Commutative Differential Algebra, Part III

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Basic assumptions.

All rings are associative, commutative, with 1, and are $\mathbb{Q}$-algebras.
Except for the 0 ring, they contain $\mathbb{Q}$.
$\Delta = \{\delta_1, \ldots, \delta_m\}$ commuting derivation operators.
$\Theta = \left\{\theta = \delta_1^{i_1} \cdots \delta_m^{i_m} : i = (i_1, \ldots, i_m) \in \mathbb{N}^m\right\}$.
$\mathcal{R}$: Δ-ring. We reserve the symbols $y, y_1, \ldots, y_n$ for Δ-indeterminates or families of such.

Summary of Part II

- Let Σ be a multiplicative set in $\mathcal{R}$. Let $\mathfrak{a}$ be a Δ-ideal avoiding Σ. $\exists$ maximal Δ-ideal $\mathfrak{p}$ containing $\mathfrak{a}$ and avoiding Σ. $\mathfrak{p}$ is prime.
- Let S be a subset of $\mathcal{R}$, $1 \notin [S]$. $\sqrt{[S]}$ is the intersection of the prime Δ-ideals containing S. $1 \in [S] \iff \sqrt{[S]}$ is not in any prime Δ-ideal of $\mathcal{R}$.
- The nil radical of $\mathcal{R}$ is the set of nilpotent elements of $\mathcal{R}$. It is a radical Δ-ideal and is the intersection of the prime ideals of $\mathcal{R}$, and of the minimal prime ideals of $\mathcal{R}$.
- Each minimal prime ideal of $\mathcal{R}$ is contained in the set of zero divisors of $\mathcal{R}$.

Theorem

Let $\mathfrak{a}$ be an ideal in a ring $\mathcal{R}$. Then, $\sqrt{\mathfrak{a}}$ is the intersection of the prime ideals containing $\mathfrak{a}$, and the minimal prime ideals containing $\mathfrak{a}$.

Proof.

Let $\pi : \mathcal{R} \to \mathcal{R}/\mathfrak{a}$ be the quotient homomorphism. $\mathfrak{n}\left(\mathcal{R}/\mathfrak{a}\right) = \pi\left(\sqrt{\mathfrak{a}}\right)$ is the intersection of all the prime ideals (and of all the minimal prime ideals) of $\mathcal{R}/\mathfrak{a}$. The mapping $\mathfrak{b} \longmapsto \pi\left(\mathfrak{b}\right)$ is an inclusion preserving bijective mapping from the set of ideals of $\mathcal{R}$ containing $\mathfrak{a}$ onto the set of ideals of $\mathcal{R}/\mathfrak{a}$ and preserves the arithmetic of ideals, radicality and primality. If, in addition, $\mathcal{R}$ and $\mathfrak{a}$ are differential, the mapping preserves stability under the set $\cdot$. □

Theorem

If $\mathcal{R}$ is any Δ-ring, then every minimal prime ideal $\mathfrak{p}$ of $\mathcal{R}$ is a Δ-ideal. (Keigher, Prime differential ideals in differential rings, 1977, Proposition 1.5, p. 242).

First, some handy symbolism.

The value of suggestive symbolism.

Notation: $\theta := \delta_1^{j_1} \cdots \delta_m^{j_m}$, The exponent vector $j = (j_1, \ldots, j_m)$, is the *multi-index of* θ.
ord $\theta = \sum_{i=1}^m j_i$.
$\delta = (\delta_1, \ldots, \delta_m)$. Write $\theta = \delta^j$.
$\theta' := \delta^{j'}$, $j' = (j_1', \ldots, j_m')$. $\theta' \leq \theta$ if $j' \leq j$ in the product order on $\mathbb{N}^m$.
Thus, $j_i' \leq j_i, i = 1, \ldots, m$.
$\binom{\theta}{\theta'} = \binom{j}{j'} = \prod_{i=1}^m \binom{j_i}{j_i'}$.

Example

$m = 2,\ \theta = \delta_1^2\delta_2 = \delta^{(2,1)}$. The lattice of multi-indices $\leq (2,1)$:

$$\begin{array}{ccccc} (0,1) & - & (1,1 & - & (2,1) \\ | & & | & & | \\ (0,0) & - & (1,0) & - & (2,0) \end{array}.$$

$$\begin{array}{cccc} j' & j-j' & \binom{j}{j'} = \binom{j_1}{j'_1}\binom{j_2}{j'_2} & \delta^{j-j'}a \cdot \delta^{j'}b \\ (0,0) & (2,1) & \binom{(2,1)}{(0,0)} = \binom{2}{0}\binom{1}{0} = 1 & \delta_1^2\delta_2 a \cdot b \\ (1,0) & (1,1) & \binom{(2,1)}{(1,0)} = \binom{2}{1}\binom{1}{0} = 2 & \delta_1\delta_2 a \cdot \delta_1 b \\ (2,0) & (0,1) & \binom{(2,1)}{(2,0)} = \binom{2}{2}\binom{1}{0} = 1 & \delta_2 a \cdot \delta_1^2 b \\ (0,1) & (2,0) & \binom{(2,1)}{(0,1)} = \binom{2}{0}\binom{1}{1} = 1 & \delta_1^2 a \cdot \delta_2 b \\ (1,1) & (1,0) & \binom{(2,1)}{(1,1)} = \binom{2}{1}\binom{1}{1} = 2 & \delta_1 a \cdot \delta_1\delta_2 b \\ (2,1) & (0,0) & \binom{(2,1)}{(2,1)} = \binom{2}{2}\binom{1}{1} = 1 & a \cdot \delta_1^2\delta_2 b \end{array}.$$

Spose $\Delta = \{\delta\}$, and we write $\delta^j a = a^{(j)}$. Then, if $\mathcal{R}$ is a Δ-ring, and $a, b \in \mathcal{R}$,

$$\begin{aligned}(ab)^{(j)} &= \sum_{0 \le j' \le j} \binom{j}{j'} a^{(j-j')} b^{(j')}. \\ \delta^j (ab) &= \sum_{0 \le j' \le j} \binom{j}{j'} \delta^{j-j'} a \cdot \delta^{j'} b.\end{aligned}$$

Lemma

Leibniz Generalized Product Rule Let $\mathcal{R}$ be a Δ-ring, $a, b \in \mathcal{R}$, $\theta = \delta^j$, $\theta' = \delta^{j'} \in \Theta$. Then,

$$\delta^j (ab) = \sum_{0 \le j' \le j} \binom{j}{j'} \delta^{j-j'} a \cdot \delta^{j'} b.$$

The proof uses induction on j. It follows from the case of one derivation operator by applying each component of the multi-index to the formula for given j. We illustrate both the statement and the proof in our example. First follow the instructions from Leibniz.

j'	$j-j'$	$\binom{j}{j'} = \binom{j_1}{j'_1}\binom{j_2}{j'_2}$	$\delta^{j-j'}a \cdot \delta^{j'}b$
$(0,0)$	$(2,1)$	$\binom{(2,1)}{(0,0)} = \binom{2}{0}\binom{1}{0} = 1$	$\delta_1^2\delta_2 a \cdot b$
$(1,0)$	$(1,1)$	$\binom{(2,1)}{(1,0)} = \binom{2}{1}\binom{1}{0} = 2$	$\delta_1\delta_2 a \cdot \delta_1 b$
$(2,0)$	$(0,1)$	$\binom{(2,1)}{(2,0)} = \binom{2}{2}\binom{1}{0} = 1$	$\delta_2 a \cdot \delta_1^2 b$
$(0,1)$	$(2,0)$	$\binom{(2,1)}{(0,1)} = \binom{2}{0}\binom{1}{1} = 1$	$\delta_1^2 a \cdot \delta_2 b$
$(1,1)$	$(1,0)$	$\binom{(2,1)}{(1,1)} = \binom{2}{1}\binom{1}{1} = 2$	$\delta_1 a \cdot \delta_1\delta_2 b$
$(2,1)$	$(0,0)$	$\binom{(2,1)}{(2,1)} = \binom{2}{2}\binom{1}{1} = 1$	$a \cdot \delta_1^2\delta_2 b$

.

$$\begin{aligned}\delta^{(2,1)}(ab) &= \delta_1^2\delta_2 a \cdot b + 2\delta_1\delta_2 a \cdot \delta_1 b + \delta_2 a \cdot \delta_1^2 b + \\ &\quad + \delta_1^2 a \cdot \delta_2 b + 2\delta_1 a \cdot \delta_1\delta_2 b + a \cdot \delta_1^2\delta_2 b.\end{aligned}$$

Now, use induction, assuming the Leibniz formula for $\delta_2\left(ab\right)$, and $\delta_1^2\left(ab\right)$.

$$\begin{aligned}
\delta_1^2\left(ab\right) &= \delta_1^2 a\cdot b+2\delta_1 a\cdot\delta_1 b+a\cdot\delta_1^2 b. \\
\delta_2\left(\delta_1^2\left(ab\right)\right) &= \delta_2\delta_1^2 a\cdot b+2\delta_2\delta_1 a\cdot\delta_1 b+\delta_2 a\cdot\delta_1^2 b+ \\
&\quad +\delta_1^2 a\cdot\delta_2 b+2\delta_1 a\cdot\delta_2\delta_1 b+a\cdot\delta_2\delta_1^2 b \\
&= \delta_1^2\delta_2 a\cdot b+2\delta_1\delta_2 a\cdot\delta_1 b+\delta_2 a\cdot\delta_1^2 b+ \\
&\quad +\delta_1^2 a\cdot\delta_2 b+2\delta_1 a\cdot\delta_1\delta_2 b+a\cdot\delta_1^2\delta_2 b.
\end{aligned}$$

Same result!

We now prove Keigher's Theorem: Every minimal prime ideal in a Δ-ring is a Δ-ideal.

Proof.

Let

$$\mathfrak{p}_{\#} = \{a \in \mathfrak{p} : \forall \theta \in \Theta \ \ \theta a \in \mathfrak{p}\}.$$

$\mathfrak{p}_{\#}$ is clearly closed under sums. Let $a \in \mathcal{R}$ and $b \in \mathfrak{p}_{\#}$. Let $\theta = \delta^{j} \in \Theta$. By Leibniz's generalized product rule,

$$\delta^{j}(ab) = \sum_{0 \le j' \le j} \binom{j}{j'} \delta^{j-j'} a \cdot \delta^{j'} b.$$

By hypothesis, $\forall j' \le j$, $\delta^{j'} b \in \mathfrak{p}$, and $\binom{j}{j'} \in \mathbb{N}$. Therefore, since $ab \in \mathfrak{p}$, and $\delta^{j}(ab) = \theta(ab) \in \mathfrak{p}$, it follows that $ab \in \mathfrak{p}_{\#}$. So, $\mathfrak{p}_{\#}$ is an ideal contained in $\mathfrak{p}$. But, it is clearly stable under Δ. For, let $a \in \mathfrak{p}_{\#}$ and let $\delta \in \Delta$. $\delta a \in \mathfrak{p}$. Let $\theta \in \Theta$. Then, $\theta\delta a = \delta\theta a \in \mathfrak{p}$. So, $\mathfrak{p}_{\#}$ is a Δ-ideal contained in $\mathfrak{p}$. □

Proof.

continued. Since $\mathfrak{p}$ is prime, the nil radical $\mathfrak{n}$ is contained in $\mathfrak{p}$. Since it is a Δ-ideal, $\mathfrak{n} \subseteq \mathfrak{p}_{\#}$. Let $\Sigma = \mathcal{R} \backslash \mathfrak{p}$. $\mathfrak{n}$ and $\mathfrak{p}_{\#}$ are Δ-ideals avoiding $\pm$. Clearly, by definition, $\mathfrak{p}_{\#}$ is a maximal Δ-ideal containing $\mathfrak{n}$ and avoiding Σ. Therefore, $\mathfrak{p}_{\#}$ is prime. By minimality of $\mathfrak{p}$, $\mathfrak{p}_{\#} = \mathfrak{p}$. □

The proof of Keigher's theorem uses nothing about the Δ-ring except that it is a $\mathbb{Q}$-algebra.

Corollary

If $\mathcal{R}$ is a Δ-ring, the nil radical $\mathfrak{n}(\mathcal{R})$, which is a Δ-ideal, is the intersection of all the prime ideals of $\mathcal{R}$, all the prime Δ-ideals of $\mathcal{R}$, and all the minimal prime ideals of $\mathcal{R}$, and each minimal prime ideal is a Δ-ideal.

Theorem

If $\mathcal{R}$ is a Δ-ring, and $\mathfrak{a}$ is a radical Δ-ideal, $\mathfrak{a}$ is the intersection of the prime Δ-ideals containing it, and is the intersection of the minimal prime ideals containing it. Each minimal prime ideal is a Δ-ideal.

Proof.

The quotient homomorphism preserves differentiation. □

Definition

We denote the set of zero divisors of a ring $\mathcal{R}$ by $\mathfrak{Z}(\mathcal{R})$.

Theorem

Let $\mathcal{R}$ be a reduced ring. Then, the union of the minimal prime ideals of $\mathcal{R}$ equals $\mathfrak{Z}(\mathcal{R})$.

Proof.

By a prior corollary, the union of the minimal prime ideals of $\mathcal{R}$ is contained in $\mathfrak{Z}(\mathcal{R})$. Let $a \in \mathfrak{Z}(\mathcal{R})$. We may assume that $a \neq 0$. There exists $b \in \mathcal{R}$ such that $b \neq 0$, and $ab = 0$. Spose a is outside every minimal prime ideal of $\mathcal{R}$. Then, b is inside every prime ideal of $\mathcal{R}$. Therefore, $b \in \mathfrak{n}(\mathcal{R})$, which is (0) since $\mathcal{R}$ is reduced. $\longrightarrow\longleftarrow$ □

- The intersection of the minimal prime ideals of $\mathcal{R}$ is equal to the nil radical of $\mathcal{R}$.
- If $\mathcal{R}$ is a Δ-ring, each minimal ideal is a Δ-ideal.
- If $\mathcal{R}$ is *reduced,* the intersection of the minimal primes of $\mathcal{R}$ is (0), and the union of the minimal prime ideals equals $\mathfrak{Z}(\mathcal{R})$. Neither of these results requires Noetherianity of the ring.
- Let $\mathfrak{a}$ be a radical Δ-ideal in a Δ-ring $\mathcal{R}$. The residue class ring $\mathcal{R}/\mathfrak{a}$ is reduced. So, (0) is the intersection of the minimal prime ideals, and their union is precisely the set of zero divisors of $\mathcal{R}$.

Recall: $\mathfrak{a} :=$ radical Δ-ideal of a Δ-ring $\mathcal{R}$. and $S \subseteq \mathfrak{a}$. $\mathcal{S}$ is a radical Δ-ideal basis of $\mathfrak{a}$ if $\mathfrak{a}$ is the smallest radical Δ-ideal in $\mathcal{R}$ containing S. $\mathcal{R}$ a $\mathbb{Q}$-algebra $\Longrightarrow \mathfrak{a} = \sqrt{[S]}$. $\mathcal{R} :=$ Δ-polynomial ring $\mathcal{F}\{y_1, \ldots, y_n\}$, $\mathcal{F}$ a Δ-field. Let

$$P = 0, P \in S$$

be the associated system of differential equations. Recall Ritt's interpretation of "all differential consequences of the system": the extended system

$$P = 0, P \in \sqrt{[S]}\,.$$

We will see that the two systems have the same solutions in any "differentially closed Δ-field." Does the second system contain a *finite* equivalent system?

The following slide contains three equivalent properties of a ring $\mathcal{R}$. First column: $\mathcal{R}$ is a ring. Second column: $\mathcal{R}$ is a Δ-ring.

- Every *ideal* $\mathfrak{a}$ in $\mathcal{R}$ has a finite ideal basis.
- Every ascending chain of *ideals* of $\mathcal{R}$ stabilizes.
- Every non-empty set of *ideals* of $\mathcal{R}$ has a maximal element.

- Every *radical* Δ-ideal $\mathfrak{a}$ in $\mathcal{R}$ has a finite radical Δ-ideal basis.
- Every ascending chain of *radical* Δ-ideals of $\mathcal{R}$ stabilizes.
- Every non-empty set of *radical* Δ-ideals of $\mathcal{R}$ has a maximal element.

Definitions

A ring $\mathcal{R}$ is *Noetherian* if it satisfies any (hence all) of the properties in the first column. A Δ-ring $\mathcal{R}$ is *Rittian* if it satisfies any (hence all) of the properties in the second column.

The equivalences for radical Δ-ideals in a Δ-ring $\mathcal{R}$:
$1 \iff 2$.

Proof.

$1 \Rightarrow 2$. Let

$$\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \cdots \subseteq \mathfrak{a}_i \subseteq \cdots$$

be an increasing sequence of radical Δ-ideals of $\mathcal{R}$. The union $\cup \mathfrak{a}_i = \mathfrak{a}$ is clearly a Δ-ideal of $\mathcal{R}$, hence has a finite Δ-ideal basis $f_1, \ldots, f_p$. For some i, $f_1, \ldots, f_p$ are in $\mathfrak{a}_i$. Thus, $\mathfrak{a} = \surd\, [f_1, \ldots, f_p] \subseteq \mathfrak{a}_i$. It follows that for all $j \geq i$, $\mathfrak{a}_j \subseteq \mathfrak{a}_i$.
$2 \Rightarrow 1$. Let $\mathfrak{a} \subseteq \mathcal{R}$ be a radical Δ-ideal. We successively choose $a_n \in \mathfrak{a}$, $n = 1, 2, \ldots$ such that either $\mathfrak{a}_n = \sqrt{[a_1, \ldots, a_n]} = \mathfrak{a}$, or $a_{n+1} \notin \mathfrak{a}_n$.. We get a chain

$$\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \cdots \subseteq \mathfrak{a}_n \subseteq \cdots$$

which must stabilize by 2. Therefore, $\mathfrak{a}$ has a finite radical Δ-ideal basis. □

$2 \implies 3.$

Proof.

Suppose 3 is false, and let S be a nonempty set of radical Δ-ideals of $\mathcal{R}$ that does not have a maximal element. Let $\mathfrak{a}_1$ be in S. There is an element $\mathfrak{a}_2$ in S such that $\mathfrak{a}_1 \subsetneqq \mathfrak{a}_2$, else $\mathfrak{a}_1$ would be maximal in S. Continuing this process, we construct an increasing chain of radical Δ-ideals of $\mathcal{R}$ that does not stabilize. □

3$\Longrightarrow$2.

Proof.

Let

$$\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \cdots \subseteq \mathfrak{a}_i \subseteq \cdots$$

be an increasing sequence of radical Δ-ideals of $\mathcal{R}$. The set $\{\mathfrak{a}_j\}_{j=1,2,\ldots}$ has a maximal element, say, $\mathfrak{a}_n$. Thus, the sequence stabilizes. □

The chess master's theorem: Lasker-Noether theorem.

The following theorem was proved by the chess master Emanuel Lasker (champion 1894-1920) for polynomial rings, and extended to arbitrary Noetherian rings by Emmy Noether. Lasker also invented the concept of “primary ideal."

Theorem

In a Noetherian ring every ideal is a finite intersection of prime ideals.

Theorem

In a Rittian Δ-ring every radical Δ-ideal is a finite intersection of prime Δ-ideals.

Call it a *finite prime decomposition* of the radical Δ-ideal.

First, we prove:

Lemma

Let $\mathfrak{b}$ be a radical Δ-ideal. $\mathfrak{b}$ is not prime $\Longleftrightarrow$ there exist radical Δ-ideals $\mathfrak{s}$, $\mathfrak{t}$ properly containing $\mathfrak{b}$ such that $\mathfrak{s} \cap \mathfrak{t} = \mathfrak{b}$.

Proof.

The condition is clearly sufficient. Now, suppose there exist $s, t \in \mathcal{R} \setminus \mathfrak{b}$ such that $st \in \mathfrak{b}$. Let $\mathfrak{s} = \sqrt{[\mathfrak{b} \cup s]}$and $\mathfrak{t} = \sqrt{[\mathfrak{b} \cup t]}$. $\mathfrak{b} \subsetneq \mathfrak{s}$, and $\mathfrak{b} \subsetneq \mathfrak{t}$. By a lemma from last week,

$$\mathfrak{s} \cap \mathfrak{t} = \sqrt{[\mathfrak{b} \cup s]} \cap \sqrt{[\mathfrak{b} \cup t]} = \sqrt{[(\mathfrak{b} \cup s)(\mathfrak{b} \cup t)]} \subseteq \mathfrak{b}.$$

Thus, $\mathfrak{s} \cap \mathfrak{t} = \mathfrak{b}$. □

We now prove the theorem.

Proof.

Spose there is a radical Δ-ideal for which the theorem is false. Then the set B of all such radical Δ-ideals is not empty. By the third property of Rittian rings, B contains a maximal element $\mathfrak{b}$. $\mathfrak{b}$ is not prime. Therefore, by the lemma, there exist radical Δ-ideals $\mathfrak{s}$, $\mathfrak{t}$ properly containing $\mathfrak{b}$ such that $\mathfrak{s} \cap \mathfrak{t} = \mathfrak{b}$. But, since $\mathfrak{b}$ is maximal in $\mathfrak{b}$, $\mathfrak{s}$ and $\mathfrak{t}$ are finite intersections of prime Δ-ideals, hence so is $\mathfrak{b}$. This is a contradiction. □

Definition

A prime decomposition

$$\mathfrak{a} = \bigcap_{i \in I} \mathfrak{p}_i$$

of a radical Δ-ideal $\mathfrak{a}$ is *irredundant* if whenever $i \neq j$, $\mathfrak{p}_i \not\supseteq \mathfrak{p}_j$.

Corollary

Every radical Δ-ideal in a Rittian Δ-ring has an irredundant finite prime decomposition.

Uniqueness of an irredundant finite prime decomposition.

We saw that prime decompositions of radical ideals are not unique. But,

Theorem

Let $\mathfrak{a}$ be a radical Δ-ideal in a Rittian Δ-ring. Then, $\mathfrak{a}$ has a unique finite irredundant prime decomposition.

$$\mathfrak{a} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r.$$

Lemma

A prime ideal $\mathfrak{p}$ *in a ring* $\mathcal{R}$ *that contains a finite intersection*

$$\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$$

of prime ideals contains one of them.

Proof.

Suppose not. Then, $\forall i = 1, \ldots, r$, $\exists a_i \in \mathfrak{p}$ such that $a_i \notin \mathfrak{p}$. Then, $\prod_{i=1}^{r} a_i \in \mathfrak{p}$, and therefore, one of the a_i is in $\mathfrak{p}$. □

Proof.

(of Theorem)Suppose there exist two finite irredundant prime decompositions

$$\begin{aligned} \mathfrak{a} &= \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r \\ &= \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s. \end{aligned}$$

Suppose $r \leq s$. Let $i = 1$. By the lemma, $\exists j$ such that $\mathfrak{q}_j \subseteq \mathfrak{p}_1$. Similarly, $\exists k$ such that $\mathfrak{p}_1 \subseteq \mathfrak{q}_k$. Thus, $\mathfrak{p}_k \subseteq \mathfrak{q}_j \subseteq \mathfrak{p}_1$. By irredundancy, $\mathfrak{p}_1 = \mathfrak{p}_k = \mathfrak{q}_j$. We re-label the indices so that $j = 1$. So, $\mathfrak{p}_1 = \mathfrak{q}_1$, and $\mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_r = \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_s$. Continuing this process, we have $\mathfrak{p}_i = \mathfrak{q}_i$, $i = 1, \ldots, r$. Spose $s > r$ Then,

$$\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s.$$

Let $r < j < s$. Then, $\mathfrak{q}_j \supseteq \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$. Therefore, $\exists i$ such that $\mathfrak{q}_j \supseteq \mathfrak{q}_i$ which contradicts irredundancy. Therefore, $r = s$, and the irredundant decompositions are equal. □

Keigher's Theorem redux.

The following theorem gives a new proof in the Rittian case of the Δ-stability of a minimal prime ideal.

Theorem

Let $\mathfrak{a}$ be a radical Δ-ideal in a Rittian Δ-ring $\mathcal{R}$. Let

$$\mathfrak{a} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$$

be an irredundant prime decomposition of $\mathfrak{a}$. Then, $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are exactly the minimal prime ideals of $\mathcal{R}$ containing $\mathfrak{a}$.

Proof.

Let $\mathfrak{M}$ be the set of minimal prime ideals of $\mathcal{R}$ containing $\mathfrak{a}$.

$$\mathfrak{a} = \bigcap_{\mathfrak{q} \in \mathfrak{M}} \mathfrak{q}.$$

Let $\mathfrak{q}$ be a minimal prime ideal containing $\mathfrak{a}$. $\mathfrak{q} \supseteq \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$. Therefore, by the lemma, there is an index i such that $\mathfrak{q} \supseteq \mathfrak{p}_i$. By the minimality of $\mathfrak{q}$, $\mathfrak{q} = \mathfrak{p}_i$. Therefore, card $\mathfrak{M} = r$, and the two irredundant prime decompositions are equal. □

The set of minimal prime ideals in a Noetherian ring is finite. But, Rittian rings are most often far from Noetherian.

Corollary

Let $\mathcal{R}$ be a Rittian Δ-ring. Then, the set of minimal prime ideals is finite.

Proof.

The nil radical is a Δ-ideal. □

Existence of Rittian differential rings.

Examples of Noetherian rings abound: for example, every finitely generated k-algebra, k a field, is Noetherian.
What about Rittian Δ-rings $\mathcal{R}$? If $\mathcal{R}$ is a Noetherian ring or a simple Δ-ring, then, of course, it is Rittian. But, these are strong conditions.

Lemma

Let $\mathcal{R}$ be a Rittian Δ-ring, and, let $\varphi : \mathcal{R} \longrightarrow \mathcal{R}'$ be a surjective Δ-homomorphism. Then, $\mathcal{R}'$ is Rittian.

Proof.

Let Σ' be a non-empty set of radical Δ-ideals in $\mathcal{R}'$. Let $\Sigma = \left\{\varphi^{-1}\mathfrak{a}' : \mathfrak{a}' \in \Sigma'\right\}$. $\varphi^{-1}\mathfrak{a}'$ is a radical Δ-ideal of $\mathcal{R}$. Since the maps $\mathfrak{a}' \longmapsto \varphi^{-1}\mathfrak{a}'$, $\mathfrak{a} \longmapsto \varphi\mathfrak{a}$, define a bijection between the set of Δ-ideals of $\mathcal{R}'$ onto the set of Δ-ideals of $\mathcal{R}$ containing $\ker\varphi$, a maximal element of Σ gives rise to a maximal element of Σ'. □

Ritt Basis Theorem redux.

Theorem

(Ritt-Raudenbush 1930's) Let $\mathcal{F}$ be a Δ-field. Every radical Δ-ideal in the differential polynomial ring $\mathcal{F}\{y_1, \ldots, y_n\}$ has a finite radical Δ-ideal basis.

Corollary

$\mathcal{F}\{y_1, \ldots, y_n\}$ is Rittian.

Proof.

The first property, column 2. □

The theorem was generalized by Kolchin, who replace the coefficient field with an arbitrary Rittian Δ-ring. Both Ritt and Kolchin use the theory of characteristic sets of differential polynomial ideals.

The defining differential ideal of a point.

Recall the substitution homomorphism from Part I:

Definitions

Let $\mathcal{F}$ be a Δ-field and $y = (y_1, \ldots, y_n)$ a family of Δ-indeterminates over $\mathcal{F}$. Let $\mathcal{A} = \mathcal{F}\{y\}$.

$$\mathcal{A}_r = \mathcal{F}[\theta y]_{\theta \in \Theta(r)}.$$

Let $\mathcal{R}$ be a Δ-$\mathcal{F}$-algebra, and $z = (z_1, \ldots, z_n) \in \mathcal{R}^n$. Then, $z \leftrightarrow (z, \delta_1 z, \ldots, \delta_m z, \ldots, \theta z, \ldots)$ in a jet space approach to differential algebra. On each polynomial ring $\mathcal{A}_r$ we have the substitution homomorphism

$$\mathcal{A}_r \longrightarrow \mathcal{R}, \quad (\theta y) \longmapsto (\theta z), \theta \in \Theta(r).$$

This defines a Δ-$\mathcal{F}$-homomorphism σ_z from $\mathcal{A}$ into $\mathcal{R}$, called the *substitution homomorphism*. For $P \in \mathcal{A}$, write $P(z)$ for $\sigma(P)(z)$, and call it the *value* of P at z. $\ker \sigma_z$ is a radical Δ-ideal of $\mathcal{A}$, called the *defining Δ-ideal* of z.

Rittian differential rings exist.

Theorem

Every Δ-ring finitely Δ-generated over a Δ-field is Rittian.

Proof.

Let $\mathcal{F}$ be a Δ-field, and let $\mathcal{R} = \mathcal{F}\{z_1, \ldots, z_n\}$. Then, $\mathcal{R}$ is the image of the differential polynomial ring $\mathcal{F}\{y_1, \ldots, y_n\}$ under the sustitution homomorphism taking $y_i \mapsto z_i$ and leaving the elements of the coefficient field fixed. □

We will see that the converse is false. However, if $\mathcal{R}$ is a Hopf algebra over a Δ-field $\mathcal{F}$, endowed with a Δ-structure, $\mathcal{R}$ is Rittian iff $\mathcal{R}$ is finitely Δ-generated.

To emphasize the implications of the Ritt Basis Theorem:

Theorem

Let $\mathcal{R}$ be Δ-ring, finitely Δ-generated over a Δ-field $\mathcal{F}$. Let $\mathfrak{a}$ be a radical Δ-ideal in $\mathcal{R}$.

1. *The set of minimal prime ideals in $\mathcal{R}$ containing $\mathfrak{a}$ is finite.*
2. *Every minimal prime ideal containing $\mathfrak{a}$ is a Δ-ideal.*
3. $$\mathfrak{a} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r,$$ *where $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are the minimal prime ideals in $\mathcal{R}$ containing $\mathfrak{a}$, and this is the unique irredundant prime decomposition of $\mathfrak{a}$.*

Corollary

1. *The set of minimal prime ideals of* $\mathcal{R}$ *is finite.*
2. *Every minimal prime ideal of* $\mathcal{R}$ *is a* Δ*-ideal.*
3. $$\mathfrak{n}\left(\mathcal{R}\right) = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r,$$
   where $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ *are the minimal prime ideals in* $\mathcal{R}$.
4. *If* $\mathcal{R}$ *is reduced,* $\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_r$ *equals* $\mathfrak{Z}\left(\mathcal{R}\right)$.