
- the Axiom of Replacement holds as originally stated in (f), and as a result the range of any function between sets is a set;
- the Axiom of Unions holds as initially stated in (g);
- $\mathbb{N}$ is a set;
- Zermelo's postulate holds as initially stated in (i);
- (6.2) remains valid; and
- Theorem 6.3 remains valid.

The advantage of this extension of the axioms, aside from now being able to treat the collection of all sets as something which actually exists (if only as a class), is that one can now approach category theory in much the same manner as one approaches elementary set theory, suddenly freed from nagging worries about mysterious and vague distinctions between sets and classes. Specifically:

- one can define a *category* $\mathcal{C}$ as an ordered pair $(\mathcal{O}_{\mathcal{C}}, \mathcal{M}_{\mathcal{C}})$ of classes, the first class being the class of *objects* (of $\mathcal{C}$), and the second being the class of *morphisms* (of $\mathcal{C}$) (or *arrows* [of $\mathcal{C}$], depending on one's preference), related by various conditions (which are assumed familiar); and

- one can define a *functor* between two categories as a pair of functions, the first between the objects and the second between the morphisms, subject to various (again assumed familiar) conditions.

- Even though by definition a pair of functions, in practice a functor is most conveniently represented by just one symbol. For example, suppose $\mathcal{C}$ and $\mathcal{D}$ are categories and $T := (t_1, t_2) : (\mathcal{O}_{\mathcal{C}}, \mathcal{M}_{\mathcal{C}}) \to (\mathcal{O}_{\mathcal{D}}, \mathcal{M}_{\mathcal{C}})$ is a functor. Then for any objects $c, c_1, c_2 \in \mathcal{C}$ and any morphism $f : c_1 \to c_2$ in $\mathcal{M}_{\mathcal{C}}$ one would most likely write $t_1(c)$ as $Tc$, and (assuming the functor is covariant) $t_2(f) : t_1(c_1) \to t_2(c_2)$ as $Tf : Tc_1 \to Tc_2$.

A few specific consequences of our reformulation of the axiomatic set theory are worth recording, several of these for little more reason than to convince readers that the axioms imply the types of results, and allow the the types of proofs thereof, that one would hope for.

**Proposition 6.4 :** *The class $U$ is not a set.*

**Proof :** If $U$ were a set the same would hold for $\{ x \in U : x \notin x \}$ by the Axiom of Specification (applied to sets), and we would again be faced with the Russell paradox. **q.e.d.**

**Corollary 6.5 :** *The $U$-complement $U \setminus X := \{ y \in U : y \notin X \}$ of set $X$ is not a set.*

**Proof :** Otherwise we see from the Axiom of Unions and $U = X \cup (X \setminus U)$ that $U$ would be a set. **q.e.d.**

**Proposition 6.6 :** *Suppose $X$ is a set and $Y \in \mathcal{P}(X)$. Then $Y$ is a set.*

Equivalently: if $X$ is a set and $Y \subset X$ then $Y$ is a set. In plain English: Subclasses of sets are sets (and therefore subsets of the given sets).

**Proof :** Since $\mathcal{P}(X)$ is a set the same must hold for $Y$ by (j) (with $X$ and $x$ replaced by $\mathcal{P}(X)$ and $Y$). **q.e.d.**

**Corollary 6.7 :** *$\emptyset$ is a set.*

**Proof :** Readers are assumed familiar with the elementary set-theoretic result that $\emptyset \subset X$ for any set $X$. Since $Y \subset X \Leftrightarrow Y \in \mathcal{P}(X)$, the result can be obtained by choosing $Y = \emptyset$ in Proposition 6.6. **q.e.d.**

Intersections of classes are defined and handled in the expected way, the intersection of two classes is a class, and the intersection of two sets is a set. What about the intersection of a set with a class?

**Corollary 6.8 :** *The intersection of a set with a class is a set.*

**Proof :** The intersection is a subclass of the given set. **q.e.d.**

Now for a more substantial result.

**Proposition 6.9 :** *Suppose $f : X \to Y$ is a function from a set $X$ into a class $Y$, and if $f(x)$ is a set for each $x \in X$, then the range $f(X)$ is a set.*

**Proof :** Since $f$ is a function the statement $(s,t) \in f$ is functional in the set $X$. The result therefore follows from the Axiom of Replacement (in the original form (f)). **q.e.d.**

Mac Lane freely admits[74] that the addition of the Axiom of the Existence of a Universe does not solve all the foundational problems posed by category theory[75], but it does allow

> ... *"ordinary" Mathematics* ... [to be] *carried out exclusively within U* ...
>
> [Mac$_2$, Chapter I, §6, p. 22],

and that is quite sufficient for our purposes.

Oddly enough, for many who work in category and/or set theory the difficulties related to the set-theoretic foundations of category theory do not seem to be of major concern.

> *It seems that no book on category theory is considered complete without some remark on its set-theoretic foundations. The well-known set theorist Andreas Blass gave a talk* ... *on the interaction between category theory and set theory in which he offered three set-theoretic foundations for category theory. One was the universes of Grothendieck* ... [;] *another was systematic use of the reflection principle, which probably does provide a complete solution to the problem; but his first suggestion, and one that he clearly thought at least reasonable, was: None* [exists].
>
> From the preface to[76] [B-W].

In essence, these practitioners seem to prefer a reworking of the foundations of mathematics.

> ... *there has been considerable discussion of a foundation for category theory (and for all of Mathematics) not based on set theory.* ... *Lawvere* ... [[Law$_1$] in the references for these notes] ... *has given axioms for the elementary* (*i.e. first-order*) *theory of the category of all sets, as an alternative to the usual axioms on membership.* [Also see [Law$_2$]].
>
> S. Mac Lane [Mac$_2$, Chapter I, §6, p. 24].

[74]See [Mac$_2$, Chapter I, the final paragraph of §6, p. 24].

[75]For example, there is no concept of the class of all classes.

[76]This quote was brought to the attention of this author by Hunter College student Philip Ross.

This last quote if from the first edition of *Categories for the Working Mathematician*; the paragraph was revised in the second edition:

> *... there has been considerable discussion of a foundation for category theory (and for all of Mathematics) not based on set theory. ... axioms for the elementary* (*i.e., first-order*) *theory of the category of all sets, as an alternative to the usual axioms on menbership can be given - as an "elementary topos"* (cf. Mac Lane-Moerdijk[1992]).
>
> S. Mac Lane [$\text{Mac}_3$, Chapter I, §6, p. 24].

Since such approaches have yet to be widely accepted in "ordinary" Mathematics, in these notes we will stick with Mac Lane's universe[77].

[77] For an in-depth discussion of sets vs. classes, which includes further references, see [$\text{Potter}_2$, Appendix C, pp. 312-6]

# 7. The Recursion Theorem Revisited

In this section we reformulate the Recursion Theorem 5.1 and Corollary 5.2 in terms of classes, and then offer a few applications, including (what this author would consider) the missing details of the previously discussed proof of the existence of the algebraic closure of a field[78]. In view of the work in §5 the reformulations can be accomplished by nothing more than word replacements; in particular, there is no reason to give proofs.

**Theorem 7.1 (The Recursion Theorem) :** *Let* $X$ *be a non-empty class, let* $x_0 \in X$*, and let* $f : X \to X$ *be any function. Then there is a unique function* $g : \mathbb{N} \to X$ *such that*

(i) $$g(0) = x_0$$

*and is such that the diagram*

(ii) $$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle g} \\ \mathbb{N} & \xrightarrow{n \mapsto n+1} & \mathbb{N} \end{array}$$

*commutes. Equivalently: there is a unique function* $g : \mathbb{N} \to X$ *such that* (i) *holds and*

(iii) $$g(n+1) = f(g(n)) \qquad \text{for all} \qquad n \geq 0.$$

**Corollary 7.2 :** *Let* $X$ *be a non-empty class, let* $x_0 \in X$*, and suppose* $\{f_n : X \to X\}_{n \in \mathbb{N}}$ *is any family of functions. Then there is a unique function* $g : \mathbb{N} \to X$ *such that*

(i) $$g(0) = x_0$$

*and such that for each* $n \in \mathbb{N}$ *the diagram*

(ii) $$\begin{array}{ccc} X & \xrightarrow{f_n} & X \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle g} \\ \mathbb{N} & \xrightarrow{m \mapsto m+1} & \mathbb{N} \end{array}$$

[78] See the last few paragraphs of the introduction.

*commutes. Equivalently: there is a unique function* $g : \mathbb{N} \to X$ *such that* (i) *holds and*

$$\text{(iii)} \qquad g(n+1) = f_n(g(n)) \qquad \textit{for all} \qquad n \geq 0.$$

**Corollary 7.3 :** *Suppose in either of* Theorem 7.1 *and* Corollary 7.2 *that every element of* $X$ *is a set. Then the resulting collection* $\{g(n)\}_{n\in\mathbb{N}}$ *is a set. In particular, the set-theoretic union* $\cup_{n\in\mathbb{N}} g(n)$ *is defined, and is a set.*

**Proof :** Apply Proposition 6.9 with $f : X \to Y$ in that statement replaced by $g : \mathbb{N} \to X$. **q.e.d.**

**Examples 7.4 :**

(a) Section 5 ends with the assertion that *one should be able to back up any definition by recursion by appealing to the Recursion Theorem.* Unfortunately, for some strange reason it seems to have slipped my mind at that point that we were not in a position to do so with the definition of the $n$-fold Cartesian product of a set $A$ given in (3.1). We now have all we need.

Let $A$ be a non-empty set, let $X$ be the class $U$ of all sets, and define $f : X \to X$ by $f : S \in X \mapsto A \times S \in X$. Then by Theorem 7.1 there is a unique function $g : \mathbb{N} \to X$ such that $g(0) = A$ and $g(n+1) = f(g(n))$ for all $n \in \mathbb{N}$, i.e. such that

$$g(n) = A^n \qquad \text{for} \qquad n = 1, 2, 3, \ldots .$$

If we now define $h : \mathbb{N} \to X$ by $h(0) := \{\emptyset\}$ and $h(n) := g(n-1)$ for $n > 0$ we uncover the sequence implict in (3.1).

(b) We claim that for any non-empty set $A$ the collection of all $n$-tuples $(a_1, a_2, \ldots, a_n)$ of elements of $A$, where $n$ varies though $\mathbb{Z}^+$ and

$$(a_1, a_2, \ldots, a_n) := a_1 \qquad \text{when} \qquad n = 1,$$

is a set. Indeed, this collection coincides with the collection $\{h(n)\}_{n\in\mathbb{N}}$ of Example (a) after removing the set $\{\emptyset\}$ from the latter. But this latter collection $\{h(n)\}_{n\in\mathbb{N}}$ is a set by Corollary 7.3, and removing one specific point does not change that status[79]. The claim follows.

[79]See the bulleted items following the statement of the Axiom of Specification in §6.

(c) We return to the proof of the existence of the algebraic closure of an arbitrary field outlined toward the end of the Introduction. At the heart of the argument was a proof[80] that every non-algebraically closed field $K$ is contained in a field $L$ such that every polynomial in $K[x]$ contains a root in $L$. In terms of the class $X$ of all fields this has the following immediate consequence: *for any* $K \in X$ *either*

- $K$ *is algebraically closed, or*
- *the class* $X_K$ *of all proper extension fields* $L$ *of* $K$ *which contain a root of every polynomial in* $K[x]$ *is non-empty.*

Write

$$X = X^{\text{ac}} \coprod X^{\text{nac}} \qquad \text{(disjoint union)},$$

wherein $X^{\text{ac}}$ is the class of algebraically closed fields and $X^{\text{mac}}$ is the class of those which are not algebraically closed. By the Axiom of Choice there is a function $c : X^{\text{nac}} \to X$ such that $c(K) \in L_K$ for each $K \in X^{\text{mac}}$, and we can therefore define a function $f : X \to X$ by

$$K \in X \qquad \mapsto \qquad \begin{cases} K & \text{if } K \in X^{\text{ac}} \\ c(K) & \text{if } K \in X^{\text{ac}}. \end{cases}$$

Now choose any $K_0 \in X$. Since the goal is to prove that $K_0$ admits an algebraic closure, we can assume $K_0 \in X^{\text{nac}}$. Theorem 7.1 then gives the existence of a function $g : \mathbb{N} \to X$ such that $g(0) = K_0$ and $g(n+1) = f(g(n))$, i.e. such that $K_{n+1}$ admits a root of every polynomial in $K_n[x]$. By Corollary 7.3 the collection $\{K_n\}_{n \in \mathbb{N}}$ is a set, and our set-theoretic "existence of the union" problem is now history.

(d) We claim that for any non-empty set $A$ the collection of all $n$-tuples $(y_1, y_2, \ldots, y_n)$ of $m$-tuples $y_j = (a_{j1}, a_{j2}, \ldots, a_{jm_{y_j}})$, of elements of $A$, where $n$ and $m$ vary through $\mathbb{Z}^+$,

(i) $$(y_1, y_2, \ldots, y_n) := y_1 \qquad \text{when} \qquad n = 1,$$

and

(ii) $$(a_{j1}, a_{j2}, \ldots, a_{jm_j}) := a_{j1} \qquad \text{when} \qquad m_j = 1,$$

is a set. To see this replace $A$ in Example (b) by the set constructed from $A$ in that example.

[80] Which was perfectly acceptable to this author, and was therefore not detailed.

(e) Let $A$ be as in the previous example and let $T$ be a UA type which (as a set) satisfies $\emptyset \neq T \subset A$. Then the collection of all $n$-tuples $(y_1, y_2, \ldots, y_n)$ of $m$-tuples $y_j = (a_{j1}, a_{j2}, \ldots, a_{jm_{y_j}})$ as in Example (d), satisfying the additional requirements that

(i) $$y_1 \in T_{n-1} \qquad \text{and} \qquad a_{j1} \in T_{m_{j-1}},$$

is a set. To see this consider the set of tuples of tuples defined from $A$ in Example (d). One then sees by means of the Axiom of Specification that those elements of this set satisfying the conditions given in (i) form a subset, and therefore a set.

(f) It should be clear that we can define subsets of the set of Example (e) by adding to the conditions given in (i) of that example and again appealing to the Axiom of Specification. In fact we will need to do just that, but we will put our detailed justifications for using the term "set" behind us, since by now the arguments are (or should be) becoming predictable.

# 8. Preliminaries on Free Objects

Let $\mathcal{C} := (\mathcal{C}_O, \mathcal{C}_M)$ and $\mathcal{D} = (\mathcal{D}_O, \mathcal{D}_M)$ be categories: the subscripts indicate the respective classes of objects and morphisms. A functor $T : \mathcal{C} \to \mathcal{D}$ is *faithful* if for every pair of distinct morphisms $f_1, f_2 : c \to c'$ in $\mathcal{C}_M$ the correponding morphisms $Tf_1, Tf_2 : Tc \to Tc'$ in $\mathcal{D}_M$ are distinct. A category $\mathcal{C}$ is *concrete* if there is a faithful functor from $\mathcal{C}$ to the category $\mathcal{S}$ of sets and set mappings. When this is the case the functor is called the (associated) *forgetful functor* (since it forgets all the structure associated with the set, leaving nothing for a latecomer to contemplate other than the set itself), and the image of an object is called the *underlying set* of the object. The category of topological spaces (and continuous mappings) is concrete; the underlying set of an object $(X, \tau)$ is the set $X$. The category of groups (and group homomorphisms) is concrete; the underlying set of an object $(G, \times)$ is the set $G$.

Let $\mathcal{C}$ be a concrete category with forgetful functor $T : \mathcal{C} \to \mathcal{S}$, let $c \in \mathcal{C}_O$ (i.e. let $c$ be an object of $\mathcal{C}$), let $X \in \mathcal{S}_O$ (i.e. let $X$ be a set), and let $\iota : X \to Tc$ be a morphism in $\mathcal{S}_M$ (i.e. a set-theoretic mapping from $X$ into the underlying set of $c$). The pair $(c, \iota)$ is *free* on $X$ if for each object $c' \in \mathcal{C}_O$ and each (set) mapping $h : X \to Tc'$ there is a unique morphism $h_{c'} : c \to c'$ which renders the diagram

$$\begin{array}{ccc} Tc & \stackrel{Th_{c'}}{\longrightarrow} & Tc' \\ \iota\uparrow & \stackrel{h}{\nearrow} & \\ X & & \end{array} \tag{8.1}$$

commutative. When this is the case and the mapping $\iota$ is clear from context one simply says that $c$ is *free* on $X$. When an object $c \in \mathcal{C}_O$ can be associated with such a (not necessarily unique) pair $(X, \iota)$ one refers to $c$ as a *free object* (of $\mathcal{C}$). The existence of the morphism $Th_{c'}$ of (8.1) for each $h : X \to Tc'$ is referred to as the *universal* (*mapping*) *property* of the free object $c$. The fundamental point behind the definition of a free object is: to specify a unique morphism $h : c \to c'$ (which in general would be subject to many restrictions, e.g. in the category of topological spaces inverse images of open sets must be open; in the category of groups the group operations must be respected) it is is enough to specify a set function $h : X \to Tc'$ (subject to no restrictions[81]). Example One: Let $R$ be a commutative ring, let

[81] In other words, one is "free" to define $h$ in any way that strikes one's fancy (or, more to the point, in any way that proves appropriate for solving whatever problem is at hand).

$\mathcal{A}_R$ be the category of $R$-algebras, and let $X = \{x_1, x_2, \ldots, x_n\}$ be algebraically independent over $R$. Then the polynomial algebra $R[x] := R[x_1, x_2, \ldots, x_n]$ is free on $X$, with the mapping $\iota : X \to R[x]$ being the inclusion $x_j \in X \mapsto x_j \in R[x]$: any $R$-algebra homomorphism from $R[x]$ into any $R$-algebra is uniquely determined by the images of the $x_j$, i.e. by "substitution." Example Two: Any vector space over a field is free on any basis. (To be precise: let $\mathcal{V}$ be the category of [finite-dimensional, if the reader so desires] vector spaces and linear transformations over a field $K$, let $V \in \mathcal{V}_O$, and let $\mathbf{e}$ be any basis of $V$. Then $V$ is free on $\mathbf{e}$ [with the mapping $\iota : \mathbf{e} \to TV$ being inclusion].)

In practice it proves convenient to remember diagram (8.1) as

(8.2)
$$\begin{array}{ccc} c & \xrightarrow{h_{c'}} & c' \\ \iota\uparrow & \nearrow^{h} & \\ X & & \end{array}$$

even though the labeling makes no sense: the actual mapping $\iota$ has image in the set $Tc$, not in the overlying object $c$ (assuming $\mathcal{C} \neq \mathcal{S}$); and $h$ is similarly misrepresented. Nevertheless, we will adopt this practice, the reason being that this last diagram quickly conveys a strengthening of the fundamental property of free objects mentioned in the previous paragraph: the mapping $h \mapsto h_{c'}$ is a bijection between set mappings of $\mathcal{S}_M$ of the form $X \to Tc'$ and morphisms of $\mathcal{C}_M$ of the form $c \to c'$.

**Theorem 8.3 (Uniquess of Free Objects) :** *Let $\mathcal{C}$ be a concrete category and let $X$ be an non-empty set. Then:*

(a) *any two objects of $\mathcal{C}$ which are free on $X$ are isomorphic; and*

(b) *if two objects of $\mathcal{C}$ are isomorphic, and if one is free on $X$, then the same holds for the other.*

Since the definition of an object $c \in \mathcal{C}$ being free on a set $X$ depends on the set-theoretic mapping $\iota : X \to Tc$, it would seem that the same $c$ could be free on $X$ in many different ways. In fact this is often the case, but is of no great concern: choose $c' := c$ and assume $\iota' \neq \iota$ in the following proof.

**Proof :**

(a) When $c, c' \in \mathcal{C}_O$ are free on $X$ one can construct the commutative diagram

$$\begin{array}{ccccc} c & \stackrel{h_{c'}}{\longrightarrow} & c' & \stackrel{h_c}{\longrightarrow} & c \\ & \stackrel{\iota}{\nwarrow} & \uparrow \iota' & \stackrel{\iota}{\nearrow} & \\ & & X & & \end{array}$$

by merging (8.2) with the analogue for $c'$. Since the triangle formed by the outer boundary remains commutative when the composition $h_c \circ h_{c'}$ is replaced by the identity morphism $\mathrm{id}_c : c \to c$, we see from the uniqueness condition in the definition of a free object that $h_c \circ h_{c'} = \mathrm{id}_c$, and a completely analogous argument gives $h_{c'} \circ h_c = \mathrm{id}_{c'}$.

(b) Suppose $c \in \mathcal{C}_O$ is free on $X$ and $h : X \to c'$ (i.e. $h : X \to Tc'$), where $c' \in \mathcal{C}_O$. Then there is a unique morphism $h_{c'}$ as in (8.2). If $p : c'' \to c$ is an isomophism of a second object $c'' \in \mathcal{C}_O$ with $c$, $c'' \in \mathcal{C}_O$ we can extend that diagram to

$$\begin{array}{ccccc} c'' & \stackrel{h_{c''}}{\longrightarrow} & c & \stackrel{h'_c}{\longrightarrow} & c' \\ & \stackrel{h_{c''}^{-1} \circ \iota}{\nwarrow} & \uparrow \iota & \stackrel{h}{\nearrow} & \\ & & X & & \end{array}$$

The composition $h_{c'} \circ h_{c''} : c'' \to c'$ is therefore a morphism which renders the diagram

$$\begin{array}{ccc} c'' & \stackrel{h_{c'} \circ h_{c''}}{\longrightarrow} & c' \\ h_{c''}^{-1} \circ \iota \uparrow & \stackrel{h}{\nearrow} & \\ X & & \end{array}$$

commutative, and it remains to prove that the composition represented by the top line is unique in this respect. To this end suppose this last diagram commutes when $h_{c'} \circ h_{c''}$ is replaced by a morphism $p : c'' \to c'$. Then the diagram

$$\begin{array}{ccc} c & \stackrel{p \circ h_{c''}^{-1}}{\longrightarrow} & c' \\ \iota \uparrow & \stackrel{h}{\nearrow} & \\ X & & \end{array}$$

also commutes, uniqueness (in diagram (8.2)) gives $p \circ h_{c''}^{-1} = h_{c'}$, and $p = h_{c'} \circ h_{c''}$ follows.

**q.e.d.**

# 9. Free $T$-Algebras

*Here $T$ is a UA type and $X$ is a non-empty set.*

A *free $T$-algebra* (*on* $X$) refers to a free object (on $X$) in the category $\mathcal{A}_T$. In this section we prove the following result.

**Theorem 9.1 :** *There is a free $T$-algebra on $X$.*

The argument we give is based on a two-page proof found in [B-M, Chapter I, §2, Theorem 2.2, pp. 4-6]. As we have already indicated in the Introduction, that proof, in the opinion of this author, appeared to contain a set-theoretic gap which requires all the preliminary work we have done to repair.

On the other hand, the basic idea behind the construction of a free $T$-algebra found in that reference is quite clever, not that difficult, and is well worth sketching as a prelude to the myriad technical details we will face.

By definition a free $T$-algebra on $X$ consists of a $T$-algebra $F$ together with a set-theoretic mapping $\iota : X \to F$ having the following property: for any $T$-algebra $A$ and any set-theoretic mapping $h : X \to A$ there is a unique morphism $h_A : F \to A$ rendering the diagram

$$\begin{array}{ccc} F & \xrightarrow{h_A} & A \\ \iota\uparrow & \overset{h}{\nearrow} & \\ X & & \end{array} \qquad (9.2)$$

commutative. The basic idea behind our construction of such an $F$ is the following observation.

**Proposition 9.3 :** *Suppose $F$ is a set with the following properties:*

(a) $T \subset F$; *and*

(b) *if* $\ell \in \mathbb{N}$, *if* $t_\ell \in T_\ell$, *and if* $y_1, y_2, \ldots, y_\ell \in F$ *are arbitrary when* $\ell > 0$, *then*

(i) $$(t_\ell, y_1, y_2, \ldots, y_\ell) \in F,$$

*where*

(ii) $$(t_\ell, y_1, y_2, \ldots, y_\ell) := t_0 \qquad \text{when} \qquad \ell = 0.$$

*Then $F$ is given the structure of a $T$-algebra by assigning $t = t_\ell \in T_\ell$ to the function*

$$\text{(iii)} \qquad t_F : (y_1, y_2, \ldots, y_\ell) \in F^\ell \mapsto (t_\ell, y_1, y_2, \ldots, y_\ell) \in F,$$

*which for $\ell = 0$ and $t_\ell = t_0 \in T_0$ is understood to denote the mapping*

$$\text{(iv)} \qquad t_F : \emptyset \in F^0 \mapsto t_0 \in F$$

*which one identifies*[82] *with the element $t_0 \in T_0$. In particular, one has*

$$\text{(v)} \qquad t_F = t \qquad \text{when} \qquad t \in T_0.$$

**Proof :** Obvious from the definitions. **q.e.d.**

The construction of a set $F$ with the properties listed in Proposition 9.3 is accomplished in [B-M, Chapter I, §2, Theorem 2.2, pp. 4-6] by defining $F$ as the union of pairwise disjoint recursively defined sets $F_0$, $F_1$, $F_2 \cdots$ which, as one would certainly expect, are endowed with properties compatible with the ultimate goal. Specifically, the procedure presented there results in $F$ being a free $T$-algebra, and simultaneously takes into account everything necessary to verify the universal mapping property summarized by (9.2).

The first step of that procedure is to define

$$(9.4) \qquad F_0 := T_0 \coprod X,$$

and to then define a mapping $\iota_0 : X \to F_0$ by

$$(9.5) \qquad \iota_0 : x \in X \mapsto x \in F_0.$$

Since the intent is to construct $F$ overlying $F_0$, this already solves the problem of constructing set-theoretic mapping $\iota : X \to F$ required of a free object on $X$: once $F$ has been constructed we can define $\iota$ to be the composition of $\iota_0$ and the inclusion mapping $F_0 \hookrightarrow F$. This is the sense in which this initial step is compatible with the ultimate goal.

Looking far ahead (with perhaps unjustifiable optimism, since we have only taken the first step), suppose we are ultimately successful in constructing our free $T$-algebra $F$, that $A$ is a second $T$-algebra, that a set-theoretic function $h : X \to A$ has been given, and that the unique morphism $h_A : F \to A$ of diagram (9.2) has been determined. Then, as can be seen from (iii) of the following result, the restriction $h_A|_{F_0} : F_0 \to A$ can be described (and therefore defined!) without reference to $h_A$.

[82] As in Example 3.2(a).

**Proposition 9.6 :** *Suppose* $F$ *is a free* $T$*-algebra on* $X$ *with associated set mapping* $\iota : X \to F$*, let* $A$ *be any* $T$*-algebra, let* $h : X \to A$ *be any set mapping, and let* $h_A : F \to A$ *be the resulting morphism indicted in* (9.2). *Assume, in the notation of the paragraph surrounding* (9.4) *and* (9.5), *that* $F_0 \subset F$ *and that* $\iota : X \to F$ *is the composition of* $\iota_0$ *with the inclusion* $F_0 \hookrightarrow F$*. Then the restriction*

(i) $$h_0 := h_A|_{F_0}$$

*must be given by*

(ii) $$h_0 : \begin{cases} t \in T_0 & \mapsto \quad t_A \in A \quad \text{(see (4.1)); } \quad \textit{and} \\ x \in X & \mapsto \quad h(x) \in A. \end{cases}$$

**Proof :** For $t \in T_0$ we must have $h_0(t_F) = h_A(t_F) = t_A$ by (i) and (4.4), and the top line in (ii) is then evident from (v) of Proposition 9.3. The bottom line follows from (i), the hypothesized composition decomposition of $\iota$, and the commutativity of (9.2): for $x \in X$ one has $h_0(x) = h_A(x) = h_A(\iota(x)) = (h \circ \iota_0)(x) = h(x)$. **q.e.d.**

We proceed to the second step. Continuing with the hypotheses and notations of Proposition 9.6 define

(9.7) $$F_1 := \{ (t, y_1, y_2, \ldots, y_\ell) : \ell \in \mathbb{Z}^+, \quad \ell \in T_\ell, \quad \text{and} \quad y_j \in F_0, \; j = 1, 2, \ldots, \ell \}.$$

By means of an Axiom of Specification argument similar to that seen in Example 7.4(d), one can see that $F_1$ must be a set[83]. Note that the definition of $F_1$ depends only on $F_0 := T_0 \coprod X$; in particular, and in spite of the fact that the symbol $F$ occurs within the notation, the set $F_1$ is independent of the $T$-algebra $F$ appearing in Proposition 9.6.

**Proposition 9.8 :** *Assuming the hypotheses of* Proposition 9.6 *the restriction*

(i) $$h_1 := h_A|_{F_1}$$

*is given by*

(ii) $$\begin{cases} h_1(t, y_1, y_2, \ldots, y_\ell) & = \quad t_A(h_A(y_1), h_A(y_2), \ldots, h_A(y_\ell)) \\ & = \quad t_A(h_0(y_1), h_0(y_2), \ldots, h_0(y_\ell)) \end{cases}$$

*for all* $(t, y_1, y_2, \ldots, y_\ell) \in F_1$.

[83]Despite the fact that the notation does not indicate an appeal that axiom.

The important thing to note, from the final equality in (ii), is that the values of $h_1$ are completely determined by the values of $h_0$. Since we have already seen that $h_0$ can be definined without reference to $h_A$, the same therefore holds for $h_1$.

**Proof :** Since $h_A : F \to A$ is a morphism in $\mathcal{A}_T$ we have

$$\begin{aligned} h_1(t, y_1, y_2, \ldots, y_\ell) &= h_A(t, y_1, y_2, \ldots, y_\ell) \\ &= h_A(t_F(y_1, y_2, \ldots, y_\ell)) \quad \text{(by (iii) of Proposition 9.3)} \\ &= t_A(h_A(y_1), h_A(y_2), \ldots, h_A(y_\ell)) \quad \text{(by (4.3))} \\ &= t_A(h_0(y_1), h_0(y_2), \ldots, h_0(h_\ell)) \end{aligned}$$

**q.e.d.**

The third step is slightly more involved, since we need to account for both $F_0$ and $F_1$ in the definition of $F_2$. We formulate that definition in a manner which easily generalizes, still maintaining the hypotheses and notations of Proposition 9.6 and still assuming (ii) of Proposition 9.3:

$$(9.9) \qquad \begin{cases} F_2 \;:=\; \{\, (t, y_1, y_2, \ldots, y_\ell) \,:\, \ell \in \mathbb{Z}^+, \;\; \ell \in T_\ell, \;\; y_j \in F_{i_j}, \\ \qquad\qquad j = 1, 2, \ldots, \ell, \;\; \text{and} \;\; \sum_{j=1}^{\ell} i_j = 1 \,\}. \end{cases}$$

(One can again argue as in Example 7.4(d) to conclude that $F_2$ is a set.) Note that the $\ell$-tuples are becoming more "complex," i.e. we can now be dealing with $\ell$-tuples of $m$-tuples, whereas in $F_1$ we were just working with $\ell$-tuples of elements single elements of $F_0 = T_0 \coprod X$. The analogue of (ii) of Proposition 9.8 for $h_2 := h_A|_{F_2}$ is easily seen to be

$$(9.10) \qquad \begin{cases} h_2(t, y_1, y_2, \ldots, y_\ell) = t_A(h_1(y_1), h_1(y_2), \ldots, h_1(y_\ell)) \\ \qquad \text{for all} \qquad (t, y_1, y_2, \ldots, y_\ell) \in F_2. \end{cases}$$

Proceeding by induction[84] we encounter "increasing complexity" in the definitions of the $F_j$, i.e. tuples of tuples of $\cdots$ of tuples, all surreptitiously camouflaged by the terse notation. Suppose $n > 1$, that $F_0, F_1, \cdots, F_{n-1}$ have been defined, that each of the restrictions $h_A|_{F_j}$ have been determined, and that each depends only on the restrictions $h_A|_{F_k}$ for $0 \leq k < j$. Define

$$(9.11) \qquad \begin{cases} F_n \;:=\; \{\, (t, y_1, y_2, \ldots, y_\ell) \,:\, \ell \in \mathbb{N}, \;\; \ell \in T_\ell, \;\; y_j \in F_{i_j}, \\ \qquad\qquad j = 1, 2, \ldots, p, \;\; \text{and} \;\; \sum_{j=1}^{p} i_j = n - 1 \,\}. \end{cases}$$

[84] And, of course, still maintaining the hypotheses and notations of Proposition 9.6, and still assuming (ii) of Proposition 9.3 .

The general version of (ii) of Proposition 9.8 for $h_n := h_A|_{F_n}$ is then evident:

$$(9.12) \qquad \begin{cases} h_n(t, y_1, y_2, \dots, y_\ell) = t_A(h_{n-1}(y_1), h_{n-1}(y_2), \dots, h_{n-1}(y_\ell)) \\ \qquad \qquad \text{for all} \qquad (t, y_1, y_2, \dots, y_\ell) \in F_n. \end{cases}$$

As several of our earlier comments were intended to suggest, a good portion of the (not very long) proof of Theorem 2.2 on pages 4-6 of [B-M] involves verifying that for a given $T$-algebra $A$ one can define functions $h_n : F_n \to A$ satisfying (9.12) *without assuming the existence of the morphism* $h_A : F \to A$. (The careful reader of these notes might already be able to see how this could be done.) The authors then define

$$(9.13) \qquad F := \cup_{n=0}^{\infty} F_n$$

(see (iii) on page 5 of [B-M]), endow $F$ with the structure of a $T$-algebra, and prove the existence and uniqueness of $h_A$ for a given $h : X \to A$, essentially by a minor modification of what we have done above, thereby establishing that the $T$-algebra $F$ is free on $X$.

As was the case with Lang's proof of the existence on an algebraic closure for a field, the problem for this author with the argument in the previous paragraph is the formation of the union in (9.13): it is simply not clear that the collection $\{F_n\}_{n \in \mathbb{N}}$ is a set, and if not the union does not make sense. However, in the case of Lang's proof we saw how to get around the problem by means of (the Recursion Theorem in the form of) Corollary 7.3. The same result will be used to eliminate the current difficulty.

**Proof of Theorem 9.1 :** The first task is to recursively define a sequence of pairwise disjoint sets analogous to the sequence of sets[85] $F_n$ appearing in (9.11). For that we will use Corollary 7.2, which requires defining a class[86] $C$ and a sequence of functions $f_n : C \to C$.

Let $S$ be any non-empty set admitting a family of pairwise disjoint subsets $\{S_k\}_{k \in \mathbb{N}}$ such that $S = \coprod_{k \in \mathbb{N}} S_k$. For each $n \in \mathbb{N}$ define $F_{S,n}$ to be the collection of all finite tuples $(t, s_1, s_2, \dots, s_\ell)$ of elements of the disjoint union $T \coprod S$ satisfying $t \in T_\ell$, where:

[85] And to keep that analogy in mind we will denote this new sequence by $\{F_n\}_{n \in \mathbb{N}}$.

[86] $C$ plays the role of the class $X$ in Corollary 7.2: in the current proof $X$ has a different meaning.

(i) $\ell$ can assume any value in $\mathbb{N}$;

(ii) $(t, s_1, s_2, \ldots, s_\ell) := t$ if $\ell = 0$; and

(iii) $s_j \in S_{i_j}$, where $\sum_{j=1}^{\ell} i_j = n - 1$ if $\ell > 0$.

One can easily see (recall Examples 7.4(e) and (f)) that each $F_{S_n}$ is a set.

Consider the class $C$ of all ordered pairs $(S, \{S_k\}_{k\in\mathbb{N}})$ as in the previous paragraph, and to ease notation abbreviate such a pair as $S$, or as $S = \coprod_{j=0}^{m} S_j$ if $S_k = \emptyset$ for all $k > m$. For each $n \in \mathbb{Z}^+$ define a mapping $f_n : C \to C$ by

(iv) $$f_n : S \mapsto \coprod_{j=1}^{n-1} S_j \coprod F_{S,n} = S_0 \coprod S_1 \coprod \cdots \coprod S_{n-1} \coprod F_{S,n}.$$

Now set

(v) $$F_0 := T_0 \coprod X \in C.$$

Then by Corollary 7.2 there is a unique function $g : \mathbb{N} \to C$ such that

(vi) $$g(0) = F_0,$$

and

(vii) $$g(n+1) = f_n(g(n)) \qquad \text{for all} \qquad n \in \mathbb{N}.$$

By Corollary 7.3 the collection $\{g(n)\}_{n\in\mathbb{N}}$ is a family of sets.

To tie this in with our preliminary discussion of the proof set

$$F_n := g(n), \qquad n \in \mathbb{N},$$

which we note is consistent with (vi). Then from (i)-(iii) we see that for all $n > 0$ we have

(viii) $$\begin{cases} F_n \;:=\; \{\, (t, y_1, y_2, \ldots, y_\ell) : \ell \in \mathbb{Z}^+, \;\; t \in T_\ell, \;\; y_j \in F_{i_j}, \\ \qquad\qquad j = 1, 2, \ldots, p, \;\text{ and }\; \sum_{j=1}^{p} i_j = n - 1 \,\}, \end{cases}$$

and that $\{F_n\}_{n\in\mathbb{N}}$ is a family of sets. We can therefore form the (set-theoretic) union

(ix) $$F := \cup_{n\in\mathbb{N}} F_n$$

(again see Corollary 7.3), and the set-theoretic difficulty experienced by your author within the oft-cited proof in [B-M] has thereby been resolved. Moreover, it then follows immediately from Proposition 9.3 that $F$ can be given the structure of a $T$-algebra, and this is the structure we henceforth assume.

Let $\iota : X \to F$ be as in the paragraph surrounding (9.4) and (9.5), i.e. the composition of the mapping $\iota_0 : x \in X \mapsto x \in T_0 \coprod X =: F_0$ with the inclusion mapping $F_0 \hookrightarrow F$. To complete the proof we will verify that $F = (F, \iota)$ is free on $X$. To this end let $A$ be an arbitrary $T$-algebra and let $h : X \to A$ be an arbitrary set-theoretic function. We must produce a $T$-algebra morphism $h_A : F \to A$ which renders the diagram

$$\text{(x)} \qquad \begin{array}{ccc} F & \xrightarrow{h_A} & A \\ {\scriptstyle\iota}\uparrow & \overset{h}{\nearrow} & \\ X & & \end{array}$$

commutative, and prove that $h_A$ is unique in this respect.

To this end first note from (v) that for each $z \in F_0$ there are two alternatives: either $z \in T_0$ or $z \in X$. On the other hand, when $n > 0$ each element $z \in F_n$ is a tuple $(t, y_1, y_2, \ldots, y_\ell)$ beginning with $t \in T_\ell$. To handle all $n \in \mathbb{N}$ simultaneously let us agree that

$$\text{(xi)} \qquad (t, y_1, y_2, \ldots, y_\ell) \ \begin{array}{l}\text{denotes either an element } t \in T_0, \text{ or} \\ \text{an element } x \in X, \text{ when } \ell = 0.\end{array}$$

Now fix some element[87] $a_0 \in A$, let $\widehat{X}$ be the set of functions $h : F \to A$, and for each $n \in \mathbb{N}$ let $\widehat{f}_n : \widehat{X} \to \widehat{X}$ be the function which assigns to any such $h$ the function with value at $(p+1)$-tuple $(t, y_1, y_2, \ldots, y_\ell) \in F$ given by (keeping (xi) in mind)

$$\text{(xii)} \qquad \begin{cases} h(t, y_1, y_2, \ldots, y_\ell) & \text{if } (t, y_1, y_2, \ldots, y_\ell) \in F_j \text{ with } j \leq n, \\ t_A(h(y_1), h(y_2), \ldots, h(y_\ell)) & \text{if } (t, y_1, y_2, \ldots, y_\ell) \in F_{n+1}, \text{ and} \\ a_0 & \text{otherwise.} \end{cases}$$

To gain a better feeling fot these constructions introduce the sets

$$\text{(xiii)} \qquad E_n := F_0 \coprod F_1 \coprod \cdots \coprod F_n, \qquad n \in \mathbb{N},$$

[87] If the $T$-algebra $A$ were to admit a "zero element," e.g. if $A$ were a ring, we would most likely, just for simplicity, choose $a_0$ to be that element. However, the existence of a zero element is not part of, nor is it implied by, the definition of a $T$-algebra. Moreover, if $T$ involves many $n$-ary operations it is conceivable that there might be no obvious candidate for "zero."

and note from (viii) that

(xiv)
$$E_0 \subset E_1 \subset E_2 \cdots \subset F$$

holds, as well as

(xv)
$$\cup_n E_n = F.$$

For each $n > 0$ one can then imagine $f_n(h)$ as a makeover of the function $h$ involving three steps: restricting $h$ to $E_n$; redefining $h|_{F_{n+1}}$ completely in terms of values depending only on this initial restriction; and redefining $h|_{\coprod_{j>n+1} F_j}$ to be the constant mapping with value $a_0$ (the only reason for this final step, or some variation thereof, is to ensure that $f_n(h)$ has the required domain to qualify as an element of $\widehat{X}$). Since $\widehat{f}_n(h)$ is completely determined by the restriction $\widehat{f}_n(h)|_{E_n}$, one tends to think of $\widehat{f}_n(h)$ as a mapping from $E_n$ into $A$.

Define $h_0 : F \to A \in \widehat{X}$ to be the function given defined in (iii) of Proposition 9.6, although with domain now extended to $F$ by defining $h_0(x) := a_0$ if $x \in \coprod_{n>0} F_n$. Then by Corollary 7.2 there is a unique function $\widehat{g} : \mathbb{N} \to \widehat{X}$ such that

(xvi)
$$\widehat{g}(0) = h_0 \in F_0$$

and

(xvii)
$$\widehat{g}(n+1) = \widehat{f}_n(\widehat{g}(n)).$$

By viewing $\widehat{g}(n)$ as a mapping from $E_n$ into $A$ (see the end of the previous paragraph) we see by construction that $\widehat{g}(n+1) : E_{n+1} \to A$ is an extension of $\widehat{g}(n) : E_n \to A$ for each $n \in \mathbb{N}$, hence in particular an extension of $f_0 : F_0 \to A$, and from (xiv) that

(xviii)
$$\widehat{g}(n+1)(t, y_1, y_2, \ldots, y_\ell) = t_A(\widehat{g}(r_1)(y_1), \widehat{g}(r_2)(y_2), \ldots, \widehat{g}(r_\ell)(y_\ell))$$

if $(t, y_1, y_2, \ldots, y_\ell) \in F_n$ (with the usual deference to (ii)), which by (iii) of Proposition 9.3 can also be written

(xix)
$$\widehat{g}(n+1)t_F(y_1, y_2, \ldots, y_\ell) = t_A\widehat{g}(r_1)(y_1), \widehat{g}(r_2)(y_2), \ldots, \widehat{g}(r_\ell)(y_\ell)).$$

From the extension property and (xv) we see that a function $h_A : F \to A$ can be unambiguously defined as follows: for $z \in F$ pick $n \in \mathbb{N}$ such that $z \in F_n$; assign $z$ to $\widehat{g}(n)(z)$. From (xvi), (xvii) and (xix) we see that $h_A$ is a morphism in $\mathcal{A}_T$.

To prove uniqueness simply note that if diagram (xi) is commutative when $h_A$ is replaced by some other $\mathcal{A}_T$ morphism $q$ then from (iii) of Proposition 9.3 we must have $q|_{F_0} = f_0$, and from (4.3) we must have

$$q(t_F(t, y_1, y_2, \ldots, y_\ell)) = t_A(q(y_1), q(y_2), \ldots, q(y_\ell))$$

must hold for any $(t, y_1, y_2, \ldots, y_\ell) \in F_n$. One then sees by induction that $q|_{F_n} = \widehat{g}(n)|_{F_n} = h_A|_{F_n}$, and $q = h_A$ follows. **q.e.d.**

By Theorem 8.3 any two objects of $\mathcal{A}_T$ which are free on $X$ must be isomorphic, and w.l.o.g. we may therefore assume, in any context in which we are dealing with a free object on $X$, that we are dealing with the free object $F$ constructed in the preceding proof. Since the $T$-algebra structure on $F$ resulted from an application of Proposition 9.3 (see the paragraph of that proof surrounding (ix)), and since the structure of an induced morphism $h_A : F \to A$ is uniquely determined by (9.12), our construction has the following corollary, which summarizes in a convenient way all the propeties of the free $T$-algebra we will need.

**Corollary 9.14 :** *There is a free $T$-algebra $F$ on $X$ which contains $T$ as a subset, in which the associated mapping $\iota : X \to F$ is inclusion, and which admits the following two structure properties:*

(i) $$t_F = t \qquad \textit{for all} \qquad t \in T_0;$$

*and*

(ii) $$\left\{ \begin{array}{c} t_F(y_1, y_2, \ldots, y_\ell) = (t, y_1, y_2, \ldots, y_\ell) \in F \\ \textit{for all } \ t \in T_\ell \ \textit{ and all } \ (y_1, y_2, \ldots, y_\ell) \in F^\ell, \end{array} \right.$$

*where*

(iii) $$(t, y_1, y_2, \ldots, y_\ell) := t \qquad \textit{if} \qquad \ell = 0.$$

*In addition, if $A$ is any $T$-algebra, and if $h : X \to A$ is any set-theoretic mapping, the induced morphism $h_A : F \to A$* (see (9.2)) *is given by*

(iv) $$\left\{ \begin{array}{c} h_A : (t, y_1, y_2, \ldots, y_\ell) \in F \mapsto t_A(h_A(y_1), h_A(y_2), \ldots, h_A(y_\ell)) \in A, \\ \textit{for all } \ t \in T_\ell \ \textit{ and all } \ (y_1, y_2, \ldots, y_\ell) \in F^\ell, \end{array} \right.$$

*which is understood to mean*

$$\text{(v)} \qquad h_A : t \in T_0 \mapsto t_A \qquad \textit{if} \qquad \ell = 0.$$

*Moreover,* $F$ *is the union of the family* $\{F_n\}_{n\in\mathbb{N}}$ *of pairwise disjoint sets which begins with*

$$\text{(vi)} \qquad F_0 := T_0 \coprod X,$$

*is defined recursively for* $0 < n \in \mathbb{N}$ *by*

$$\text{(vii)} \qquad \begin{cases} F_n & := \ \{ (t, y_1, y_2, \ldots, y_\ell) \,:\, \ell \in \mathbb{N}, \quad \ell \in T_\ell, \quad y_j \in F_{i_j}, \\ & \qquad j = 1, 2, \ldots, p, \quad \textit{and} \quad \sum_{j=1}^{p} i_j = n - 1 \}, \end{cases}$$

*and is related to the morphism* $h_A$ *of* (iv) *as follows:*

$$\text{(viii)} \qquad h_A|_{F_0} : \begin{cases} t \in T_0 & \mapsto \quad t_A \in A \quad \textit{and} \\ x \in X & \mapsto \quad h(x) \in A; \end{cases}$$

*and for* $n > 0$ *one has*

$$\text{(ix)} \qquad h_A|F_n : (t, y_1, y_2, \ldots, y_\ell) \in F_n \mapsto t_A(h_A(y_1), h_A(y_2), \ldots, h_A(y_n)) \in A.$$

*In particular, for each* $n > 0$ *the restriction* $h_A|_{F_n}$ *is completely determined by the prior restrictions* $h_A|_{F_j}$, $0 \leq j < n$.

The final assertion is evident from (iv) and (vii).

When an object $F$ of $\mathcal{A}_T$ is free on a set $X$ one refers to the elements of $X$ as (the) $T$-*algebra variables* (of $F$), or, when $X$ is understood, simply as "the variables." The tuples $(y_1, y_2, \ldots, y_\ell) \in F^\ell$ encountered in (ii) of Corollary 9.14 are referred to as *words* in these variables, although when that terminology is used one generally thinks of a "string" of symbols $ty_1y_2\cdots y_\ell$ rather than a tuple $(t, y_1, y_2, \ldots, y_\ell)$. Moreover, one is apt to reposition the $t$, e.g. if $\ell = 2$ and $t = +$ one would most likely write $(+, y_1, y_2)$ as $y_1 + y_2$ than as $+y_1y_2$. Similary, if $T = \{+, \cdot\}$ one would most likely write $(y_1 + y_2) \cdot y_3$ in place of the 3-tuple $(\cdot\,, (+\,, y_1, y_2), y_3)$.

# 10. Proposition Algebras

*In this section* $T = \{t', t''\}$ *is a UA type in which the operation* $t'$ *is unary and the operation* $t''$ *is binary. We will eventually replace the notations* $t'$ *and* $t''$ *with* $\neg$ *and* $\Rightarrow$ *respectively, but the initial choices are helpful for keeping in mind that we are always dealing with sets*[88].

Any object of the category $\mathcal{A}_T$, where $T$ is as above, is called a *proposition algebra* or, when confusion confusion might otherwise result, a *proposition* $T$*-algebra*.

**Examples 10.1 :** In working through these examples one needs to keep in mind the italicized statement ending the paragraph containing (4.1).

(a) Let $K$ denote the field $\mathbb{Z}/2\mathbb{Z}$ and for each $k \in \mathbb{Z}$ write the coset $k + 2\mathbb{Z} \in K$ as[89] *by* $[k]$. Then $K$ becomes a proposition algebra by defining $t'_K : K \to K$ by

(i) $$t'_K : [k] \mapsto [k] + [1]$$

and $t''_K : K \times K \to K$ by

(ii) $$t''_K : ([k_1], [k_2]) \mapsto [1] + [k_1]([k_2] + [1]) = [1] + [k_1] \cdot t'_K([k_2]).$$

For example, $t''_K([1],[1]) = [1] + [1]([1] + [1]) = [1] + [1]([1 + 1]) = [1] + [1][2] = [1] + [2] = [1 + 2] = [3] = [1]$.

(b) (Boolean algebras) The power set $\mathcal{P}(X)$ of a non-empty set $X$ is given the structure of a $T$-algebra if we define $t'_{\mathcal{P}(\mathcal{X})} : \mathcal{P}(X) \to \mathcal{P}(X)$ by $S \mapsto X \setminus S$ and $t''_{\mathcal{P}(X)} : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathcal{P}(X)$ by $(S_1, S_2) \in \mathcal{P}(X) \times \mathcal{P}(X) \mapsto (X \setminus S_1) \cup S_2 \in \mathcal{P}(X)$.

[88]Although it may not yet be evident, the tension in working with proposition algebras is between what we would like to think we are dealing with, e.g. logical constructs such as $\neg p$ (read "not p") and $p \Rightarrow q$ (read "p implies q"), and whether such real-world entities can be identified with (or, if the reader prefers, "interpreted as") sets in the mathematical world. If so we could deal with them (as do model theorists) in standard mathematical fashion.

[89]The usual custom is to use the same letter $k$ to denote both an element of $\mathbb{Z}$ and the corresponding coset $k+2\mathbb{Z}$ of $\mathbb{Z}/2\mathbb{Z}$. This seldom causes confusion, but this author prefers not to use the same symbol within a specific context to denote two different entities, and to always use brackets [ ] to denote equivalence classes, no matter what the relation.

(c) The *proposition algebra* $P(X)$ of a non-empty set $X$ is by definition the free $T$-algebra the set $X$, and is denoted $P(X)$. In this context the $T$-algebra variables (i.e. the elements of $X$) are called *atomic statements*. In view of Corollary 9.14 the assignments of the operations are pre-ordained (and easily described):

(i) $$t'_{\mathcal{P}(X)} : p \in \mathcal{P}(X) \mapsto (t', p) \in \mathcal{P}(X)$$

and

(ii) $$t''_{\mathcal{P}(X)} : (p_1, p_2) \in \mathcal{P}(X) \times \mathcal{P}(X) \mapsto (t'', p_1, p_2) \in \mathcal{P}(X).$$

Every proposition algebra $A$ admits two induced binary operations of particular importance[90]:

(10.2) $$t_A^{\vee} : (y_1, y_2) \in A \times A \mapsto t''_A\big(t'_A(y_1), y_2\big) \in A,$$

and

(10.3) $$t_A^{\wedge} : (y_1, y_2) \in A \times A \mapsto t'_A(t_A^{\vee}\Big(t'_A(y_1), t'_A(y_2)\Big)) \in A$$

**Examples 10.4 :** We will only illustrate the induced operations (10.2) and (10.3) for Examples 10.1(a) and (b). The discussion of the analogues for Examples 10.1(c) will be delayed, but the alert reader has probably already guessed the results in that case given the $\vee$ and $\wedge$ notations we have employed.

(a) In Example 10.1(a) one sees from the calculation

$$\begin{aligned} t_K^{\vee}([k_1],[k_2]) &= t''_K(t'_K([k_1]),[k_2]) \\ &= t''_K([k_1]+[1],[k_2]) \\ &= [1] + ([k_1]+[1])([k_2]+[1]) \\ &= [1] + [k_1][k_2] + [k_1] + [k_2] + [1] \\ &= [2] + [k_1][k_2] + [k_1] + [k_2] \\ &= [k_1k_2] + [k_1] + [k_2] \end{aligned}$$

that $t_K^{\vee} : K \times K \to K$ is given by

(i) $$t_K^{\vee} : \begin{cases} ([1],[1]) \mapsto [1], \\ ([1],[0]) \mapsto [1], \\ ([0],[1]) \mapsto [1], \quad \text{and} \\ ([0],[0]) \mapsto [0]. \end{cases}$$

[90]Read $t_A^{\vee}$ as "tee vee (sub) A" and $t_A^{\wedge}$ as "tee wedge (sub) A."

and from

$$\begin{aligned}
t_K^{\wedge}([k_1],[k_2]) &= t'_K(t_K^{\vee}\big(t'_K([k_1]), t'_K([k_2])\big) \\
&= t'_K\big(t_K^{\vee}([k_1]+[1],[k_2]+[1])\big) \\
&= t'_K\big(([k_1]+[1])([k_2]+[1]) + ([k_1]+[1]) + ([k_2]+[1])\big) \\
&= t'_K\big([k_1][k_2] + ([k_1]+[1]) + ([k_2]+[1]) + [1] \\
&\qquad + ([k_1]+[1]) + ([k_2]+[1])\big) \\
&= t'_K\big([k_1k_2] + [1] + 2([k_1]+[1]) + 2([k_2]+[1])\big) \\
&= t'_K\big([k_1k_2]+[1]\big) \\
&= ([k_1k_2]+[1]) + [1] \\
&= [k_1k_2],
\end{aligned}$$

that $t_K^{\wedge} : K \times K \to K$ is given by

$$\text{(ii)} \qquad t_K^{\wedge} : \begin{cases} ([1],[1]) & \mapsto & [1], \\ ([1],[0]) & \mapsto & [0], \\ ([0],[1]) & \mapsto & [0], \quad \text{and} \\ ([0],[0]) & \mapsto & [0]. \end{cases}$$

One could also express (i) and (ii) in analogy with the "group multiplication tables" seen in introductory modern algebra courses, i.e. by

| $t_K^{\vee}$ | [0] | [1] |
|---|---|---|
| [0] | [0] | [1] |
| [1] | [1] | [1] |

Table I

and by

| $t_K^{\wedge}$ | [0] | [1] |
|---|---|---|
| [0] | [0] | [0] |
| [1] | [0] | [1] |

Table II

However, it would be more in keeping with our later work to express (i) and (ii) by the respective tables

| $[k_1]$ | $[k_2]$ | $t_K^{\vee}([k_1],[k_2])$ |
|---|---|---|
| [1] | [1] | [1] |
| [1] | [0] | [1] |
| [0] | [1] | [1] |
| [0] | [0] | [0] |

Table III

and

| $[k_1]$ | $[k_2]$ | $t_K^{\wedge}([k_1],[k_2])$ |
|---|---|---|
| [1] | [1] | [1] |
| [1] | [0] | [0] |
| [0] | [1] | [0] |
| [0] | [0] | [0] |

Table IV

(b) For Example 10.1(c) one verifies, by means of analogous elementary calculations, that $t_{\mathcal{P}(X)}^{\vee} : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathcal{P}(X)$ is given by

(i) $$t_{\mathcal{P}(X)}^{\vee} : (S_1, S_2) \in \mathcal{P}(X) \times \mathcal{P}(X) \mapsto S_1 \cup S_1 \in \mathcal{P}(X).$$

and that $t_{\mathcal{P}(X)}^{\wedge} : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathcal{P}(X)$ is given by

(ii) $$t_{\mathcal{P}(X)}^{\wedge} : (S_1, S_2) \in \mathcal{P}(X) \times \mathcal{P}(X) \mapsto S_1 \cap S_1 \in \mathcal{P}(X).$$

**Remark 10.5 :** We will be working with many arrays such as Table IV. Each has a "labeling box" at the top, e.g.

| $[k_1]$ | $[k_2]$ | $t_K^{\wedge}([k_1],[k_2])$ |
|---|---|---|

in Table IV, followed by several rows. When we refer to the "rows" of such a table *the labeling box will not be counted as a row.* The "second row" of Table IV would therefore refer to

| | | |
|---|---|---|
| [1] | [0] | [0] |

and the "final row" would be

| | | |
|---|---|---|
| [0] | [0] | [0] |

Moreover, when we refer to the "columns" of such a table *the labeling box entry which sits at the top of the designated column will not be regarded as an entry of that column.* For example, $[k_2]$ would not be considered an entry of column two of Table IV.

Let $X$ be a non-empty set and let $\mathbb{Z}/2\mathbb{Z}$ be the proposition algebra defined in Example 10.1(a). An $\mathcal{A}_T$-morphism $\nu : P(X) \to \mathbb{Z}/2\mathbb{Z}$ is called a[91] *valuation* on $P(X)$. Since there are only two arity operations involved the morphism condition (4.3) required of a valuation is easy to write down explicitly: for $t'$ one sees from (i) of Example 10.1(a) that

$$\nu(t'_{P(X)}(p)) = t'_{\mathbb{Z}/2\mathbb{Z}}(\nu(p)) = \nu(p) + [1], \qquad \text{for all} \qquad p \in P(X), \tag{10.6}$$

and from (ii) of Example 10.1(a) that

$$\nu(t''_{P(X)}(p,q)) = t''_{\mathbb{Z}/2\mathbb{Z}}(\nu(p), \nu(q)) = [1] + \nu(p)(\nu(q) + [1]),$$

hence that

$$\nu(t''_{P(X)}(p,q)) = \nu(p)\nu(q) + \nu(p) + [1] \qquad \text{for all} \qquad p, q \in P(X). \tag{10.7}$$

Since the image of $\nu$ consists of at most two values, the compositions implicit in (10.6) and (10.7) are easily summarized in a form analogous to that seen in Tables III and IV of Example 10.4(a), i.e. by

| $\nu(p)$ | $\nu(t'_{P(X)}(p))$ |
|---|---|
| [1] | [0] |
| [0] | [1] |

Table V

together with

| $\nu(p)$ | $\nu(q)$ | $\nu(t''_{P(X)}(p,q))$ |
|---|---|---|
| [1] | [1] | [1] |
| [1] | [0] | [0] |
| [0] | [1] | [1] |
| [0] | [0] | [1] |

Table VI

[91]Not to be confused with the valuations one encounters in number theory. Fortunately, the context generally renders this a non-problem.

For example, the third row[92] in Table VI records the fact that when $[0]$ and $[1]$ are substituted for $\nu(p)$ and $\nu(q)$ respectively in (10.7) the result, since $[0][1]+[0]+[1] = [1]$, is the entry $[1]$ which ends that line.

Again let $X$ be any non-empty set and let $P(X)$ be the associated proposition algebra. The fact that $P(X)$ is free makes it very easy to define valuations: for each set-theoretic mapping $h : X \to \mathbb{Z}/2\mathbb{Z}$ there is a unique valuation $\nu_h : P(X) \to \mathbb{Z}/2\mathbb{Z}$ which makes the diagram

$$(10.8) \qquad \begin{array}{ccc} P(X) & \xrightarrow{\nu_h} & \mathbb{Z}/2\mathbb{Z} \\ {\scriptstyle \mathrm{inc}}\uparrow & \overset{h}{\nearrow} & \\ X & & \end{array}$$

commute, and all valuations on $P(X)$ arise in this manner. Put another way: a valuation is uniquely determined by the values of atomic statements. The calculation of $\nu$-values of non-atomic statements is then reduced to a mechanical process. Specifically, one sees from Corollary 9.14 (with $F := P(X)$, $A = \mathbb{Z}/2\mathbb{Z}$ and $h_A = \nu$) that $F(X)$ can be viewed in terms of pairwise disjoint "levels" $F_n$, and that observation enables one to compute any specific value of $\nu$ on a step-by-step basis by means of tables. For example, if one has determined (or is given) the values $\nu(p)$ and $\nu(q)$ of elements $p, q \in P(X)$ then one can determine the values of $t'_{P(X)}(p)$ and $t''_{P(X)}(p,q)$ by referring to Tables V and VI. To simplify such procedures it is convenient to have several additional tables at hand: those which will prove important for our purposes are constructed in the following examples.

**Examples 10.9 :**

(a) We claim that the table corresponding to Table IV for the induced binary opeation $r := t^{\vee}_{P(X)}(p,q)$ initially defined in (10.2) is

| $\nu(p)$ | $\nu(q)$ | $\nu(t^{\vee}_{P(X)}(p,q))$ |
|---|---|---|
| $[1]$ | $[1]$ | $[1]$ |
| $[1]$ | $[0]$ | $[1]$ |
| $[0]$ | $[1]$ | $[1]$ |
| $[0]$ | $[0]$ | $[0]$ |

Table VII

[92] Which by Remark 10.5 would mean the second-to-last row.

The third row indicates, for example, that if $p$ and $q$ have been assigned the respective values [0] and [1] then the value assigned to $t^{\vee}_{P(X)}(p,q)$, which is at a "higher level" than either of $p$ and $q$, must be [1].

Note that Table VII results by removing the third column from the table

| $\nu(p)$ | $\nu(q)$ | $\nu(t'_{P(X)}(p))$ | $\nu(t^{\vee}_{P(X)}(p,q))$ |
|---|---|---|---|
| [1] | [1] | [0] | [1] |
| [1] | [0] | [0] | [1] |
| [0] | [1] | [1] | [1] |
| [0] | [0] | [1] | [0] |

,

and since by (10.2) we have

$$t^{\vee}_{P(X)}(p,q) := t''_{P(X)}((t'_{P(X)}(p),q),$$

we can keep the definition of the operation $t^{\vee}_{P(X)}$ in mind if we replace this last table with

| $\nu(p)$ | $\nu(q)$ | $\nu(t'_{P(X)}(p))$ | $\nu(t''_{P(X)}((t'_{P(X)}(p),q))$ |
|---|---|---|---|
| [1] | [1] | [0] | [1] |
| [1] | [0] | [0] | [1] |
| [0] | [1] | [1] | [1] |
| [0] | [0] | [1] | [0] |

Table VIII

In fact this is precisely how is how Table VII was constructed, and we can therefore forget about that table and concentrate on Table VIII.

The first two entries of each row of[93] Table VIII comprise all the possibilities for the pairs $(\nu(p),\nu(q))$. The third column records the value for $t'_{P(X)}(p)$ corresponding to the value $\nu(p)$ for $p$ which appears in the first column of the same row, and can be determined directly from Table V. (In particular, column three of Table VIII has nothing to do with column two.) Each entry in he fourth column is obtained from the respective entries in the third and second columns of the same row by means of Table VI. For example, the entries from the third and second columns in the final row of Table VIII are [1] and [0]

[93]Recall Remark 10.5.

respectively, and the second row in Table V gives $\nu((t''_{P(X)}(u,v)) = [0]$ when $\nu(u) = [1]$ and $\nu(v) = [0]$. The lower right entry in Table VIII must therefore be [0].

(b) The table analogous to VII for the binary operation $t^{\wedge}_{P(X)}$ is

| $\nu(p)$ | $\nu(q)$ | $\nu(t^{\wedge}_{P(X)}(p,q))$ |
|---|---|---|
| [1] | [1] | [1] |
| [1] | [0] | [0] |
| [0] | [1] | [0] |
| [0] | [0] | [0] |

Table IX

but the analogue of Table VIII is a bit more complicated, i.e. it is

| $\nu(p)$ | $\nu(q)$ | $\nu(t'(p))$ | $\nu(t'(q))$ | $\nu(t^{\vee}(t'(p),t'(q)))$ | $\nu(t'\big(t^{\vee}(t'(p),t'(q))\big))$ |
|---|---|---|---|---|---|
| [1] | [1] | [0] | [0] | [0] | [1] |
| [1] | [0] | [0] | [1] | [1] | [0] |
| [0] | [1] | [1] | [0] | [1] | [0] |
| [0] | [0] | [1] | [1] | [1] | [0] |

Table X

(wherein we have dropped the $P(X)$ subscripts to avoid spacing problems). In this instance the entries in the third and fourth columns are obtained from those of the first and second, respectively, using Table V. The entries of the fifth column are then obtained from those of the third and fourth using Table VII, i.e. by replacing $p$ and $q$ in the labeling box of that table by $t'_{P(X)}(p)$ and $t'_{P(X)}(q)$ respectively. The entries of the sixth column are then obtained from those of the fifth using Table V.

(c) For any $p \in P(X)$ the table for the element $t^{\wedge}_{P(X)}(t'_{P(X)}(p),p) \in P(X)$ is given by

| $\nu(p)$ | $\nu(t^{\wedge}_{P(X)}(t'_{P(X)}(p),p))$ |
|---|---|
| [1] | [0] |
| [0] | [0] |

Table XI

as can be seen by verifying the column entries of

| $\nu(p)$ | $\nu(t'_{P(X)}(p))$ | $\nu(t^{\wedge}_{P(X)}((t'_{P(X)}(p), p))$ |
|---|---|---|
| [1] | [0] | [0] |
| [0] | [1] | [0] |

starting with column two, by means of Tables V and XI.

(d) For any $p \in P(X)$ the table for the element $t'_{P(X)}\big(t^{\wedge}_{P(X)}(t'_{P(X)}(p), p)\big) \in P(X)$ is given by

| $\nu(p)$ | $\nu(t'_{P(X)}(t^{\wedge}_{P(X)}(t'_{P(X)}(p), p)))$ |
|---|---|
| [1] | [1] |
| [0] | [1] |

Table XII

Here one uses Tables V and XI.

**Proposition 10.10 (The Existence of Contradictions and Tautologies) :** *Let $X$ be a non-empty set and let $P(X)$ be the free $T$-algebra on $X$. Then any element $q \in P(X)$ of the form*

(i) $$q := t^{\wedge}_{P(X)}(r, t'_{P(X)}(r)) \qquad \textit{for some} \qquad r \in P(X)$$

*is such that*

(ii) $$\nu(q) = [0] \qquad \textit{for all valuations} \qquad \nu : P(X) \to \mathbb{Z}/2\mathbb{Z}.$$

*Moreover, when* (ii) *holds the point*

(iii) $$p := t'_{P(X)}(q)$$

*satisfies*

(iv) $$\nu(p) = [1] \qquad \textit{for all valuations} \qquad \nu : P(X) \to \mathbb{Z}/2\mathbb{Z}.$$

Any point $q \in P(X)$ having the form given in (i) is called a *contradiction* (*in* $P(X)$). Any point $p \in P(X)$ having the property stated in (iv) (but not necessarily having the form seen in (iv)) is called a *tautology* (*in* $P(X)$), or is said to be (*logically*) *valid* (*in* $P(X)$). Any point $q \in P(X)$ satisfying (ii) is said to be (*logically*) *invalid*. In particular, all contradictions are logically invalid.

**Proof :** Immediate from Table XI of Example 10.9(c) and Table XII of Example 10.9(d). **q.e.d.**

Assume the notation of Proposition 10.10 and let[94] $A \subset P(X)$ and $q \in P(X)$. One says that[95] $q$ is a *consequence* of $A$, or that $A$ *semantically implies* $q$, and writes

$$A \models q, \tag{10.11}$$

if $\nu(q) = [1]$ for all valuations $\nu : P(X) \to \mathbb{Z}/2\mathbb{Z}$ such that $\nu(p) = [1]$ for all[96] $p \in A$. When $A = \emptyset$ the condition that $\nu(p) = [1]$ for all $p \in A$ holds vacuously, and the condition on $q$ therefore becomes: $\nu(q) = [1]$ for all valuations $\nu : P(X) \to \mathbb{Z}/2\mathbb{Z}$. In other words,

$$\emptyset \models q \qquad \text{if and only if} \qquad q \text{ is a tautology.} \tag{10.12}$$

Suppose $n \geq 2$. The *concatenation* $t^{\wedge}_{P(X)}(p_1, p_2, \ldots, p_n)$ of (not necessarily distinct) elements $p_1, p_2, \ldots, p_n \in P(X)$ is defined inductively[97] by

$$t^{\wedge}_{P(X)}(p_1, p_2, \ldots, p_n) := t^{\wedge}_{P(x)}(p_1, t^{\wedge}_{P(X)}(p_2, t^{\wedge}_{P(X)}(p_3 \cdots t^{\wedge}_{P(X)}(p_{n-1}, p_n) \cdots ))). \tag{10.13}$$

One sees by induction and Table IX of Example 10.9(b) that for any valuation $\nu : P(X) \to \mathbb{Z}/2\mathbb{Z}$ one has

$$\nu(t^{\wedge}_{P(X)}(p_1, p_2, \ldots, p_n)) = [1] \quad \Leftrightarrow \quad \nu(p_j) = [1] \ \text{ for } \ j = 1, 2, \ldots, n. \tag{10.14}$$

**Proposition 10.15 :** *Suppose* $A \subset P(X)$ *is non-empty and finite, say* $A = \{p_1, p_2, \ldots, p_n\}$, *suppose* $q \in P(X)$, *and suppose* $t''_{P(X)}(t^{\wedge}_{P(X)}(p_1, p_2, \ldots, p_n), q)$ *is a tautology. Then*

$$A \models q.$$

**Proof :** From Table VI (the second table following (10.7)) we have

[94] A model theorist would refer to $A$ as a "set of formulas."

[95] The definition and notation are from [B-M, Chapter II, §3, Definition 3.2, p. 31].

[96] In [Hamilton, Chapter 1, §1.6, Definition 1.28, p. 23] this is the definition of a *valid argument form*, and the notation introduced in (10.11) is not employed in this context, although it does appear later in connection with predicate calculus (see [Hamilton, Chapter 3, §3.4, Definition 3.24, p. 63]). Indeed, my impression is that most authors avoid using the symbol $\models$ until predicate calculus has been introduced, but I found an earlier introduction very helpful to my (mis)understanding.

[97] From this point onward I am not going to worry about proving that my inductive definitions can be made rigorous by means of the Recursion Theorem. I have hopefully made my point, and we can therefore return to normal practice. Readers upset by this are invited, in each case, to supply their own proofs.

| $\nu(t^{\wedge}_{P(X)}(p_1, p_2, \ldots, p_n))$ | $\nu(q)$ | $\nu(t''_{P(X)}(t^{\wedge}_{P(X)}(p_1, p_2, \ldots, p_n), q)$ |
|---|---|---|
| [1] | [1] | [1] |
| [1] | [0] | [0] |
| [0] | [1] | [1] |
| [0] | [0] | [1] |

If $\nu(p_j) = [1]$ for $j = 1, 2, \ldots, n$ then $\nu(t^{\wedge}_{P(X)}(p_1, p_2, \ldots, p_n)) = [1]$ by (10.14) and we can therefore ignore the final two lines in this table. On the other hand, given that $t''_{P(X)}(t^{\wedge}_{P(X)}(p_1, p_2, \ldots, p_n), q)$ is a tautology we see (from the final column) that we can also ignore line two, $\nu(q) = [1]$ follows, and $A \models q$ is thereby established. **q.e.d**.

## 11. A Short Course on Sentential Inference

In this section we take a short vacation from the mathematical world, returning to the real world[98] for rest and relaxation. Upon arrival a gregarious stranger suggests that for amusement as well as edification we should learn the game of "Sentential Inference," which on the deluxe edition box is called the "Theory of Sentential Inference." We are told it has applications to the dissection of legal arguments, to switching circuits, and to all sorts of other practical things. Since we are in dire need of a break from pure mathematics we locate a store which has the game in stock and make the purchase, only to learn that, other than a few printed pages of instructions, the box contains nothing more than blank paper and a few pencils. We angrily demand our money back, but are told, by an employee with a somewhat disingenuous smile, that store policy prohibits refunds. Nor are we allowed to swap for the standard edition and pocket the difference. (He shakes his head no while maintaining that smile.) So we decide to accept the loss and see if we can at least enjoy the game.

The instructions state that it is played with pencil and paper[99] and employs the symbols $\neg$, $\wedge$, $\vee$, and $\Rightarrow$ (read 'not', 'and', 'or', and 'implies' respectively), known as (*logical*) *connectives*. (One is allowed to write these down repeatedly; after all, this is the real world, and we are well supplied with paper and pencils *which we paid dearly for*.) The symbols $T$ and $F$, known as *truth values*, are also used, along with parentheses, and along with additional symbols, such as $A, B, C, \ldots$ (etc.), known as[100] *atomic statements*. In the instructions we are told that we should think of atomic statements as sentences conveying just one thought, e.g. "The sky is blue," as opposed to compound sentences such as "The sky is blue and Platonism is a thoroughly discredited philosophy."

Unfortunately, the game is generally played alone, and it is not clear what "winning" means. But it does while away the time, and keeps one out of trouble[101].

As for the rules: The first one specifies the arrangements of these symbols which are to be regarded as '(valid) statements', which one could think of as meaningful sentences.

---

[98]This is accomplished by being "beamed up."

[99]Computer apps are apparently available, but after our initial experience we decide against that option.

[100]If this terminology seems vaguely familiar, see Example 10.1(c).

[101]On the other hand, this game is not quite what we had in mind when we applied for R&R.

**Logical Rules 11.1 - The Rules for Statement Formation :** *Within the theory of sentential inference only the following qualify as 'statements'*:

(a) *an atomic statement* ; *or*

(b) *a string of symbols having one of the following forms, where* $A$ *and* $B$ *are statements* :

| | | |
|---|---|---|
| (i) | $(A)$ | |
| (ii) | $\neg(A)$ | (the *negation* of $A$ ) |
| (iii) | $(A) \wedge (B)$ | (the *conjunction* of $A$ and $B$ ) |
| (iv) | $(A) \vee (B)$ | (the *disjunction* of $A$ and $B$ ) |
| (v) | $(A) \Rightarrow (B)$ | (the *conditional* of $A$ and $B$ ). |

*Nothing else is a statement.*

We are told that statements constructed as in (ii)-(v) of 11.1(b) are called *non-atomic*, and $A$ in (i)-(v), as well as $B$ in (iii)-(v), are the *component parts*. We are also informed that (ii) is read as 'not $A$', (ii) as '$A$ and $B$', (iv) as '$A$ or $B$', and (v) as '$A$ implies $B$', or as 'if $A$ then $B$', or as '$A$ is a sufficient condition for $B$', or as '$B$ is a necessary condition for $A$'.

In addition we are told that when confusion cannot result parentheses can be freely omitted, e.g., $\neg(A)$ would usually appear as $\neg A$, and $(A) \vee (B)$ as $A \vee B$, etc.

Last but not least, we are told that statements of the form $A \Rightarrow B$ are called *implications*, and that $B \Rightarrow A$ is the *converse* (*implication*) of (the implication) $A \Rightarrow B$. The game designer was kind enough to supply us with examples[102].

**Examples 11.2 - Examples of Statement Formation :** Let $A$ and $B$ be statements.

(a) $(\neg A)$ *is a statement*; use 11.1(ii), omitting parentheses, and then 11.1(i).

(b) $\neg(\neg A)$ *is a statement*; again use 11.1(ii), but applied to $\neg A$ rather than $A$.

(c) $A\neg$ *is not a statement*, since it is not formed according to Rules 11.1.

(d) $\neg A \Rightarrow \neg B$, i.e., $(\neg A) \Rightarrow (\neg B)$, *is a statement.* It is called the *inverse* (*implication*) of the implication $A \Rightarrow B$.

[102] One wonders if this is the case with the standard edition.

(e) $\neg B \Rightarrow \neg A$, i.e., $(\neg B) \Rightarrow (\neg A)$, *is a statement.* It is called the *contrapositive* (*implication*) of the implication $A \Rightarrow B$.

We next encounter the rules for assigning 'truth' or 'falsity' to a statement when the truth or falsity of the component parts has already been specified.

**Logical Rules 11.3 - The Rules for Assigning Truth Values :** *Let $S, S'$ and $S''$ be statements. Then one is permitted to assign a truth value $T$ or $F$ to $S$, $\neg S$, $S' \wedge S''$, $S' \vee S''$ and $S' \Rightarrow S''$, e.g., to write a $T$ or $F$ under each, only according to the following rules*:

(a) *If $S$ is atomic either of $T$ or $F$ can be assigned.*

(b) *If truth values have been assigned to $S$, $S'$ and $S''$, then a truth value can be assigned to $\neg S$, $S' \wedge S''$, $S' \vee S''$ and $S' \Rightarrow S''$ only in accordance with the following tables.*

| **Negation** | |
|---|---|
| $S$ | $\neg S$ |
| $T$ | $F$ |
| $F$ | $T$ |

| **Conjunction** | | |
|---|---|---|
| $S'$ | $S''$ | $S' \wedge S''$ |
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ |
| $F$ | $T$ | $F$ |
| $F$ | $F$ | $F$ |

| **Disjunction** | | |
|---|---|---|
| $S'$ | $S''$ | $S' \vee S''$ |
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $T$ |
| $F$ | $T$ | $T$ |
| $F$ | $F$ | $F$ |

| **Conditional** | | |
|---|---|---|
| $S'$ | $S''$ | $S' \Rightarrow S''$ |
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ |

For example, the line $T\ F\ F$ in the table for conjunction indicates that $F$ must be assigned to $S' \wedge S''$ if $T$ has been assigned to $S'$ and $F$ has been assigned to $S''$.

The instruction sheet suggests the following informal summary: $\neg A$ is true only when $A$ is false; $A \wedge B$ is true only when both $A$ and $B$ are true; $A \vee B$ is false only when both $A$ and $B$ are false; $A \Rightarrow B$ is false only when $A$ is true and $B$ is false.

Bells start to jangle in our heads: except for the notation used, these tables seem oddly familiar. The Negation table looks awfully close to, but has a much simpler form than, Table V in the section on Proposition Algebras; the Conjunction table looks like Table IV; the Disjunction table looks like Table III; and the Conditional table looks like Table VI. Have we played this game before, perhaps in a different world? This game is starting to become interesting, and we decide to continue reading the instructions.

They go on to state that, with the possible exception of the last two lines of the Conditional Table, the rules have been designed to conform with everyday notions of the 'truth' and 'falsity' of statements. To motivate these final cases we are told to think of $A \Rightarrow B$ as "If $A$ happens then $B$ must happen". If we agree that this statement carries the same information as "Either $B$ happens or else $A$ did not happen", i.e., $B \vee (\neg A)$, then the choices in those last two lines are dictated by the choices in the Negation and Disjunction Tables. Indeed, we are told that in Example 11.5(a) we will see that these two statements are 'logically equivalent'.

The instructions note that, as a result of Rule 11.3(b), the assignment of a truth value to a statement is automatic once truth values have been assigned to the atomic parts. (Why do "valuations" and "Proposition Algebras" suddenly spring to mind?) We are furnished with a nice example (which seems far easier to grasp than Table X).

**Example 11.4 :** Consider the statement $((A \Rightarrow B) \wedge B) \Rightarrow A$. Suppose $T$ has been assigned to $A$, and $F$ to $B$. Then from the line $T$ $F$ $F$ in the Conditional Table we see that $F$ must be assigned to $A \Rightarrow B$; whence from the line $F$ $T$ $F$ in the Conjunction Table (using $A \Rightarrow B$ for $S'$ and $B$ for $S''$) that $F$ must be assigned to $(A \Rightarrow B) \wedge B$; whence from the line $F$ $T$ $T$ in the Conditional Table (using $(A \Rightarrow B) \wedge B$ for $S'$ and $A$ for $S''$) that $T$ must be assigned to the complete statement. This sequence of steps is easily summarized by means of the table

| $A$ | $B$ | $A \Rightarrow B$ | $(A \Rightarrow B) \wedge B$ | $((A \Rightarrow B) \wedge B) \Rightarrow A$ |
|---|---|---|---|---|
| $T$ | $F$ | $F$ | $F$ | $T$ |

.

Of course there are other possible assignments of $T$'s and $F$'s to $A$ and $B$; a complete summary is given next.

| $A$ | $B$ | $A \Rightarrow B$ | $(A \Rightarrow B) \wedge B$ | $((A \Rightarrow B) \wedge B) \Rightarrow A$ |
|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $T$ | $F$ |
| $F$ | $F$ | $T$ | $F$ | $T$ |

As we will only have need of the first two and last columns, we abbreviate this to

| $A$ | $B$ | $((A \Rightarrow B) \wedge B) \Rightarrow A$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $T$ |
| $F$ | $T$ | $F$ |
| $F$ | $F$ | $T$ |

and we call this last configuration the *truth table* of the statement $((A \Rightarrow B) \wedge B) \Rightarrow A$.

Suddenly we encounter something we haven't seen in Propositional Algebra, but which we could easily formulate in that framework.

The instructions state that two statements are (*logically*) *equivalent* if their truth tables become indistinguishable when the statements are erased, i.e., when the entries in the upper right hand corners are covered up[103].

**Examples 11.5 - Examples of Equivalent Statements :**

(a) $A \Rightarrow B$ *and* $B \vee (\neg A)$ *are equivalent.* Indeed, simply compare (and in the second case verify) the following truth tables.

| $A$ | $B$ | $A \Rightarrow B$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ |

| $A$ | $B$ | $B \vee (\neg A)$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ |

(b) $A$ *and* $\neg(\neg A)$ *are equivalent.*

(c) $A \Rightarrow B$ *and the contrapositive* $\neg B \Rightarrow \neg A$ *are equivalent.*

Just as abruptly we return to familiar territory. The instructions tell us that a statement is a *tautology* if the right-hand column of the associated truth table consists only of $T$'s. this is clearly *not* the case for the statement $((A \Rightarrow B) \wedge B) \Rightarrow A$ discussed above. A correct example would be $(A \wedge (A \Rightarrow B)) \Rightarrow B$, as can be seen directly from the next truth table (which the instructions suggest the reader should verify). This particular implication occurs so often it carries a name: *modus ponens.*

[103]How's that for a rigorous mathematical definition? Remember, we are currently in the real world!

| $A$ | $B$ | $(A \wedge (A \Rightarrow B)) \Rightarrow B$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $T$ |
| $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ |

We then read: Notice that when $A$ and $B$ are logically equivalent statements both $A \Rightarrow B$ and $B \Rightarrow A$ are tautologies.

Now the terminology begins to change, but not the concept. We read that a statement $B$ is *logically implied* by statements $A_1, A_2, \ldots, A_n$, or is said to be a *logical consequence* of these statements, if the implication $(A_1 \wedge (A_2 \wedge \cdots \wedge (A_n) \cdots)) \Rightarrow B$ is a tautology. For example, the truth table above shows that $B$ is a logical consequence of $A$ and $A \Rightarrow B$. Notice that each of tow logically equivalent statements is logically implied by the other. This is what in the mathematical world we called "semantically implies."

We are back in familiar territory when we read that a statement of the form $A \wedge (\neg A)$ is called a *contradiction*, or is said to be *inconsistent*. It is noted that, in contrast to a tautology, the right-hand column of the associated truth table contains only $F$'s.

Now we reach something really quite new.

**Logical Rules 11.6 - The Rules of Sentential Inference :** A (logically) (valid) argument *consists of a collection of statements, called the* hypotheses *or* premises (of the argument)*, an additional formula called the* conclusion (of the argument)*, together with a finite listing $A, B, \ldots$ of statements, ending with the conclusion, such that for each statement $S$ on the list*:

(a) *$S$ is a(n) hypothesis ; or*

(b) *$S$ is a logical consequence of the conjunction of some collection of prior statements on the list ; or*

(c) *$S$ is a tautology ; or*

(d) *$S$ is logically equivalent to a prior statement $S'$ on the list ; or*

(e) *a contradiction is a logical consequence of $\neg S$ in conjunction with some permissible combination of prior statements on the list.*

*The conclusion of a valid argument is said to be a* logical consequence of the hypotheses*, or to be* (logically) implied *by the hypotheses. When some statement in an* *argument is justified by* (e) *the argument is said to be* by contradiction, indirect*, or by* reductio ad absurdum*. Otherwise the argument is* direct*.*

**Example 11.7 :** If $A$, $A \Rightarrow B$, and $B \Rightarrow C$ are hypotheses then

$$\begin{array}{l} A \\ A \Rightarrow B \\ B \\ B \Rightarrow C \\ C \end{array}$$

is a valid argument with conclusion $C$. Indeed, $A$ and $A \Rightarrow B$ are hypotheses, $B$ is a logical consequence of $A$ and $A \Rightarrow B$ (modus ponens), $B \Rightarrow C$ is a hypothesis, and $C$ is a logical consequence of $B$ and $B \Rightarrow C$ (again modus ponens). The argument is direct.

This real-world visit was beneficial far beyond expectations. We can now see that the proposition algebra $P(X)$ is a set-theoretically based analogue of sentential inference, that "statements"' in the Theory of Sentential Inference have set-theoretical counterparts, and that the notation we have used with the proposition algebra $P(X)$ is terrible, the only virtue being that is reminds us that we are actually working with sets. Our first task upon our return[104] will be to change that notation.

[104]I.e. after being "beamed down."

# 12. Proposition Algebras Revisited

*In this section* $T = \{\neg, \Rightarrow\}$ *is the UA type which in* §10 *was denoted* $\{t', t''\}$. *Specifically, the unary operation* $t'$ *will now be denoted* $\neg$, *and the binary operation* $t''$ *will now be denoted* $\Rightarrow$. *Throughout* $X$ *denotes a non-empty set, and* $P(X)$ *denotes the free proposition* ($T$-)*algebra on* $X$.

The notation associated with $t'$ and $t''$ is changed accordingly, so as to conform with what is seen in sentential inference. Specifically, for $p, q \in P(X)$:

(12.1)
$$\begin{cases} \text{(negation)} & t'_{P(X)}(p) & \text{now becomes} & \neg p & \text{(read "not p")} \\ \text{(disjunction)} & t^{\vee}_{P(X)}(p,q) & \text{now becomes} & p \vee q & \text{(read "p or q")} \\ \text{(conjunction)} & t^{\wedge}_{P(X)}(p,q) & \text{now becomes} & p \wedge q & \text{(read "p and q"), and} \\ \text{(implication)} & t''_{P(X)}(p,q) & \text{now becomes} & p \Rightarrow q & \text{(read "p implies q").} \end{cases}$$

The notations for other induced operations are modified in predictable ways. For example, the definition of the concatenation of elements $p_1, p_2, \ldots, p_n \in P(X)$ (see (10.13)) now assumes the far more palatable form

$$t^{\wedge}_{P(X)}(p_1, p_2, \ldots, p_n) := p_1 \wedge p_2 \wedge \cdots \wedge p_n, \tag{12.2}$$

whereupon Proposition 10.15 becomes

**Proposition 12.3 :** *Suppose* $A \subset P(X)$ *is non-empty and finite, say* $A = \{p_1, p_2, \ldots, p_n\}$, *suppose* $q \in A$, *and suppose* $p_1 \wedge p_2 \wedge \cdots \wedge p_n \Rightarrow q$ *is a tautology. Then*

$$A \models q.$$

When there is a need to distinguish an operation in $T$ with the associated arity operation on $P(X)$ a subscript will be added to denote the latter, e.g. $\wedge_{P(X)}$ would denote the binary operation $(p,q) \in (P(X))^2 \mapsto p \wedge q \in P(X)$ corresponding to the operation $\wedge \in T_2$.

Readers are no doubt already comparing this switch in notation to a sudden cool breeze on a hot, humid summer day. It is even better than that: it vastly simplifies the appearance and construction of tables, particularly if we omit the $\nu$ symbol in the top boxes and replace all occurrences of [1] and [0] by[105] $T$ (read "true") and $F$ (read "false") For example, Table VI, which appears shortly after (10.7), i.e.

[105]Since we are dealing with "$T$-algebras," this introduces a conflict of notation. However, in practice the meaning of $T$ is almost always clear from context.

| $\nu(p)$ | $\nu(q)$ | $\nu(t''_{P(X)}(p,q))$ |
|---|---|---|
| [1] | [1] | [1] |
| [1] | [0] | [0] |
| [0] | [1] | [1] |
| [0] | [0] | [1] |

,

now becomes the table for implication[106] i.e.

| $p$ | $q$ | $p \Rightarrow q$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ |

,

and Table X of Example 10.9(b), i.e.

| $\nu(p)$ | $\nu(q)$ | $\nu(t'(p))$ | $\nu(t'(q))$ | $\nu(t^{\vee}(t'(p), t'(q)))$ | $\nu(t'\big(t^{\vee}(t'(p), t'(q))\big))$ |
|---|---|---|---|---|---|
| [1] | [1] | [0] | [0] | [0] | [1] |
| [1] | [0] | [0] | [1] | [1] | [0] |
| [0] | [1] | [1] | [0] | [1] | [0] |
| [0] | [0] | [1] | [1] | [1] | [0] |

,

now becomes

| $p$ | $q$ | $\neg p$ | $\neg q$ | $\neg p \vee \neg q$ | $\neg(\neg p \vee \neg q)$ |
|---|---|---|---|---|---|
| $T$ | $T$ | $F$ | $F$ | $F$ | $T$ |
| $T$ | $F$ | $F$ | $T$ | $T$ | $F$ |
| $F$ | $T$ | $T$ | $F$ | $T$ | $F$ |
| $F$ | $F$ | $T$ | $T$ | $T$ | $F$ |

.

As another example of how this notation simplifies matters note from the first two rows of the table

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $T$ |
| $F$ | $T$ | $T$ |
| $F$ | $F$ | $F$ |

,

[106]Technically, for the "conditional."

for disjunction that $\{p\} \models p \vee q$, but from the final column that $p \rightarrow p \vee q$ is not a tautology. The converse of Proposition 10.15 is therefore false.

Hopefully readers can now understand what is going on: a "mathematical world" entity has been determined[107], involving only sets, but sets which, from a mathematical viewpoint, are amenable to study just as if[108] they were sentences within the "real world" construct of Sentential Inference. In particular, even if it we totally confine our mathematics to the mathematical world it now makes perfect sense to speak of "sets of sentences," or, perhaps more appropriately, "sets of propositions."

It is fairly easy to formulate Logical Rules 11.6 in this setting, and thereby justify speaking of a "logically valid argument" in a purely set-theoretic context. To formulate model theory from that perspective one has to push beyond what we have done so as to include quantifiers. The result is called "first order logic." Readers interested in the that extension of our work thus far should see, e.g. [B-M, Chapters IV and V].

[107] Or constructed, whatever your preference.

[108] Or "essentially as if." I cannot go so far as to that say the systems are "isomorphic," because that would imply both belong to the mathematical world.

# 13. Topological Considerations

*When $X$ is a topological space and $Y \subset X$ the closure of $Y$ will be denoted* $\mathrm{cl}(Y)$*, and when $Y = \{y\}$ is a singleton we will abbreviate* $\mathrm{cl}(\{y\})$ *as* $\mathrm{cl}(y)$*.*

Since propositional algebra has been reduced to set theory, the tools of topology now become available for use[109]. We do very little with this, only offering a few hints as to how the ideas we have seen thus far can be tied in with algebraic and differential algebraic geometry, and a few definitions which may help readers decipher some of the literature.

Let $X$ be a non-empty set equipped with a pre-order relation[110] $\mathcal{R} \subset X \times X$. Write $x \to y$ in place of $(x, y) \in \mathcal{R}$; write $x \leftrightarrow y$ to indicate the two conditions $x \to y$ and $y \to x$. The pre-order conditions are: $x \leftrightarrow x$ (reflexivity); $x \to y$ and $y \to z \Rightarrow x \to z$ (transitivity).

**Examples 13.1 :**

(a) Let $Y$ be a set[111] and let $X := P(Y)$ be the free proposition algebra on $Y$, i.e. the free $T$-algebra on $Y$, where $T = \{\neg, \Rightarrow\}$. Then a pre-order relation on $X$ is defined by declaring that $p \to q$ if $p \models q$, where $p \models q$ is being used as an abbreviation for $\{p\} \models q$. With less formality: $p \to q$ if $q$ is true whenever $p$ is true[112]. Reflexivity and transitivity are obvious.

(b) Let $R$ be a commutative ring with unity, let $X := \mathrm{Spec}(R)$ be the collection of all prime ideals of $R$, and for $\mathfrak{p}, \mathfrak{q} \in X$ let $\mathfrak{p} \to \mathfrak{q}$ mean $\mathfrak{p} \subset \mathfrak{q}$.

(c) Let $R = (R, \delta)$ be a commutative differential[113] ring, i.e. a commutative ring $R$ with unity together with a derivation $\delta : R \to R$. Call an ideal $\mathfrak{i} \subset R$ a *differential ideal* if $\mathfrak{i}$ is closed under the derivation, i.e. if $r \in \mathfrak{i} \Rightarrow \delta r \in \mathfrak{i}$, and let $X := \mathrm{diffSpec}(R)$ denote the collection of differential prime ideals[114].

---

[109] On the other hand, the fact that so many arrows are in use suggests that we might look at things categorically. There is, however, a limit to the scope of these notes.

[110] The definition is recalled in the next two sentences.

[111] To conform with previous notation we should use $X$, but in this section $X$ is reserved for topological spaces.

[112] Note how we have suddenly lapsed into the sort of language a model theorist would use. What $p \to q$ "really" means is that $\nu(q) = [1]$ for all valuations $\nu : P(Y) \to \mathbb{Z}/2\mathbb{Z}$ such that $\nu(p) = [1]$. We say "with less formality" since there are no such valuations when $p$ is a contradiction.

[113] I was feeling guilty about mentioning derivation only once thus far in these notes (see Example 3.2(d)). After all, this talk is being prepared for a differential algebra seminar!

[114] That is, the prime ideals which are also differential ideals.

For $\mathfrak{p}, \mathfrak{q} \in X$ let $\mathfrak{p} \to \mathfrak{q}$ mean $\mathfrak{p} \subset \mathfrak{q}$. (In other words, restrict the relation of Example (b) to diffSpec$(R)$.)

For each $p \in X$ let

$$D(p) := \{ q \in X \, : \, p \not\to q \}. \tag{13.2}$$

Note that $p \notin D(p)$. The *pre-order topology* on $X$ is the topology generated by these sets; a neighborhood basis is provided by the collection of finite intersections of sets of the form $D(p)$. For each $p \in X$ let $V(p)$ denote the (closed) complement of $D(p)$, i.e.,

$$V(p) := \{ q \in X \, : \, p \to q \}. \tag{13.3}$$

Note that

$$p \in V(p). \tag{13.4}$$

The pre-order topology in Example 13.1(b) is called the *Zariski topology*; that in Example 13.1(c) is the *Kolchin topology*.

**Examples 13.5 :**

(a) Let $X = P(Y)$ be as in Example 13.1(a) and let $p \in Y$. Then

$D(p)$ consists of all those $q \in X$ such that $p \not\to q$, i.e. all those $q$ such that for some valuation $\nu = \nu(q) : X \to \mathbb{Z}/2\mathbb{Z}$ one has

$$\nu(p) = T \qquad \text{and} \qquad \nu(q) = F.$$

and

$V(p)$ consists of all those $q \in X$ such that $p \to q$, i.e. all those $q$ such that for all valuations $\nu : X \to \mathbb{Z}/2\mathbb{Z}$ one has

$$\nu(q) = T \qquad \text{if} \qquad \nu(p) = T.$$

(b) Suppose $R$ is a commutative ring with unity and the Zariski topology is assumed on $X := \text{Spec}(R)$. Then for any prime ideal $\mathfrak{p} \in X$ the collection $D(\mathfrak{p})$ is the set of prime ideals which do not contain $\mathfrak{p}$ and $V(\mathfrak{p})$ is the collection of prime ideals which do contain $\mathfrak{p}$.

(c) Suppose $R = (R, \delta)$ is a commutative differential ring and the Zariski topology is assumed on $X := \text{diffSpec}(R)$. Then for any differential prime ideal $\mathfrak{p} \in X$ the collection $D(\mathfrak{p})$ is the set of differential prime ideals which do not contain $\mathfrak{p}$ and $V(\mathfrak{p})$ is the collection of differential prime ideals which do contain $\mathfrak{p}$.

**Proposition 13.6 :** *Let $X$ be a non-empty set with a pre-order $\rightarrow$ and assume $X$ has been endowed with the pre-order topology. Then for any $p, q \in X$ the following statements hold:*

(a) $V(p) = \text{cl}(p)$, *i.e.* $V(p)$ *is the closure of the point* $p$.

(b) $p \rightarrow q$ *if and only if* $V(q) \subset V(p)$*;*

(c) $p \leftrightarrow q$ *if and only if* $V(p) = V(q)$.

In classical geometry the closure of a point was referred to as the *locus* of the point. One still encounters this terminology in algebraic and differential algebraic geometry.

**Proof :**

(a) The closure $\text{cl}(p)$ is certainly contained in $V(p)$, and if the result is false there is a point $q \in V(p)\backslash\text{cl}(p)$ (necessarily) contained in an open set $U$ having empty intersection with $\text{cl}(p)$. It follows from the neighborhood basis comments that there is a finite collection $r_1, \ldots, r_n \in X$ such that $q \in \cap_{j=1}^{n} D(r_j) \subset U$, and since $p \notin U$ at least one $r_j =: r$ is such that $p \notin D(r)$. But $p \notin D(r) \Rightarrow r \rightarrow p$ and $q \in V(p) \Rightarrow p \rightarrow q$. Transitivity then guarantees $r \rightarrow q$, thereby contradicting $q \in D(r)$.

(b) One has

$$\begin{aligned} p \rightarrow q \quad &\Leftrightarrow \quad q \in V(p) \\ &\Leftrightarrow \quad q \in \text{cl}(p) \qquad \text{(by (a))} \\ &\Leftrightarrow \quad \text{cl}(q) \subset \text{cl}(p) \quad \text{(by the minimal property of closures)} \\ &\Leftrightarrow \quad V(q) \subset V(p) \quad \text{(again by (a)).} \end{aligned}$$

(c) Immediate from (b).

**q.e.d.**

Combining Proposition 13.6 with (13.2) and (13.3) we see that for $p, q \in X$ we have both

$$q \in D(p) \quad \Leftrightarrow \quad p \not\to q \quad \Leftrightarrow \quad q \in X \setminus \mathrm{cl}(p) \tag{13.7}$$

and

$$q \in V(p) \quad \Leftrightarrow \quad p \to q \quad \Leftrightarrow \quad q \in \mathrm{cl}(p). \tag{13.8}$$

Let $X$ be any topological space.

- A point $x \in X$ is *closed*, or is a *closed point*, if $\mathrm{cl}(x) = \{x\}$.
- A point $x$ is (*a*) *generic* (*point*) (*of*) $X$ if $\mathrm{cl}(x) = X$.
- Any point $y \in \mathrm{cl}(x)$ is a *specialization of* $x$, and to indicate this one writes[115] $x \to y$ (read: "x specializes to y").
- A specialization $y$ of $x$ is a *generic specialization* (of $x$) if $y \to x$, and when then is the case we write $x \leftrightarrow y$ (read: "x generically specializes to y"). Note that $x \leftrightarrow y$ is an equivalence relation; this is a useful property of generic specializations which is not shared by specializations.
- A specialization $y$ of $x$ is a *non-generic specialization* (of $x$) if $y \not\to x$. The set of all (if any) non-generic specializations of $x$ is denoted $NGS_x$.
- The point $x$ is *constrained* if $NGS_x$ is a closed set.
- An element $x \in X$ is *isolated* if $\mathrm{cl}(x)$ is minimal (w.r.t. inclusion) among all closed subsets of $X$ of the form[116] $\mathrm{cl}(y)$.
- A subset $A \subset X$ is *locally closed* if $A = U \cap C$, where $U \subset X$ is open and $C \subset X$ is closed.
- A subset $A \subset X$ is *constructible* if it can be expressed as the finite disjoint union of locally closed sets[117].

[115]The use of the symbol $\to$ does *not* imply that the pre-order topology is being assumed, but that possibility is also not excluded.

[116]One is reminded of the following result for commutative rings with identity: an element $a$ of the ring is irreducible if and only if the principal ideal $(a)$ is maximal among all proper principal ideals of the ring. See, e.g., [Hun, Theorem 4.3(ii), p. 136].

[117]We take this definition from [Hart, Chapter II, Exercise 3.18, p. 94].

- $X$ is a *Stone space* if it is a compact, Hausdorff, and admits a neighborhood basis consisting of "clopen" sets, i.e. sets which are both open and closed[118].

**Examples 13.9 :**

(a) Let $X$ be a set and endow the associated proposition algebra $P(X)$ with the pre-order topology. Then the closure of any tautology is the collection of all tautologies in $P(X)$.

> Let $p \in P(X)$ be a tautology. If $q \in P(X)$ is not a tautology there is a valuation $\nu$ such that $\nu(q) = F$, whereas $\nu(p) = T$, and $p \models q$ is therefore false. On the other hand, $p \models q$ is obviously true when $q$ is a tautology.

(b) Let $P(X)$ (and the topology on $P(X)$) be as in (a). Then then each logically invalid statement is generic.

> If $p \in P(X)$ is logically invalid and $q \in P(X)$ the implication $p \Rightarrow q$ is true regardless of the truth value of $q$, hence $p \models q$, and $q \in \mathrm{cl}(p)$ therefore holds.

(c) Let $R$ be a commutative integral domain with unity and endow $\mathrm{Spec}(R)$ with the Zariski topology. Then the zero ideal $(0)$ is generic in $\mathrm{Spec}(R)$. (One needs the integral domain hypothesis to guarantee that $(0) \in \mathrm{Spec}(R)$, i.e. that $(0)$ is a prime ideal.)

**Proposition 13.10 :** *Let $X$ be a non-empty set with a pre-order $\to$ and assume $X$ has been endowed with the pre-order topology. Then for any $p, q \in X$ the following statements are equivalent:*

(a) *$p$ is isolated;*

(b) $p \to q \Rightarrow V(p) = V(q)$; *and*

(c) $p \to q \Rightarrow p \leftrightarrow q$.

Condition (c) is my interpretation of Kolchin's definition of "isolated" [Kol, p. 386].

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[118] The definition is from [Roth, Chapter 5, §6, p. 62].

**Proof :**

(a) $\Rightarrow$ (b) : For any $q \in X$ we have

$$\begin{aligned} p \to q \ &\Leftrightarrow\ V(q) \subset V(p) \quad \text{(by Proposition 13.6(b))} \\ &\Rightarrow\ V(p) = V(q) \quad \text{(by the minimality of } V(p)\text{).} \end{aligned}$$

(b) $\Rightarrow$ (c) : By Proposition 13.6(c).

(c) $\Rightarrow$ (a) : Suppose $q \in X$ is such that $\text{cl}(q) \subset \text{cl}(p)$, which by Proposition 13.6(a) is equivalent to $V(q) \subset V(p)$. The equality $\text{cl}(q) = \text{cl}(p)$ needed to establish (a) is then seen from

$$\begin{aligned} V(q) \subset V(p) \ &\Leftrightarrow\ p \to q &&\text{(by Proposition 13.6(b))} \\ &\Rightarrow\ p \leftrightarrow q &&\text{(by (c))} \\ &\Leftrightarrow\ V(p) = V(q) &&\text{(by Proposition 13.6(c))} \\ &\Leftrightarrow\ \text{cl}(p) = \text{cl}(q) &&\text{(by Proposition 13.6(a)).} \end{aligned}$$

**q.e.d.**

Recall that a pre-order $\to$ on a set $X$ is a *partial order relation* if for $p, q \in X$ one has $p \leftrightarrow q$ if and only if $p = q$.

**Corollary 13.11 :** *When the relation* $\to$ *in* Proposition 13.10 *is a partial order relation and* $p \in X$ *the following statements are equivalent:*

(a) $p$ *is isolated; and*

(b) $p$ *is a closed point.*

For example, the maximal ideals are the isolated points of $\text{Spec}(R)$ for any commutative ring $R$ with unity.

Note from the bulleted definitions that

$$\text{cl}(x) = GS_x \cup NGS_x \qquad \text{(disjoint union).} \tag{13.12}$$

Two easy consequences to keep in mind are

**Proposition 13.13 :** *A point* $x \in X$ *is constrained if and only if* $GS_x \subset X$ *is locally closed.*

and

**Proposition 13.14 :** *Let* $x \in X$. *Then an open subset of* $X$ *contains* $x$ *if and only if it contains* $GS_x$.

# 14. Closure Operations

Let $X$ be a non-empty set with a pre-order relation $\to$. In [B-M, Chapter I, §3, Lemma 3.9, p. 14][119], a mapping $\kappa : X \to X$ is called a *closure operation* if the following three conditions are satisfied for any $x, y \in X$:

I. $x \to \kappa(x)$;

II. $\kappa(\kappa(x)) = \kappa(x)$; and

III. $A \to B \quad \Rightarrow \quad \kappa(A) \to \kappa(B)$.

When $\kappa : X \to X$ is such a mapping and $x \in X$ we call $\kappa(x)$ the *closure* of $x$, and an element $x \in X$ satisfying $x = \kappa(x)$ is said to be[120] *closed*.

**Examples 14.1 :**

(a) Let $R$ be a commutative ring with unity, let $X := \text{Spec}(R)$, and let $\to$ be inclusion. Then the mapping sending a prime ideal $\mathfrak{p} \in X$ to its radical $\sqrt{\mathfrak{p}} \in X$ is a closure operation.

(b) Let $Y$ be a topological space and let $X$ be the power set of $Y$ with inclusion as the pre-order. Then the mapping sending $A \in \mathcal{P}(A)$ to $\text{cl}(A)$ is a closure operation.

(c) Examples of closure operations arise in proposition algebras, but we have not covered enough background to explain them[121]. See [B-M, Chapter II, §3, Lemma 3.9, p. 14 and §4, Lemma 4.4, pp. 15-6].

[119] In fact this reference only considers the case in which $X$ is a power set and the pre-order relation is inclusion.

[120] One must be careful with this terminology since it is defined without reference to a topology.

[121] Nor do we have any intention of doing so.

**Proposition 14.2 :** *Suppose* $\kappa : X \to X$ *is a closure operation on a set* $X$ *with a partial order relation* $\to$, *and suppose* $x, y \in X$. *Then the following results hold.*

(a) *If* $x \to y$, $y \to \kappa(x)$ *and* $y$ *is closed then* $y = \kappa(x)$.

(b) *Suppose* $(x, y) \mapsto x \vee y$ *is a binary operation on* $X$ *such that for all* $x, y \in X$ *one has*

(i) $$x \to x \vee y \qquad \text{and} \qquad y \to x \vee y,$$

(ii) $$x \vee y \to \kappa(x) \vee \kappa(y),$$

*and*

(iii) $x \vee y \to z$ *whenver* $z \in X$ *is closed and* $x \to z$ *and* $y \to z$.

*Then*

(iv) $$\kappa(x \vee y) = \kappa(\kappa(x) \vee \kappa(y)) \qquad \text{for all} \qquad x, y \in X.$$

**Proof :**

(a) This is seen from the chain of implications

$$\begin{aligned} x \to y \to \kappa(x) \quad &\Rightarrow \quad \kappa(x) \to \kappa(y) \to \kappa(\kappa(x)) \quad &&\text{(by III)} \\ &\Rightarrow \quad \kappa(x) \to \kappa(y) \to \kappa(x) &&\text{(by II)} \\ &\Rightarrow \quad \kappa(x) \to y \to \kappa(x) &&\text{(because } y \text{ is closed)} \\ &\Rightarrow \quad y = \kappa(x). \end{aligned}$$

(b) In this case we see from (III) that

$$x \to x \vee y \qquad \Rightarrow \qquad \kappa(x) \to \kappa(x \vee y)$$

and that

$$y \to x \vee y \qquad \Rightarrow \qquad \kappa(y) \to \kappa(x \vee y).$$

Since $\kappa(x \vee y)$ is closed it follows from (i) and (ii) that

$$\kappa(x) \vee \kappa(y) \to \kappa(x \vee y),$$

whereas from (iii) we have

$$x \vee y \subset \kappa(x) \vee \kappa(y).$$

This gives

$$x \vee y \subset \kappa(x) \vee \kappa(y) \rightarrow \kappa(x \vee y)$$

whereupon (iv) is seen to follow from (a).

**q.e.d.**

Closure operations also arise in topology, e.g. see [Kelly, Chapter I, p. 43], but only in the context of Example 14.1(b), and the definition in that case is not equivalent to the one we have given. Instead one assumes the existence of a mapping $\kappa : \mathcal{P}(Y) \rightarrow \mathcal{P}(Y)$ such that for all subsets $A, B \subset Y$:

I. $A \subset \kappa(A)$; (as before)

II. $\kappa(\kappa(A)) = \kappa(A)$; (as before)

IV. $\kappa(A \cup B) = \kappa(A) \cup \kappa(B)$; and

V. $\kappa(\emptyset) = \emptyset$.

One then proves that these conditions are necessary and sufficient for the collection $\{\kappa(A)\}_{A \in \mathcal{P}(Y)}$ to constitute the closed sets of a topology[122] on $X$. Note that IV is fairly close[123] to (iv) of Proposition 14.2, but there seems no obvious generalization of V (although integral domains come to mind if one is working with $\mathrm{Spec}(R)$ and one thinks of $\emptyset$ as representing the zero ideal).

[122] See the reference to [Kelly] given above, where the definition and result are attributed to Kuratowski.

[123] But no cigar.

**Notes and Comments**

§6. Our discussion of the distinction between sets and classes is a bit more formal than most mathematicians would require in practice. For a brief but very clear presentation at a less formal level I recommend [Roth, Chapter 7, §7.4, pp. 90-5].

§11. The presentation of this material owes a great deal to [Su].

§12. "Order topologies" on linearly ordered spaces are standard (see, e.g. [Kelly, Chapter I, Exercise I, pp. 57-8], and I am confident that the pre-order topology has also been throughly studied, although perhaps under a different name. The idea of using that topology in connection with differential algebra arose from conversations with Jerry Kovacic and Peter Landesman, and some of the results stated here about this topology appear in [Landes].

§13. I found [Bour, Chapter II, §4, no1-no3, pp. 94-104] a nice, compact (and, as one would expect, complete) source of information about the Zariski topology.

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Writing these notes turned out to be enjoyable far beyond expectations, and in the process I learned quite a bit of elementary set theory. I would like to thank all those who attended the talk in KSDA, in particular those who know far more about both set-theory and model theory than I ever will, for subsequent discussions on various foundational items which have long confused me.

The organization of this material was heavily influenced by preliminary discussions with Phyllis Cassidy and William Sit. In particular, the argument I given in Example 7.4(c) was strongly influenced by an observation made by the latter.

Conversations with Robin Cockett concerning the set-theoretical foundations of category theory were extremely helpful in "steering me in the right direction," as were conversations concerning universal algebras with Florian Lengyel.

Errors, of course, are my responsibility.

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R.C. Churchill
Department of Mathematics
Hunter College and the Graduate Center, CUNY
New York, NY
March, 2013

e-mail rchurchi@hunter.cuny.edu