Differentially Closed Fields

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Some Preliminary Language

$\Delta = \{\delta_1, \ldots, \delta_m\}$, commuting derivation operators.
Unless otherwise indicated, all fields are $\mathbb{Q}$-algebras.
$\mathcal{F}$, $\mathcal{G}$:= Δ-fields, $\mathcal{F} \subseteq \mathcal{G}$.
Δ-$\mathcal{F}$-homomorphisms leave the elements of $\mathcal{F}$ fixed.
Extension means Δ-extension field. Finitely generated means as a Δ-extension field.
$y_1, \ldots, y_n :=$ Δ-indeterminates. $\mathcal{F}\{y_1, \ldots, y_n\} :=$ Δ-polynomial ring over the Δ-field $\mathcal{F}$.
An n-tuple η is *rational over* $\mathcal{F}$ if its coordinates lie in $\mathcal{F}$. Let $S \subseteq \mathcal{G}^n$.
$S(\mathcal{F})$ is the set of elements of S that are rational over $\mathcal{F}$.

Universal Differential Fields

In1953, Kolchin proved the existence of *universal differential fields.*

Definition

Let $\eta \in \mathcal{G}^n$. The *defining ideal of* η over $\mathcal{F}$ is the prime Δ-ideal

$$\mathfrak{p} = \{P \in \mathcal{F}\{y_1, \ldots, y_n\} \mid P(\eta) = 0\}.$$

η is said to be *generic* for $\mathfrak{p}$.

$$
\begin{array}{ccc}
 & & \mathcal{U} \\
\mathcal{H} & \longrightarrow & \\
 & & | \\
 & \diagdown & \\
 & & \mathcal{G} \\
 & & | \\
 & & \mathcal{F}
\end{array}
$$

Definition

Let $\mathcal{U}$ be an extension of $\mathcal{F}$, and let $\mathcal{G}$ be a finitely generated extension in $\mathcal{U}$ of $\mathcal{F}$. $\mathcal{U}$ is *universal* over $\mathcal{F}$ if every finitely generated extension $\mathcal{H}$ of $\mathcal{G}$ can be embedded over $\mathcal{G}$ in $\mathcal{U}$. Equivalently, $\forall n$ every prime Δ-ideal in $\mathcal{G}\{y_1, \ldots, y_n\}$ has a generic zero in $\mathcal{U}^n$. $\mathcal{U}$ has infinite differential transcendence degree over $\mathcal{F}$.

Constrained Points

Let $\eta \in \mathcal{G}^n$, $\varsigma \in \mathcal{H}^n$, $\mathcal{H}$ an extension of $\mathcal{F}$. Let $\mathfrak{p}$ be the defining ideal of η, and $\mathfrak{q}$ the defining ideal of ς, in $\mathcal{F}\{y_1, \ldots, y_n\}$.

Definition

η *specializes* to ς over $\mathcal{F}$ if $\mathfrak{p} \subseteq \mathfrak{q}$. The specialization is *generic* if $\mathfrak{p} = \mathfrak{q}$.

Definition

η is *constrained* over $\mathcal{F}$ if there exists $C \in \mathcal{F}\{y_1, \ldots, y_n\}$ such that $C(\eta) \neq 0$, and $C(\varsigma) = 0$ $\forall$ nongeneric specializations ς of η over $\mathcal{F}$.

Let $\mathcal{U}$ be universal over $\mathcal{F}$. Equip affine n-space $\mathbb{A}^n(\mathcal{U})$ with the Kolchin $\mathcal{F}$-topology. The Kolchin closed sets are the zero sets of differential polynomial ideals in $\mathcal{F}\{y_1, \ldots, y_n\}$.

Theorem

The points in a Kolchin closed set that are constrained over $\mathcal{F}$ are dense in the Kolchin topology.

Definition

$\mathcal{G}$ is constrained over $\mathcal{F}$ if every finite family of elements of $\mathcal{G}$ is constrained over $\mathcal{F}$.

Definition

(Abraham Robinson, 1959) $\mathcal{F}$ is *differentially closed* if for all $n, r \in \mathbb{Z}_{>0}$, and $P_1, \ldots, P_r, Q \in \mathcal{F}\{y_1, \ldots, y_n\}$, if the system

$$P_1 = \cdots = P_r = 0, Q \neq 0$$

has a solution rational over an extension, then it has a solution rational over $\mathcal{F}$.

Robinson considered only ordinary differential fields.

Definition

(E. R. Kolchin, 1960's) $\mathcal{F}$ is *differentially closed* if it has no proper constrained extensions.

A differentially closed field is algebraically closed.

Corollary

A universal extension $\mathcal{U}$ of $\mathcal{F}$ is differentially closed.

Theorem

Robinson's definition of DCF is equivalent to Kolchin's definition.

Blum's Simplified Definition

Lenore Blum simplified Robinson's definition, using language that parallels the definition of algebraically closed field:

Definition

(Blum, 1968) Let $\mathcal{F}$ be an *ordinary* differential field. Let y be a differential indeterminate. $\mathcal{F}$ is *differentially closed* if the system

$$P = 0, \quad Q \neq 0, \quad P, Q \in \mathcal{F}\{y\}, \quad \text{ord}\ (Q) < \text{ord}\ (P),$$

has a solution in $\mathcal{F}$.

Differential Closures

Since universal extensions are differentially closed, every Δ-field has a differentially closed extension field.

Definition

A *differential closure* of $\mathcal{F}$ is a differentially closed extension $\mathcal{F}^\dagger$ that can be embedded over $\mathcal{F}$ in every differentially closed extension.

Theorem

(Morley, 1965, Blum, 1968, Shelah, 1972, – ordinary differential fields, Kolchin, 1974–any Δ-field) Every Δ-field has a differential closure $\mathcal{F}^\dagger$. It is unique up to Δ-$\mathcal{F}$-isomorphism.

Kolchin's Characterization of Differential Closures

Theorem

(Kolchin 1974)

1. $\mathcal{F}^\dagger$ *is a constrained extension of* $\mathcal{F}$*, and card* $\mathcal{F}^\dagger = \text{card } \mathcal{F}$.
2. $\mathcal{F}^\dagger$ *contains the algebraic closure* $\mathcal{F}^a$ *of* $\mathcal{F}$.
3. *The field of constants of* $\mathcal{F}^\dagger$ *is the algebraic closure of the field of constants of* $\mathcal{F}$.

$\mathcal{F}^\dagger$ is a maximal constrained extension of $\mathcal{F}$.

The non-minimality of differential closures

An algebraic closure of a field $\mathcal{K}$ is not isomorphic to any proper subfield containing $\mathcal{K}$.

(1971, Harrington) The ordinary differential field $\mathbb{Q}^\dagger$ is computable

(1972, Sacks conjecture) The ordinary differential field $\mathbb{Q}^\dagger$ is minimal.

(1974, Kolchin, Rosenlicht, Shelah) The ordinary differential field $\mathbb{Q}^\dagger$ is not minimal. There exists an infinite strictly decreasing sequence of differential closures of $\mathbb{Q}$ in $\mathbb{Q}^\dagger$.

Leaving Kolchin's Universe: Varying the DCF

In the remaining sections, $\mathcal{F} = \mathbb{C}(t)$, t transcendental over $\mathbb{C}$. $\delta = \frac{d}{dt}$.
Let $\mathcal{G}$ be a differentially closed extension of $\mathcal{F}$, with field $\mathcal{C}$ of constants.

$$\mathcal{F} \subseteq \mathcal{F}^{\mathrm{a}} \subseteq \mathcal{F}^{\dagger} \subseteq \mathcal{G}.$$

Equip affine n-space $\mathbb{A}^n(\mathcal{G})$ and projective n-space $\mathbb{P}^n(\mathcal{G})$ with the *Zariski* and *Kolchin* topologies.)

To extend either topology from affine space to projective space, start with a closed subset V of $\mathbb{A}^n(\mathcal{G})$, and homogenize its defining polynomials.Also call closed sets "varieties."

x, y, z are differential indeterminates.

Examples

Example

V is the irreducible affine Zariski closed subset of $\mathbb{A}^2(\mathcal{G})$,defined by the Legendre equation

$$y^2 = x(x-1)(x-t).$$

Its projective closure, the *elliptic curve* E, is defined by

$$yz^2 = x(x-z)(x-tz).$$

The algebraic point $\left(t+1, \sqrt{t^2+t}, 1\right)$ lies on E.

Example

$V = \mathbb{A}^2(\mathcal{C})$ is Kolchin closed in $\mathbb{A}^n(\mathcal{G})$. V has defining equations

$$x' = 0, \qquad y' = 0.$$

Its projective closure $\mathbb{P}^2(\mathcal{C})$ has defining equations

$$zx' - z'x = 0, \quad zy' - z'y = 0.$$

The point $(t, 2t, t)$ is in $\mathbb{P}^2(\mathcal{C})$.

Let $E = E(\mathcal{G}) \subseteq \mathbb{P}^2(\mathcal{G})$ be the elliptic curve. E is both Zariski $\mathcal{F}$-closed and Kolchin $\mathcal{F}$-closed. E is a commutative algebraic group by the chord-tangent construction. This means that it is a commutative group whose group laws are rational functions with coefficients in $\mathbb{C}(t)$. We write the group law additively.

The j-invariant $j(t)$ is an element of $\mathbb{C}(t)$ that is an invariant of the isomorphism class of E.

E descends to constants $\Longleftrightarrow j(t) \in \mathbb{C}$.

The j-invariant of E is

$$2^8 \frac{t^2 - t + 1}{t^2(t-1)^2},$$

which is non-constant.

Periodic Groups

Definition

A group is *periodic* (torsion) if all its elements have finite order.

Definition

A commutative group A is a *free abelian group* if $\exists S \subset A$ such that every element a in A can be written uniquely

$$a = \sum_{s \in S} n_s s, \quad n_s \in \mathbb{Z}, \quad n_s = 0 \text{ for all but a finite number of } s.$$

card S is called the *rank* of G. The rank need not be countable. We denote A by $\mathbb{Z}(S)$

Definition

Every commutative group G can be written as a direct sum

$$G = \mathbb{Z}_G(S) \oplus G_{\text{tors}} ,$$

Mordell-Weil Theorem for Function Fields

Theorem

Lang-Néron (1952) Let E be an elliptic curve defined over $\mathbb{C}(t)$ that does not descend to constants. Let $\mathcal{K}$ be a finite algebraic extension of $\mathbb{C}(t)$. $E(\mathcal{K})$ is a finitely generated abelian group.

So, r is a natural number, and $E(\mathcal{K})_{\text{tors}}$ is a finite group, whose order varies with $\mathcal{K}$.

Invariance of the Torsion group Under Change of Algebraically Closed Field

Theorem

(see J.S. Milne, Elliptic Curves) Let $\mathcal{K}$ be an algebraically closed field of characteristic 0, let $\mathcal{L}$ be an algebraically closed extension of $\mathcal{K}$, and let E be an elliptic curve in $\mathbb{P}^2(\mathcal{L})$, that is defined over $\mathcal{K}$. Then, $E(\mathcal{L})_{tors} = E(\mathcal{K})_{tors}$.

Corollary

If $\mathcal{K}$ is any algebraically closed subfield of $\mathcal{G}$ containing $\mathcal{F}$, then

$$E(\mathcal{K})_{tors} = E(\mathcal{F}^a)_{tors}.$$

A point of finite order in $E = E(\mathcal{G})$ is an algebraic function.

The Rank of the Elliptic Curve in an Algebraically Closed Field

Theorem

(Frey, Jarden, 1972). If $\mathcal{K}$ is any algebraically closed field that is not the algebraic closure of a finite field, the rank of $E(\mathcal{K})$ is the cardinality of $\mathcal{K}$.

Therefore,

$$E(\mathcal{F}^a) = \mathbb{Z}_G(S) \oplus E(\mathcal{F}^a)_{\text{tors}} ,$$

where $r =$card $\mathbb{C}$, and $E(\mathcal{F}^a)_{\text{tors}} = E(\mathcal{K})_{\text{tors}}$ is an infinite periodic group.

A Burnside Theorem

Theorem

Let $G = G(\mathcal{K})$ be any algebraic group whose points are rational over an algebraically closed field $\mathcal{K}$ of characteristic 0. If G is periodic, then it is finite.

Let $\mathcal{G}$ be a differentially closed field. A *differential algebraic group* is a subgroup of an algebraic group $G(\mathcal{G})$ that is closed in the Kolchin topology.

Theorem

(Kolchin 1986) Let $\mathcal{U}$ be a universal differential field. If a differential algebraic subgroup of an algebraic group $G(\mathcal{U})$ is periodic, then it is finite.

Russian Dolls: Elliptic Curves Defined over Constants–The Kolchin Kernel

Let $\mathcal{G}$ be a differentially closed field containing $\mathbb{C}(t)$, and let $\mathcal{C}$ be the field of constants of $\mathcal{G}$. Let $E = E(\mathcal{G})$ be an elliptic curve The Kolchin closure $E^{\#}$ of E_{tors} is a differential algebraic subgroup of E.
Suppose $E = E(\mathcal{G})$ is defined over $\mathbb{C}$ (the associated elliptic surface splits over $\mathbb{C}$). For example, E is defined by the equation

$$y^2 = (x - e_1)(x - e_2)(x - e_3),$$

e_1, e_2, e_3 distinct complex numbers.The logarithmic derivative map is a surjective homomorphism

$$\begin{aligned} \ell\delta \quad &: \quad E(\mathcal{G}) \longrightarrow \mathbb{G}_a(\mathcal{G}), \quad (x, y) \longmapsto \frac{x'}{y}, \\ \ker \ell\delta \quad &= \quad E(\mathcal{C}), \end{aligned}$$

the *Kolchin closure* of $E(\mathcal{G})_{\text{tors}}$. The kernel $E^{\#}(\mathcal{G}) = E(\mathcal{C})$ is a replica of $E(\mathcal{G})$ inside $E(\mathcal{G})$. See addendum.

An Exotic Differential Algebraic Group: The Manin Kernel

Let $\mathcal{F} = \mathbb{C}(t)$, $\delta = \frac{d}{dt}$, $\mathcal{G}$ a differentially closed extension of $\mathcal{F}$, $E = E(\mathcal{G})$, defined by the equation

$$y^2 = x(x-1)(x-t),$$

which does not descend to constants.

Theorem

(Manin, 1963, "Rational points of algebraic curves over function fields" -Buium, 1991, "Geometry of differential polynomial functions: algebraic groups") There is a surjective homomorphism of differential algebraic groups

$$\mu : E \longrightarrow \mathbb{G}_a(\mathcal{G})$$

(a differential rational function of x, y) such that the kernel of μ is $E^{\#}$.

μ is the analogue of $\ell\delta$ for elliptic curves not descending to constants.

In affine coordinates,

$$\mu\left(x,y\right)=\frac{-y}{\left(x-t\right)^{2}}+\left(2t\left(t-1\right)\frac{x'}{y}\right)'+\frac{t\left(t-1\right)}{x-t}\frac{x'}{y}.$$

(Marker-Pong correction of Manin). Manin calculated μ using de Rham cohomology.

The connected *differential algebraic group* $E^{\#}$ is called the *Manin kernel*. It is a "modular strongly minimal set"(Hrushovski). (Its proper Kolchin closed subsets are finite.) $E^{\#}$ has finite dimension 2 (tr deg over the ground field of its field of diff rat fcns).and finite Morley rank 1, but is not isomorphic to an algebraic subgroup of $\mathbb{P}^n(\mathbb{C})$.
The Manin morphism is constructed from the hypergeometric equation $(a = b = \frac{1}{2},\ \ c = 1)$

$$\frac{d^2\omega}{dt^2} + \frac{2t-1}{t(t-1)}\frac{d\omega}{dt} + \frac{1}{4t(t-1)}\omega = 0,$$

called the *Picard-Fuchs equation*. The periods ω_1, ω_2 of E are a fundamental system of solutions.

Theorem

Manin's Theorem of the Kernel (for Elliptic Curves) Let $\mathcal{K}$ be a finite algebraic extension of $\mathbb{C}(t)$. Then, $E^{\#}(\mathcal{K})$ is finite.

Theorem

(Follows from the Buium-Pillay Gap Theorem 1997) Let $\mathcal{G}$ be the differential closure $\mathcal{F}^{\dagger}$ of $\mathcal{F} = \mathbb{C}(t)$. Then

$$E^{\#}\left(\mathcal{F}^{\dagger}\right) = E\left(\mathcal{F}^{\dagger}\right)_{tors} = E^{\#}\left(\mathcal{F}^{a}\right).$$

The differential algebraic group $E^{\#}\left(\mathcal{F}^{\dagger}\right)$ contains no point of infinite order!

Historical Note

In a 1958 paper, "Algebraic curves over fields with differentiation," Manin introduced fields of definition with distinguished derivation δ, in order to prove Mordell's conjecture over function fields. In 1990, Robert Coleman found a crucial linear algebra error on p. 214 of the 1963 paper that invalidated the entire proof of the Theorem of the Kernel. He proved a weaker theorem, using Manin's techniques that implied Mordell. The next year, Ching-Li Chai showed that Manin was essentially correct, and was able to rescue the proof, using theorems of Deligne on monodromy theory of linear differential equations.

The Manin Kernel under Variation of the DCF

Theorem

Let $\mathcal{U}$ be a universal extension of $\mathcal{F} = \mathbb{C}(t)$.

$$\mathcal{F} \subseteq \mathcal{F}^{\dagger} \subseteq \mathcal{U}$$

$E^{\#}(\mathcal{F}^{\dagger}) = E\left(\mathcal{F}^{\dagger}\right)_{tors}$. $E^{\#}(\mathcal{U}) \neq E(\mathcal{U})_{tors}$.

Corollary

The Kolchin Burnside Theorem is not invariant under change of the underlying DCF.

Proof.

The differential algebraic group $E^{\#}(\mathcal{F}^{\dagger})$ is an infinite periodic group. □

Kolchin needed the presence in the group of a point generic over $\mathcal{F}$..

The Rank of the Kernels under Variation of the DCF

Let $\mathcal{G} \supseteq \mathbb{C}(t)$ be differentially closed with field $\mathcal{C}$ of constants, and let $E(\mathcal{G})$ be an elliptic curve.
How does the rank of $E^{\#}(\mathcal{G})$ = Kolchin closure of $E(\mathcal{G})_{\text{tors}}$ vary with $\mathcal{G}$?

- If E descends to constants, rank $E^{\#}(\mathcal{G}) =$ card $\mathcal{G}$, which is infinite.
- If E does not descend to constants, rank $E^{\#}\left(\mathbb{C}(t)^{\dagger}\right) = 0$, and rank $E^{\#}(\mathcal{U}) \neq 0$. Is rank $E^{\#}(\mathcal{U})$ infinite?

Note the startling difference geometrically between groups descending to constants and those that do not descend.