

The Kolchin Topology

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Conventions.

\mathcal{F} is a Δ -field, and $\mathcal{A}_{\mathcal{F}} = \mathcal{F}\{y_1, \dots, y_n\}$ y_1, \dots, y_n Δ -indeterminates. When no ambiguity results, we will omit the prefix Δ -. All rings are Δ - \mathbb{Q} -algebras. All homomorphisms are differential. Unless otherwise stated, field = Δ -field, extension field = extension Δ -field, homomorphism = Δ -homomorphism.

Universal differential fields.

$\mathcal{G} :=$ extension field of \mathcal{F} .

Definition

Let $\eta = (\eta_1, \dots, \eta_n) \in \mathcal{G}^n$. The *defining ideal* \mathfrak{p} of η over \mathcal{F} is the set of all polynomials $P \in \mathcal{A}_{\mathcal{F}}$ such that $P(\eta) = 0$.

- \mathfrak{p} is a prime Δ -ideal.
- Say η is *generic* for \mathfrak{p} .
- $\mathfrak{I}(\eta) := \mathfrak{p}$.

$$\mathcal{F}\{\eta\} \underset{\mathcal{F}}{\approx} \mathcal{A}_{\mathcal{F}}/\mathfrak{p},$$

the residue class ring. The isomorphism maps $y_i + \mathfrak{p}$ to η_i .

Definition

An extension field \mathcal{S} of \mathcal{F} is *semiuniversal over \mathcal{F}* if whenever \mathcal{G} is finitely Δ -generated over \mathcal{F} , \exists an \mathcal{F} -isomorphism from \mathcal{G} into \mathcal{U} .

So, every finitely Δ -generated extension field of \mathcal{F} can be *embedded over \mathcal{F}* in \mathcal{U} .

Theorem

\mathcal{S} semiuniversal over $\mathcal{F} \implies \forall n$, every prime Δ -ideal \mathfrak{p} in $\mathcal{A}_{\mathcal{F}}$ has a generic zero in \mathcal{S}^n .

Proof.

The quotient field of the residue class ring $\mathcal{F} \{\overline{y_1}, \dots, \overline{y_n}\} = \mathcal{A}_{\mathcal{F}}/\mathfrak{p}$ is finitely Δ -generated over \mathcal{F} . □

Definition

An extension field \mathcal{U} of \mathcal{F} is *universal over \mathcal{F}* if \mathcal{U} is semiuniversal over every finitely Δ -generated extension field of \mathcal{F} in \mathcal{U} .

If \mathcal{U} is universal over \mathbb{Q} , it is called a *universal differential field*.

Theorem

\mathcal{U} universal over $\mathcal{F} \implies \forall n$, and $\forall \mathcal{G}$ finitely Δ -generated over \mathcal{F} , every prime Δ -ideal in $\mathcal{G} \{y_1, \dots, y_n\}$ has a generic zero in \mathcal{U}^n .

Corollary

Let \mathcal{U} be a universal Δ -field. If \mathcal{F} is any subfield of \mathcal{U} that is finitely Δ -generated over the prime field \mathbb{Q} , then, \mathcal{U} is universal over \mathcal{F} .
Therefore, every prime Δ -ideal in $\mathcal{F} \{y_1, \dots, y_n\}$ has a generic zero in \mathcal{U}^n .

Specializations.

$\mathcal{R} := \Delta\text{-}\mathcal{F}$ -algebra, and an integral domain.

Definition

Let $\eta = (\eta_1, \dots, \eta_n) \in \mathcal{R}^n$ and let $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathcal{G}^n$. η specializes to ζ over \mathcal{F} if \exists a $\Delta\text{-}\mathcal{F}$ -homomorphism

$$\varphi : \mathcal{F}\{\eta\} \longrightarrow \mathcal{F}\{\zeta\}, \quad \eta_i \longmapsto \zeta_i, \quad i = 1, \dots, n$$

Let $\mathfrak{J}(\eta) = \mathfrak{p}$, and $\mathfrak{J}(\zeta) = \mathfrak{q}$.

- \mathfrak{p} and \mathfrak{q} are prime.
- η specializes to ζ over $\mathcal{F} \iff \mathfrak{p} \subseteq \mathfrak{q}$.

Definition

ζ is a *generic specialization* of η over \mathcal{F} if

- 1 η specializes to ζ over \mathcal{F} and
- 2 The above homomorphism

$$\varphi : \mathcal{F}\langle\eta\rangle \longrightarrow \mathcal{F}\langle\zeta\rangle, \quad \eta_i \longmapsto \zeta_i, \quad i = 1, \dots, n$$

is an isomorphism.

- The specialization $\eta \longrightarrow \zeta$ is generic $\iff \eta \longrightarrow \zeta$ and $\zeta \longrightarrow \eta$.
- $\iff \mathfrak{p} = \mathfrak{q} \iff \eta$ and ζ are generic for the same prime Δ -ideal.
- $\iff \exists$ a Δ - \mathcal{F} -isomorphism from the field $\mathcal{F}\langle\eta\rangle$ onto $\mathcal{F}\langle\zeta\rangle$, $\eta_i \longmapsto \zeta_i, \quad i = 1, \dots, n$.

The Lefschetz-Seidenberg Principle.

If \mathcal{F} is finitely Δ -generated over \mathbb{Q} , then:

$x_1, \dots, x_m :=$ complex variables.

$\Omega :=$ connected open region of \mathbb{C}^m .

$\mathcal{M}(\Omega) :=$ the field of functions meromorphic in Ω .

Δ .acts on $\mathcal{M}(\Omega)$. δ_i acts as ∂_{x_i} .

- 1 \exists connected open region Ω of \mathbb{C}^m and a Δ -isomorphism from \mathcal{F} into $\mathcal{M}(\Omega)$.
- 2 \mathcal{G} finitely Δ -generated over $\mathcal{F} \implies \exists$ a connected open region $\Omega' \subseteq \Omega$, and a Δ - \mathcal{F} -isomorphism from \mathcal{G} into $\mathcal{M}(\Omega')$.

More generally: Suppose $\mathcal{F} = \mathcal{M}(\Omega)$, $\Delta = \{\partial_{x_1}, \dots, \partial_{x_m}\}$, \exists a universal field extension \mathcal{U} of \mathcal{F} . Elements $\eta \in \mathcal{F}$ are meromorphic functions $\eta(x_1, \dots, x_m)$. In general, elements of \mathcal{U} can be thought of as generalized meromorphic functions.

From Part I,

Example

$\Delta = \{\delta_1, \delta_2\}$, Let $\mathcal{F} = \mathbf{C}(t, x)$, $\mathcal{G} = \mathcal{M}(\mathbb{D}_x \times \mathbb{D}_t)$, $\mathbb{D}_x =$ right half plane of \mathbf{C} , $\mathbb{D}_t = \mathbf{C} \setminus \mathbb{Z}_{\leq 0}$, and $\delta_1 = \partial_x$, $\delta_2 = \partial_t$.

$\gamma = \int_0^x s^{t-1} e^{-s} ds \in \mathcal{F}$. $\mathfrak{I}(\gamma)$ is the prime Δ -ideal

$$\mathfrak{p} =: \sqrt{\left[\partial_x^2 y - \frac{t-1-x}{x} \partial_x y, \partial_x y \partial_t^2 \partial_x y - (\partial_t \partial_x y)^2 \right]} : \partial_x y$$

in $\mathbf{C}(x, t) \{y\}$. This is a correction from earlier lectures. For a discussion of the meromorphicity of the incomplete gamma in the region described, see Frank Olver, *Asymptotics and Special Functions*, 1997.

We can rank the derivatives of y by order. Note that the highest order derivative of y in $Q = \partial_{xy} \partial_t^2 \partial_{xy} - (\partial_t \partial_{xy})^2$ is $\theta y = \partial_t^2 \partial_{xy}$. It is called the *leader of Q* . The partial derivative $\frac{\partial Q}{\partial(\theta y)}$ is called the *separant S_Q of Q* . The leader of L with respect to this ranking by order is 1.

$$\mathfrak{p} = \left\{ P \in \mathcal{F}\{y\} \mid S_Q P \in \sqrt{\left[\partial_{xy}^2 - \frac{t-1-x}{x} \partial_{xy}, \partial_{xy} \partial_t^2 \partial_{xy} - (\partial_t \partial_{xy})^2 \right]} \right\}$$

If ζ is a zero of \mathfrak{p} , then $S_Q(\zeta) = 0$ if and only if ζ is a ∂_x -constant.

Let \mathcal{U} be a universal extension of \mathcal{G} . Then, the prime Δ -ideal \mathfrak{p} in $\mathbb{Q}(x, t)\{y\}$ has the generic zero $\gamma(x, t)$ in $\mathcal{G} \subseteq \mathcal{U}$. Any generalized meromorphic function η in \mathcal{U} with defining Δ -ideal \mathfrak{p} is equivalent to the incomplete Gamma function.

Definition

A family $(\eta_i)_{i \in I}$ of elements of an extension field \mathcal{G} of \mathcal{F} is *Δ -algebraically dependent* over \mathcal{F} if the family $(\theta\eta_i)_{i \in I}$ is algebraically dependent over \mathcal{F} .

Definition

\mathcal{G}/\mathcal{F} is *pure Δ -transcendental* over \mathcal{F} if it is generated over \mathcal{F} by a Δ -algebraically independent family. $\mathcal{G}^\Delta = \mathcal{F}^\Delta$ (Rosenlicht).

Definition

\exists a family $(\eta_i)_{i \in I}$ of elements of \mathcal{G} , called a *Δ -transcendence basis*, such that $\mathcal{F}_0 = \mathcal{F} \langle (\eta_i)_{i \in I} \rangle$ is pure Δ -transcendental over \mathcal{F} and \mathcal{G} is Δ -algebraic over \mathcal{F}_0 . All Δ -transcendence bases have the same cardinality – the *Δ -transcendence degree* of \mathcal{G}/\mathcal{F} .

A universal extension field \mathcal{U} of \mathcal{F} is HUGE. Every finitely Δ -generated extension field of \mathcal{F} can be embedded in \mathcal{U} . So, $\forall n$, the Δ -polynomial ring $\mathcal{F}\{y_1, \dots, y_n\}$ can be embedded in \mathcal{U} . So, the Δ -transcendence degree of \mathcal{U} over \mathbb{Q} is infinite. Also, its constant field \mathcal{K} has infinite transcendence degree over the field of constants of \mathcal{F} and is algebraically closed. It is a universal field in the sense of Weil. \mathcal{U} is the analogue in differential algebra of Weil's universal fields.

The Kolchin topology on \mathcal{U}^n

More precisely, we are defining the Kolchin \mathcal{F} -topology.

$\mathcal{A}_{\mathcal{F}} := \mathcal{F} \{y_1, \dots, y_n\}$. In this section, we will consider only *nonempty* subsets S of $\mathcal{A}_{\mathcal{F}}$.

Definition

Let $S \subseteq \mathcal{A}_{\mathcal{F}}$. $V(S) = \{\eta \in \mathcal{U}^n \mid P(\eta) = 0, \forall P \in S\}$.

$V(S) = V([S])$.

Theorem

- 1 $V(0) = \mathcal{U}^n$; $V(1) = \emptyset$.
- 2 Let $(S_i)_{i \in I}$ be a family of subsets of $\mathcal{A}_{\mathcal{F}}$. Then,

$$V\left(\bigcup_{i \in I} S_i\right) = V\left(\sum_{i \in I} [S_i]\right) = \bigcap_{i \in I} V(S_i).$$

- 3 If S and T are subsets of $\mathcal{A}_{\mathcal{F}}$, with $S \subseteq T$, then

$$V(S) \supseteq V(T).$$

- 4 If $S \subseteq \mathcal{A}_{\mathcal{F}}$, then, $V(S) = V\sqrt{[S]}$.
- 5 If \mathfrak{a} and \mathfrak{b} are Δ -ideals of $\mathcal{A}_{\mathcal{F}}$, then

$$V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b}).$$

Proof.

The first statement follows from the definition. The second follows from the fact that $[\bigcup_{i \in I} S_i] = \sum_{i \in I} [S_i]$. The third statement follows from the definition, and implies that $V(S) \supseteq V(\sqrt{[S]})$. Now, let $F \in \sqrt{[S]}$. Then, for some positive integer k , $F^k \in [S]$. Therefore, F^k vanishes on $V(S)$, whence F vanishes on $V(S)$. Therefore, $V(S) \subseteq V(\sqrt{[S]})$. We know that $\mathfrak{a}\mathfrak{b}$ is a Δ -ideal, and that $\sqrt{(\mathfrak{a} \cap \mathfrak{b})} = \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}} = \sqrt{(\mathfrak{a}\mathfrak{b})}$. Therefore,
$$V(\mathfrak{a} \cap \mathfrak{b}) = V(\sqrt{(\mathfrak{a} \cap \mathfrak{b})}) = V(\sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}) = V(\sqrt{(\mathfrak{a}\mathfrak{b})}) = V(\mathfrak{a}\mathfrak{b})$$
 by the fourth statement. □

Definitions

We say that a subset $V \subseteq \mathcal{U}^n$ is *Kolchin closed* (or, simply, *closed*) if there is a subset S of $\mathcal{A}_{\mathcal{F}}$ with $V = V(S)$. The Kolchin closed subsets of \mathcal{U}^n are the closed sets of a topology called the *Kolchin topology*. $V = V(S)$ is also called a Δ -variety.

Notes:

- 1 The Zariski topology on \mathcal{U}^n is coarser than the Kolchin topology. For example, all proper Zariski closed subsets of \mathcal{U} are finite. There are infinitely many proper Kolchin closed subsets.
- 2 If $X \subseteq \mathcal{U}^n$, X is a topological space in the induced Kolchin topology.

Definition

A subspace X of a topological space is *reducible* if X is the union of two proper closed subsets. Otherwise X is *irreducible*.

If $X \subseteq \mathcal{U}^n$, denote the closure of X by \bar{X} .

Lemma

Let X be a topological space.

- 1 X is irreducible if and only if every nonempty open subset U of X is dense.
- 2 X is irreducible if and only if two nonempty open subsets have nonempty intersection.
- 3 If Y is an irreducible subset of X , then \overline{Y} is irreducible.

Proof.

1. Suppose U is a nonempty open subset of X , and U is not dense. Then, $X_1 = \overline{U}$ is a proper closed subset of X . Let $X_2 = X \setminus U$. Then, X_2 is a proper closed subset of X . $X = X_1 \cup X_2$ is reducible. Suppose $X = X_1 \cup X_2$, where X_1 and X_2 are two proper closed subsets. Then, $U = X_1 \setminus (X_1 \cap X_2) \neq \emptyset$. $X \setminus U = X_2$. Therefore, U is open in X , and is not dense.
2. Suppose X is irreducible. Let U_1 and U_2 be nonempty open subsets of X . $U_1 \cap U_2$ is open. Suppose $U_1 \cap U_2 = \emptyset$. Then, $X = X \setminus (U_1 \cap U_2) = (X \setminus U_1) \cup (X \setminus U_2)$. If $X \setminus U_1 = \emptyset$, $X = U_1$. So, the intersection can't be empty. Similarly, $X \setminus U_2 \neq \emptyset$. So, $U_1 \cap U_2 \neq \emptyset$. Suppose X is reducible. $X = X_1 \cup X_2$, where X_1 and X_2 are proper closed subsets. Let $U_1 = X_1 \setminus X_2$, and $U_2 = X_2 \setminus X_1$. Then, U_1 and U_2 are open in X and $U_1 \cap U_2 = \emptyset$.
3. Suppose \overline{Y} is reducible. There exist two proper closed subsets Y_1, Y_2 of \overline{Y} such that $\overline{Y} = Y_1 \cup Y_2$. Since Y is dense in \overline{Y} , $Y \cap Y_1$ and $Y \cap Y_2$ are proper closed subsets of Y , and $Y = (Y \cap Y_1) \cup (Y \cap Y_2)$. Therefore, Y is reducible. □

Definition

Let $X \subseteq \mathcal{U}^n$. Set $\mathfrak{I}(X) = \{P \in \mathcal{A}_{\mathcal{F}} \mid P(\eta) = 0 \quad \forall \eta \in X\}$. If $X = \{\eta\}$, $\mathfrak{I}(\eta) := \mathfrak{I}(X)$.

Theorem

Let $X \subseteq \mathcal{U}^n$.

- 1 $\mathfrak{I}(X)$ is a radical Δ -ideal.
- 2 $\overline{X} = V(\mathfrak{I}(X))$.

Proof.

The first statement is obvious. By definition, there is a subset S of $\mathcal{A}_{\mathcal{F}}$ such that $\overline{X} = V(S)$. Clearly, $S \subseteq \mathfrak{I}(X)$. Therefore, $\overline{X} = V(S) \supseteq V(\mathfrak{I}(X)) \supseteq X$. Since $V(\mathfrak{I}(X))$ is closed, $V(\mathfrak{I}(X)) \supseteq \overline{X}$. □

If V is closed, $\mathfrak{I}(V)$ is called the *defining* Δ -ideal of V .

Corollary

Let \mathfrak{p} be a prime Δ -ideal in $\mathcal{A}_{\mathcal{F}}$, and let η be a generic zero for \mathfrak{p} in \mathcal{U}^n . Let $X = \{\eta\}$. Then, $\overline{X} = V(\mathfrak{p})$.

\overline{X} is called the *locus* of η .

Example

Note that for fixed t , $\gamma = \gamma(x, t) = \int_0^x s^{t-1} e^{-s} ds$ is analytic in $D_x =$ right half plane of the x -plane. It is multi-valued on $\mathbb{C} \setminus 0$. For fixed x , it is analytic on $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, with simple poles at the non-positive integers. Let

$$p =: \sqrt{\left[\partial_x^2 y - \frac{t-1-x}{x} \partial_x y, \partial_x y \partial_t^2 \partial_x y - (\partial_t \partial_x y)^2 \right]} : \partial_x y,$$

in $\mathcal{F} = \mathbb{C}(t, (e^{ct})_{c \in \mathbb{C}}(x))$. Let \mathcal{U} be a universal extension of $\mathcal{G} = \mathcal{M}(D_x \times D_t)$. Let $\delta_1 = \partial_x$, $\delta_2 = \partial_t$, with generic zero $\gamma(x, t)$ in $\mathcal{G} \subset \mathcal{U}$.

The locus of γ is very interesting.

$$\begin{aligned}\gamma(x, t) &= \int_0^x s^{t-1} e^{-s} ds. \\ \partial_x \gamma &= x^{t-1} e^{-x}.\end{aligned}$$

Note that $\partial_x \gamma \neq 0$. So, I call the differential polynomials

$$\begin{aligned}L &= L(y) = \partial_x^2 y - \frac{t-1-x}{x} \partial_x y, \\Q &= Q(y) = \partial_x y \partial_t^2 \partial_x y - (\partial_t \partial_x y)^2.\end{aligned}$$

the defining differential polynomials of γ . $V(L)$ is the vector space over \mathcal{U}^{∂_x} with basis $(1, \gamma)$.

$$L = L(y) = \partial_x^2 y - \frac{t-1-x}{x} \partial_x y,$$

$$Q = Q(y) = \partial_x y \partial_t^2 \partial_x y - (\partial_t \partial_x y)^2.$$

Let

$$\zeta = c_1 + c_2 \gamma \in V(L). \quad c_i \in \mathcal{U}^{\partial_x}. \quad \partial_x \zeta = c_2 \partial_x \gamma = c_2 x^{t-1} e^{-x}.$$

$$\partial_x \zeta = 0 \iff c_2 = 0 \iff \zeta = c_1, c_i \in \mathcal{U}^{\partial_x}.$$

Suppose $c_2 \neq 0$. Then

$$Q(\zeta) = 0 \Leftrightarrow \partial_t \left(\frac{\partial_t c_2}{c_2} \right) = 0.$$

$$\zeta = c_1 + c_2\gamma \in V(\mathfrak{p}) \iff c_2 = 0 \quad \text{or} \quad c_2 \neq 0 \quad \text{and} \quad \partial_t \left(\frac{\partial_t c_2}{c_2} \right) = 0.$$

The nonzero coefficients c_2 satisfying this logarithmic equation form a subgroup of the multiplicative group G of $\mathbb{G}_m(\mathcal{U}^{\partial_x})$. Thus, although $V(\mathfrak{p})$ is not a subspace over the field of constants of ∂_x it does have the superposition principle:

$$(c_1 + c_2\gamma) \odot (d_1 + d_2\gamma) = (c_1 + d_1) + (c_2 d_2)\gamma.$$

The open subset defined by the condition $c_2 \neq 0$, is a torsor under a differential algebraic group relative to the universal ∂_x -field \mathcal{U}^{∂_x} . It is the group

$$G = \mathbb{G}_a(\mathcal{U}^{\partial_x}) \rtimes G,$$

where $\mathbb{G}_a(\mathcal{U}^{\partial_x})$ is the additive group of \mathcal{U}^{∂_x} .

In this section, we do not assume that Δ -fields are contained in \mathcal{U} . Every Δ -field \mathcal{G} has a proper Δ -algebraic extension field. Let x be transcendental over \mathcal{G} . On the polynomial ring $\mathcal{G}[x]$ define $\delta x = 0$ $\forall \delta \in \Delta$. $\mathcal{G}(x)$ is a proper Δ -algebraic extension field of \mathcal{G} . A universal Δ -field is not Δ -algebraically closed. The property of being Δ -algebraic over \mathcal{G} is replaced by the stronger property of being “constrained over \mathcal{G} .” (Shelah, 1972, Kolchin, 1974)

Definition

Let \mathcal{G} be a Δ -field, and \mathcal{H} an extension field of \mathcal{G} . $\mathcal{A}_{\mathcal{G}} := \mathcal{G}\{y_1, \dots, y_n\}$. $\eta \in \mathcal{H}^n$ is *constrained* over \mathcal{G} if there exists $C \in \mathcal{A}_{\mathcal{G}}$ such that $C(\eta) \neq 0$, and $C(\zeta) = 0$ for every non-generic specialization ζ of η .

C is called a *constraint* for η . η is said to be *C -constrained* over \mathcal{G} . $C \notin \mathfrak{p} = \mathfrak{I}_{\mathcal{G}}(\eta)$ and $C \in \mathfrak{q}$ for all prime Δ -ideals in $\mathcal{A}_{\mathcal{G}}$ properly containing \mathfrak{p} .

During the discussion, Jerry Kovacic pointed out that Kolchin had defined constrained families before 1972. Jerry pointed out that Kolchin might well have gotten the idea from Picard-Vessiot theory. A Picard-Vessiot extension is generated by a fundamental system of solutions of a homogeneous linear ordinary differential equation with coefficients in a base field \mathcal{F} . This fundamental system has constraint the Wronskian determinant.

Kolchin writes in “Constrained Extensions of Differential Fields” (1974)” that the model theorist Lenore Blum (1968) “proved that every ordinary differential field of characteristic 0 has what she called a ‘differential closure’ (or prime ‘differentially closed extension’), that is, a ‘differentially closed’ extension that can be embedded in every differentially closed extension.” Kolchin showed in the 1974 paper that what he called the “constrained closure” of a differential field and Blum’s differential closure were the same, thus providing a concrete realization of differential closures. He used techniques developed by Shelah in his 1972 paper, “Uniqueness and characterization of prime models over sets for totally transcendental first order theories.” It is interesting to note that the concept of constrained family was explored by Emile Picard in t. 3 of his *Traité d’Analyse*, (1896), where he credits the idea to Jules Drach. Picard’s approach to differential Galois theory differed markedly from that of Vessiot.