Constrained Zeros and the Ritt Nullstellensatz

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$\mathcal{G} :=$ Δ-field, $\mathcal{H} :=$ extension field of $\mathcal{G}$. Neither field is assumed to be contained in a universal Δ-field.

$\mathcal{A}_{\mathcal{G}} := \mathcal{G}\left\{y_1 \ldots, y_n\right\}.$

$\mathfrak{p} \subset \mathcal{A}_{\mathcal{G}}$ is a prime Δ-ideal.

$\eta = (\eta_1, \ldots, \eta_n) \in \mathcal{H}^n$.

$\sigma_\eta : \mathcal{A}_{\mathcal{G}} \longrightarrow \mathcal{G}\{\eta\}, \quad y_i \longmapsto \eta_i,\ i = 1, \ldots, n.$

η is *generic* for $\mathfrak{p}$ over $\mathcal{G} \Longleftrightarrow \mathfrak{I}_{\mathcal{G}}(\eta) = \mathfrak{p} \Longleftrightarrow \sigma_\eta$ is a Δ-$\mathcal{F}$-isomorphism.

Suppose $\mathcal{H} = \mathcal{U}$, universal over $\mathcal{G}$. Then, η & $\varsigma \in \mathcal{U}^n$ are generic for the same prime Δ-ideal $\mathfrak{p} \subset \mathcal{A}_{\mathcal{G}} \Longleftrightarrow \overline{\{\eta\}} = \overline{\{\varsigma\}} \Longleftrightarrow \mathcal{G}\langle\eta\rangle \underset{\Delta\text{-}\mathcal{G}}{\approx} \mathcal{G}\langle\varsigma\rangle$,.

$\eta_i \longmapsto \varsigma_i$, $i = 1, \ldots, n$.

Constrained families.

Definition

$\eta \in \mathcal{H}^n$ is *constrained* over $\mathcal{G}$ if there exists $C \in \mathcal{A}_{\mathcal{G}}$ such that $C(\eta) \neq 0$, and $C(\varsigma) = 0$ for every *non-generic* specialization ς of η.

C is called a *constraint for* η. η is said to be *C-constrained* over $\mathcal{G}$.
η is C-constrained $\Longleftrightarrow C \notin \mathfrak{p}$ & $C \in \mathfrak{q}$ $\forall$ prime Δ-ideals in $\mathcal{A}_{\mathcal{G}}$ properly containing $\mathfrak{p}$.

Suppose $\mathcal{H} = \mathcal{U}$, universal over $\mathcal{G}$. Let $\eta \in \mathcal{U}^n$ be C-constrained over $\mathcal{G}$, with defining ideal $\mathfrak{p}$. $\overline{\{\eta\}} = V(\mathfrak{p})$. $\varsigma \in \mathcal{U}^n$ is generic for $\mathfrak{p} \Longleftrightarrow C(\varsigma) \neq 0 \Longleftrightarrow \overline{\{\varsigma\}} = \overline{\{\eta\}} = V(\mathfrak{p})$ The set of generic points of $V(\mathfrak{p})$ is the open subset of $V(\mathfrak{p})$ consisting of the non-zeros of C. $\mathcal{G}\langle\eta\rangle$ and $\mathcal{G}\langle\varsigma\rangle$ are Δ-$\mathcal{G}$-isomorphicby an isomorphism sending η to ς. They are "conjugate" over $\mathcal{G}$ – Shelah's "indiscernibles."

The Ritt basis theorem implies that $\mathcal{A}_{\mathcal{G}}$ is Rittian (Property 1). Therefore, every non-empty set of radical Δ-ideals in $\mathcal{A}_{\mathcal{G}}$ has a maximal element.

Lemma

Let $\mathfrak{p}$ be a prime Δ-ideal in $\mathcal{A}_{\mathcal{G}}$, and let $C \in \mathcal{A}_{\mathcal{G}} \setminus \mathfrak{p}$. $\exists$ a prime Δ-ideal $\mathfrak{q} \supseteq \mathfrak{p}$ such that $C \notin \mathfrak{q}$, and C is in every prime Δ-ideal properly containing $\mathfrak{q}$.

Proof.

Let S be the set of all prime Δ-ideals containing $\mathfrak{p}$ and excluding C. Set $\mathfrak{q}$ equal to the maximal element of S. □

Corollary

Let η be generic for a prime Δ-ideal $\mathfrak{p}$ of $\mathcal{A}_{\mathcal{G}}$. Then, $\exists$ a specialization ς of η such that ς is constrained over $\mathcal{G}$ with constraint C.

The Ritt Nullstellensatz.

Until further notice, $\mathcal{U}$ is a universal extension of $\mathcal{F}$.
In algebraic geometry, if V is a Zariski closed set, defined over k, the points in extension fields that are algebraic over k are dense in V.
Reminder: If $S \subseteq \mathcal{A}_{\mathcal{F}}$,
$V(S) = \{\eta = (\eta_1, \ldots, \eta_n) \in \mathcal{U}^n \mid F(\eta) = 0 \quad \forall F \in S\}.$

The prime decomposition theorem.

We now invoke the third property of Rittian Δ-rings. Let $\mathcal{G}$ be any extension Δ-field of $\mathcal{F}$ in $\mathcal{U}$.

Theorem

Let $\mathfrak{a}$ be a proper radical Δ-ideal of $\mathcal{A}_{\mathcal{G}}$. There exists a unique finite set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ of prime Δ-ideals of $\mathcal{A}_{\mathcal{G}}$ containing $\mathfrak{a}$, none of which contains any other, such that

$$\mathfrak{a} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r.$$

This set is the set of minimal prime ideals containing $\mathfrak{a}$.

Proof.

See Notes, Talk II, posted on KSDA website. □

We now set $\mathcal{G} = \mathcal{F}$.

Theorem

The Ritt Nullstellensatz. Let S be a subset of $\mathcal{A}_{\mathcal{F}}$, and let F be an element of $\mathcal{A}_{\mathcal{F}}$.

1. *If $F \in \sqrt{[S]}$, then $F(\varsigma) = 0$ for all ς in $V(S)$.*
2. *If $F(\varsigma) = 0$ for all constrained points $\varsigma \in V(S)$, then $F \in \sqrt{[S]}$.*

Proof.

The first statement is obvious.

To establish the second statement, suppose $F \notin \sqrt{[S]}$. Then, there is a prime component $\mathfrak{p}$ of $\sqrt{[S]}$.such that $F \notin \mathfrak{p}$. By the lemma on the existence of constrained zeros, there is a point $\varsigma \in V(\mathfrak{p})$ that is constrained with constraint F. In particular, $F(\varsigma) \neq 0$. □

Corollary

Let $\mathfrak{a}$ be a radical Δ-ideal in $\mathcal{A}_{\mathcal{F}}$. The set of points in $V(\mathfrak{a})$ that are constrained over $\mathcal{F}$ are dense in $V(\mathfrak{a})$.

Theorem

1. *The map* $V \longmapsto \mathfrak{I}(V)$ *from the set of all closed subsets of* $\mathcal{U}^n$ *to the set of all radical* Δ*-ideals of* $\mathcal{A}_{\mathcal{F}}$ *is bijective, with inverse the map* $\mathfrak{a} \longmapsto V(\mathfrak{a})$.
2. $V \subseteq W$ *if and only if* $\mathfrak{I}(V) \supseteq \mathfrak{I}(W)$.
3. *If* V *and* W *are closed subsets of* $\mathcal{U}^n$*, then*

$$\mathfrak{I}(V \cup W) = \mathfrak{I}(V) \cap \mathfrak{I}(W).$$

4. *If* $(V_i)_{i \in I}$ *is a family of closed subsets of* $\mathcal{U}^n$*,*

$$\mathfrak{I}\left(\bigcap_{i \in I} V_i\right) = \sum_{i \in I} \mathfrak{I}(V_i).$$

Theorem

A closed set V is irreducible if and only if $\mathfrak{I}(V)$ is prime.

Proof.

Spose V is irreducible, and $F_1 F_2 \in \mathfrak{I}(V)$. Then, $V(F_1 F_2) = V(F_1) \cup V(F_2) \supseteq V(\mathfrak{I}(V)) = V$. Therefore,

$$V = V \cap (V(F_1) \cup V(F_2)) = (V \cap V(F_1) \cup (V \cap V(F_2).$$

So,

$$V = V \cap V(F_1) \quad \text{or} \quad V = V \cap V(F_2).$$

$$V \subseteq V(F_1) \quad \text{or} \quad V \subseteq V(F_2).$$

Say $V \subseteq V(F_1)$. Then, $\mathfrak{I}(V) \supseteq \mathfrak{I}(V(F_1))$, whence $F_1 \in \mathfrak{I}(V)$. Therefore, $\mathfrak{I}(V)$ is prime. The proof of the converse is similar. □

Noetherian topological spaces.

$\mathcal{A}_{\mathcal{F}}$ is Rittian, but not Noetherian.

Definition

A topological space X is Noetherian if every descending sequence of closed subsets stabilizes. Equivalently, every strictly descending sequence of closed subsets is finite.

Clearly, every subspace of a Noetherian space is Noetherian.

Corollary

$\mathcal{U}^n$, equipped with the Kolchin topology, is Noetherian.

Proof.

There is a bijective inclusion reversing correspondence between the set of closed subsets of $\mathcal{U}^n$ and the set of radical Δ-ideals of $\mathcal{A}_{\mathcal{F}}$. □

The component theorem.

Theorem

Let V be a closed subset of $\mathcal{U}^n$. Then, we can write V uniquely as a finite union

$$V = V_1 \cup \cdots \cup V_r,$$

where $V_1, \ldots, V_r$ are the distinct maximal irreducible subsets of V, called the components of V. If W is an irreducible closed subset of V, $\exists i$ with $W \subseteq V_i$.

Proof.

Let $\mathfrak{p}_j = \mathfrak{I}(V_j), j = 1, \ldots, n$. By the Ritt Nullstellensatz, and the prime decomposition theorem for radical differential ideals, the decomposition of V into components is clear. Spose W is an irreducible closed subset of V. Then, $\mathfrak{p} = \mathfrak{I}(W)$ is prime, and contains $\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$. Spose $\forall j$ $\mathfrak{p} \not\supseteq \mathfrak{p}_j$. Then, $\forall j \; \exists F_j \in \mathfrak{p}_j, \quad F_j \notin \mathfrak{p}$. So, $F_1 \cdots F_r \in \mathfrak{p}$. Thus, $\exists j$ such that $F_j \in \mathfrak{p}$.--→←-- □

Fields of definition of radical differential ideals.

The reference for this section is Kolchin, *Differential Algebra and Algebraic Groups (1973), Chapter IV.*
Let $\mathcal{G}$ be a Δ-field.

Definition

Let $\mathfrak{a}_{\mathcal{G}}$ be a radical Δ-ideal in $\mathcal{A}_{\mathcal{G}}$. Let $\mathcal{F}$ be a Δ-subfield of $\mathcal{G}$ such that there is a subset S of $\mathcal{A}_{\mathcal{F}}$ with

$$\mathfrak{a}_{\mathcal{G}} = \sqrt{[S]}.$$

$\mathcal{F}$ is called a *field of definition* of $\mathfrak{a}_{\mathcal{G}}$.

Theorem

Let $\mathfrak{a}_{\mathcal{G}}$ be a radical Δ-ideal in $\mathcal{A}_{\mathcal{G}}$. Then, $\mathfrak{a}_{\mathcal{G}}$ has a field of definition that is finitely Δ-generated over the prime field $\mathbb{Q}$.

Proof.

There exists a finite subset of $\mathfrak{a}_{\mathcal{G}}$ such that $\mathfrak{a}_{\mathcal{G}} = \sqrt{[S]}$. □

Theorem

The mapping

$$\mathfrak{a}_{\mathcal{G}} \longmapsto \mathfrak{a}_{\mathcal{G}} \cap \mathcal{A}_{\mathcal{F}}$$

from the set of radical Δ*-ideals of* $\mathcal{A}_{\mathcal{G}}$ *with field of definition* $\mathcal{F}$ *to the set of radical* Δ*-ideals of* $\mathcal{A}_{\mathcal{F}}$ *is bijective, with inverse*

$$\mathfrak{a}_{\mathcal{F}} \longmapsto \mathcal{G}\mathfrak{a}_{\mathcal{F}}.$$

If $\mathfrak{p}_{\mathcal{G}}$ *is prime, every generic zero of* $\mathfrak{p}_{\mathcal{G}}$ *rational over an extension field of* $\mathcal{G}$ *is generic for* $\mathfrak{p}_{\mathcal{F}}$ *over* $\mathcal{F}$.

The next theorem is used heavily in the proof of the theorem that states three equivalent conditions for a Δ-field to be differentially closed.

Theorem

(See Kolchin, 1973.) Let $\mathfrak{p}_\mathcal{G}$ be a prime Δ-ideal in $\mathcal{A}_\mathcal{G}$ with field $\mathcal{F}$ of definition, let $\mathcal{H}$ be an extension field of $\mathcal{G}$, and let $\eta \in \mathcal{H}^n$ be generic for $\mathfrak{p}_\mathcal{G}$. Every specialization of η over $\mathcal{F}$ lifts to a specialization of η over $\mathcal{G}$.

Differentially closed fields – Kolchin's definition.

Definition

An extension field $\mathcal{H}$ of $\mathcal{G}$ is constrained over $\mathcal{G}$ if every finite family of elements is constrained over $\mathcal{G}$.

Theorem

Let $\mathcal{H} = \mathcal{G}\langle \eta_1, \ldots, \eta_n \rangle$ be a finitely Δ-generated extension field of $\mathcal{G}$. Then, it is constrained over $\mathcal{G}$ if and only if $\eta = (\eta_1, \ldots, \eta_n)$ is constrained over $\mathcal{G}$.

A field is algebraically closed if it has no proper algebraic extensions.

Definition

(Kolchin) A Δ-field is differentially closed if it has no proper constrained extensions.

Theorem

Let n be a positive integer, and let $\mathcal{U}$ be a universal Δ-field that is an extension field of $\mathcal{G}$ ($\mathcal{U}$ need not be universal over $\mathcal{G}$). The following conditions on $\mathcal{G}$ are equivalent:

1. *$\mathcal{G}$ is differentially closed.*
2. *For every n, every point in $\mathcal{U}^n$ that is constrained over $\mathcal{G}$ is rational over $\mathcal{G}$.*
3. *For every prime Δ-ideal $\mathfrak{p}$ in $\mathcal{A}_{\mathcal{G}} = \mathcal{G}\{y_1, \ldots, y_n\}$ and every $C \in \mathcal{A}_{\mathcal{G}}$, with $C \notin \mathfrak{p}$, there is a zero η of $\mathfrak{p}$, rational over $\mathcal{G}$, such that $C(\eta) \neq 0$.*

Proof.

We first show that the second and third statements are equivalent. □

$2 \Longrightarrow 3$. Let $\mathfrak{p}$ and C be as in 3. There exists η with coordinates in some extension Δ-field of $\mathcal{G}$ such that η is a C-constrained zero of $\mathfrak{p}$. Set $\mathfrak{q}_{\mathcal{G}} = \mathfrak{I}_{\mathcal{G}}(\eta)$. There is a Δ-subfield $\mathcal{F}$ of $\mathcal{G}$, with $\mathcal{F}$ finitely Δ-generated over $\mathbb{Q}$ such that $\mathfrak{q}_{\mathcal{G}}$ has field $\mathcal{F}$ of definition, and $\mathcal{A}_{\mathcal{F}}$ contains C. It follows that $\mathcal{U}$ is universal over $\mathcal{F}$. Let $\mathfrak{q}_{\mathcal{F}} = \mathfrak{q}_{\mathcal{G}} \cap \mathcal{A}_{\mathcal{F}}$. Then, $\mathcal{U}^n$ contains a point η' that is generic for $\mathfrak{q}_{\mathcal{F}}$ over $\mathcal{F}$. Since $C \notin \mathfrak{q}_{\mathcal{F}}$, $C(\eta') \neq 0$. Now, η is also generic for $\mathfrak{q}_{\mathcal{F}}$ over $\mathcal{F}$. Since $\mathcal{F}$ is a field of definition for $\mathfrak{I}_{\mathcal{G}}(\varsigma)$, η specializes over $\mathcal{G}$ to η'. Since $C(\eta') \neq 0$, this extended specialization is generic. Thus, $\eta' \in \mathcal{U}^n$ is constrained over $\mathcal{G}$ with constraint C. $2 \Longrightarrow \eta'$ is rational over $\mathcal{G}$. Therefore, η is rational over $\mathcal{G}$. So, $2 \Longrightarrow 3$.

Proof.

$3 \implies 2.$
Let $\eta \in \mathcal{U}^n$ be constrained over $\mathcal{G}$ with constraint C. Let $\mathfrak{p} = \mathfrak{I}_{\mathcal{G}}(\eta)$. Then, $\mathfrak{p}$ is prime and $C \notin \mathfrak{p}$. Therefore, by 3, $\exists \varsigma \in \mathcal{G}^n$ such that $\varsigma \in V(\mathfrak{p})$ and $C(\varsigma) \neq 0$. Since η is generic for $\mathfrak{p}$ over $\mathcal{G}$, ς is a specialization of η over $\mathcal{G}$. Since $C(\varsigma) \neq 0$, ς is generic for $\mathfrak{p}$.. Therefore, since $\varsigma \in \mathcal{G}^n$, so is η. So, now, 2 and 3 are equivalent. □

Clearly, $1 \implies 2$.
$2 \implies 1$.

Proof.

If an element $\eta \in \mathcal{U}$ is constrained over $\mathcal{G}$ then, so is $(\eta, \ldots, \eta) \in \mathcal{U}^n$. By 2, $(\eta, \ldots, \eta) \in \mathcal{G}^n$, and, thus, $\eta \in \mathcal{G}$. Therefore, 2 holds when n is replaced by 1, and therefore, so does 3. Now, 3 is clearly independent of $\mathcal{U}$, and, therefore, so is 2. Aha! Let $\mathcal{H}$ be a constrained extension of $\mathcal{G}$. We know that there is a universal extension field of $\mathcal{H}$. So, every element of this universal extension field of $\mathcal{H}$ that is constrained over $\mathcal{G}$ is in $\mathcal{G}$. Therefore, $\mathcal{G}$ is differentially closed. □

Corollary

Let $\mathcal{U}$ be a universal Δ-field. Then, $\mathcal{U}$ is differentially closed.

Proof.

Let $\mathfrak{p}$ be a prime Δ-ideal in $\mathcal{A}_{\mathcal{U}}$, and let $C \in \mathcal{A}_{\mathcal{U}}$, $C \notin \mathfrak{p}$. Let $\mathcal{F}$ be a field of definition of $\mathfrak{p}$ that is finitely Δ-generated over the prime field. Then, there is a generic zero η of $\mathfrak{p}_{\mathcal{F}} = \mathfrak{p} \cap \mathcal{A}_{\mathcal{F}}$ in $\mathcal{U}^n$. Of course, $C(\eta) \neq 0$ (this doesnt require the Nullstellensatz). But, $\mathfrak{p} = \mathcal{U}\ \mathfrak{p}_{\mathcal{F}}$. So, η is a zero of $\mathfrak{p}$. □