EVALUATION OF THE STRENGTH OF SYSTEMS OF PARTIAL DIFFERENTIAL AND DIFFERENCE EQUATIONS VIA DIFFERENTIAL AND DIFFERENCE DIMENSION POLYNOMIALS

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Let K be a differential (respectively, difference) field with a basic set $\Delta = \{\delta_1, \dots, \delta_m\}$ of mutually commuting mappings of K into itself such that each δ_i is a derivation of K (respectively, each δ_i is an injective endomorphism of K also called a translations). Then K will be called a Δ-field.

If $\delta_1, \dots, \delta_m$ are automorphisms of a difference field K, we say that K is an *inversive difference field* with the basic set Δ or a Δ^*-field. (In this case we set $\Delta^* = \{\delta_1, \dots, \delta_m, \delta_1^{-1}, \dots, \delta_m^{-1}\}$).

If K is a Δ-field, then Θ will denote the free commutative semigroup generated by $\delta_1, \dots, \delta_m$:
$\Theta = \{\delta_1^{k_1} \dots \delta_m^{k_m} \mid k_1, \dots, k_m \in \mathbf{N}\}$). If K is a Δ^*-field, then Γ will denote the free commutative group generated by Δ:
$\Gamma = \{\delta_1^{k_1} \dots \delta_m^{k_m}\} \mid k_1, \dots, k_m \in \mathbf{Z}\}$).

If K is a Δ-field and $\theta = \delta_1^{k_1} \dots \delta_m^{k_m} \in \Theta$ ($k_1, \dots, k_m \in \mathbf{N}$), then the number $ord\, \theta = \sum_{\nu=1}^{m} k_\nu$ is called the *order* of θ.

For any $r \in \mathbf{N}$, we set $\Theta(r) = \{\theta \in \Theta \mid ord\, \theta \le r\}$.

If K is a Δ^*-field and $\gamma = \delta_1^{k_1} \dots \delta_m^{k_m} \in \Gamma$ ($k_1, \dots, k_m \in \mathbf{Z}$), then the integer $ord\, \gamma = \sum_{\nu=1}^{m} |k_\nu|$ s called the *order* of γ.

If $r \in \mathbf{N}$, we set $\Gamma(r) = \{\gamma \in \Gamma \mid ord\, \gamma \le r\}$.

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Let K be a Δ-field K of zero characteristic and let $L = K\langle\eta_1, \dots, \eta_s\rangle$ be a Δ-field extension of K generated by a finite set $\eta = \{\eta_1, \dots, \eta_s\}$.

As a field, $L = K(\{\theta\eta_j | \theta \in \Theta, 1 \leq j \leq s\})$.

The following is a unified version of E. Kolchin's theorem on differential dimension polynomial and the speaker's theorem on the dimension polynomial of a difference field extension.

Theorem 1

With the above notation, there exists a polynomial $\phi_{\eta|K}(t) \in \mathbf{Q}[t]$ such that

(i) $\phi_{\eta|K}(r) = trdeg_K K(\{\theta\eta_j | \theta \in \Theta(r), 1 \leq j \leq s\})$ for all sufficiently large $r \in \mathbf{Z}$;

(ii) $\deg \phi_{\eta|K} \leq m$ and $\phi_{\eta|K}(t)$ can be written as

$\phi_{\eta|K}(t) = \sum_{i=0}^{m} a_i \binom{t+i}{i}$ where $a_0, \dots, a_m \in \mathbf{Z}$.

(iii) $d = \deg \phi_{\eta|K}$, a_m and a_d do not depend on the set of Δ-generators η of L/K ($a_d \neq a_m$ iff $d < m$). Moreover, a_m is equal to the Δ-transcendence degree of L over K (denoted by Δ-$trdeg_K L$), that is, to the maximal number of elements $\xi_1, \dots, \xi_k \in L$ such that the family $\{\theta\xi_i | \theta \in \Theta, 1 \leq i \leq k\}$ is algebraically independent over K.

$\phi_{\eta|K}(t)$ is called the Δ- (*differential* or *difference* depending on the nature of Δ) *dimension polynomial of the* Δ*-field extension* L *of* K *associated with the system of* Δ*-generators* η. The integers $d = deg\,\phi_{\eta\,|\,K}(t)$ and a_d are called, respectively, the Δ-*type* and *typical* Δ*-transcendence degree* of L over K. These invariants of $\phi_{\eta|K}(t)$ are denoted by Δ-$type_K L$ and Δ-$t.trdeg_K L$, respectively.

If $\eta_1, \dots, \eta_s$ are Δ-algebraically independent over K, then $\phi_{\eta|K}(r) = trdeg_K K(\{\theta\eta_j \mid \theta \in \Theta,\, 1 \leq j \leq s\}) = s \cdot Card\,\Theta(r) = s\binom{r+m}{m}$ for all sufficiently large $r \in \mathbf{N}$

($Card\,\Theta(r) = Card\{((k_1, \dots, k_m) \in \mathbf{N}^m \mid k_1 + \cdots + k_m \leq r\} = \binom{r+m}{m}$.) Therefore, in this case $\phi_{\eta|K}(t) = s\binom{t+m}{m}$.

Theorem 2

Let K be a Δ^*-field where $\Delta \subseteq Aut(K)$. Then there exists a polynomial $\psi_{\eta|K}(t) \in \mathbf{Q}[t]$ with the following properties.

(i) $\psi_{\eta|K}(r) = trdeg_K K(\{\gamma\eta_j \,|\, \gamma \in \Gamma(r), 1 \leq j \leq s\})$ for all sufficiently large $r \in \mathbf{N}$.

(ii) $deg\, \psi_{\eta|K}(t) \leq m$ and $\psi_{\eta|K}(t)$ can be written as

$$\psi_{\eta|K}(t) = \frac{2^m a}{m!} t^m + o(t^m)$$

where $a \in \mathbf{Z}$ and $o(t^m)$ is a numerical polynomial of degree less than m.

(iii) The integers a, $d = deg\, \psi_{\eta|K}(t)$ and the coefficient of t^d in the polynomial $\psi_{\eta|K}(t)$ do not depend on the choice of a system of generators η.Furthermore, $a = \Delta$-$trdeg_K L$.

(iv) If $\eta_1, \dots, \eta_s$ are Δ-algebraically independent over K, then

$$\psi_{\eta|K}(t) = s \sum_{k=0}^{m} (-1)^{m-k} 2^k \binom{m}{k} \binom{t+k}{k}.$$

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Let us consider a system of algebraic Δ- (differential or difference) or Δ^*-(inversive difference) equations

$$A_i(y_1, \dots, y_s) = 0 \qquad (i = 1, \dots, p) \qquad (1)$$

where $A_i(y_1, \dots, y_s)$ are Δ- (or Δ^*-) polynomials in the ring $R = K\{y_1, \dots, y_s\}$ (respectively, in $R = K\{y_1, \dots, y_s\}^*$).

($K\{y_1, \dots, y_s\}$ is the ring of Δ-polynomials in s Δ-indeterminates over K; as a ring, it coincides with $K[\{\theta y_i \,|\, \theta \in \Theta, 1 \le i \le s\}]$. $K\{y_1, \dots, y_s\}^*$ is the ring of Δ^*-polynomials; as a ring it coincides with $K[\{\gamma y_i \,|\, \gamma \in \Gamma, 1 \le i \le s\}]$.)

System (1) is called *prime* if the Δ-ideal (respectively, Δ^*-ideal if we deal with difference or inversive difference cases) P of R is prime. (For example, this is the case if system (1) is linear.)

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Let η_i be the canonical image of y_i in R/P ($1 \leq i \leq s$) and for every $r \in \mathbf{N}$, let $R_r = K[\Theta(r)y_1 \bigcup \cdots \bigcup \Theta(r)y_s]$ (respectively, $R_r = K[\Gamma(r)y_1 \bigcup \cdots \bigcup \Gamma(r)y_s]$ in the case of inversive difference equations) then $P \cap R_r$ is a prime ideal of the ring R_r and the quotient field of $R_r/P \cap R_r$ are isomorphic to $K(\{\theta(\eta_j) \,|\, \theta \in \Theta(r),\, 1 \leq j \leq s\})$ (respectively, $K(\{\gamma(\eta_j) \,|\, \gamma \in \Gamma(r),\, 1 \leq j \leq s\})$).

Applying Theorem 1 in the case of differential equations we obtain that there exists a polynomial $\phi_P(t) \in \mathbf{Q}[t]$ such that

$$\phi_P(r) = trdeg_K K(\{\theta(\eta_j) \,|\, \theta \in \Theta(r),\, 1 \leq j \leq s\}) = trdeg_K(R_r/P \cap R_r)$$

for all sufficiently large $r \in \mathbf{Z}$, $deg\, \psi(t) \leq m$ and the polynomial $\phi_P(t)$ can be written as $\phi_P(t) = \sum_{i=0}^{m} a_i \binom{t+i}{i}$ where $a_0, \ldots, a_m \in \mathbf{Z}$ and $a_m = \Delta\text{-}trdeg_K K\langle \eta_1, \ldots, \eta_s \rangle$.

In the case of difference or inversive difference equations Theorem 2 shows the existence of a polynomial $\psi_P(t) \in \mathbf{Q}[t]$ such that

$$\psi_P(r) = trdeg_K K(\{\gamma(\eta_j) \,|\, \gamma \in \Gamma(r),\, 1 \leq j \leq s\}) = trdeg_K(R_r/P \cap R_r)$$

for all sufficiently large $r \in \mathbf{Z}$, $deg\,\psi(t) \leq m$ and the polynomial $\psi_P(t)$ can be written as $\psi_P(t) = \dfrac{2^m a_m}{m!} t^m + o(t^m)$ where $a_m = \Delta\text{-}trdeg_K K\langle \eta_1, \dots, \eta_s\rangle^*$.

The polynomial $\phi_P(t)$ (respectively, $\psi_P(t)$) is called the Δ- (respectively, Δ^*-) *dimension polynomial* of system of differential (respectively, difference) equations (1).

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The differential (Δ-) dimension polynomial, as it was first noticed in [Mikhalev, A. V.; Pankratev, E. V. Differential dimension polynomial of a system of differential equations. *Algebra*, Moscow State Univ., 1980, 57-67], expresses the strength of a system of PDEs in the sense of A. Einstein who introduced this characteristic as a measure for the size of the solution space of such a system (see [Einstein, A. The Meaning of Relativity. Appendix II (Generalization of gravitation theory), 4th edn. Princeton, 1953, 133 - 165]). His description of the strength of such a system governing a physical field is as follows:

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"... the system of equations is to be chosen so that the field quantities are determined as strongly as possible. In order to apply this principle, we propose a method which gives a measure of strength of an equation system. We expand the field variables, in the neighborhood of a point $\mathcal{P}$, into a Taylor series (which presupposes the analytic character of the field); the coefficients of these series, which are the derivatives of the field variables at $\mathcal{P}$, fall into sets according to the degree of differentiation. In every such degree there appear, for the first time, a set of coefficients which would be free for arbitrary choice if it were not that the field must satisfy a system of differential equations. Through this system of differential equations (and its derivatives with respect to the coordinates) the number of coefficients is restricted, so that in each degree a smaller number of coefficients is left free for arbitrary choice. The set of numbers of "free" coefficients for all degrees of differentiation is then a measure of the "weakness" of the system of equations, and through this, also of its "strength"."

Calculating by hand A. Einstein found out that the potential and field formulations of Maxwell equations have different strengths. However, he did not obtain an exact expression of the mentioned number of free coefficients as a function of the degree of differentiation. Even though there were several works by different authors on the strength of a system of differential equations (in particular, on its relation to Cartan characters), there was no method of evaluating such a function until the technique of dimension polynomials was developed. As one can see from the following list of such publications, most of the works were written by people who worked in mathematical physics.

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1. Mariwalla, K. H. Applications of the concept of strength of a system of partial differential equations. *J. Math. Phys.*, 15 (1974), 468-473.
2. Matthews, N. On the strength of Maxwell's equations. *J. Math. Phys.*, 28 (1987), 810-814.
3. Schutz, B. On the strength of a system of partial differential equations. *J. Math. Phys.*, 16 (1974), 855-856.
4. Seiler, W. On the arbitrariness of the general solution of an involutive partial differential equation. *J. Math. Phys.*, 35 (1994), 486-498.
5. Seiler, W. Arbitrariness of the general solution and symmetries. In *Acta Appl. Math.*, 41 (1995), 311-322.
6. Seiler, W. Involution. The Formal Theory of Differential Equations and its Applications in Computer Algebra. *Springer*, 2010.
7. Sué, M. Involutive systems of differential equations: Einstein's strength versus Cartan's degré d'arbitraire. *J. Math. Phys.*, 32 (1991), 392-399.

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Considering a system of equations in finite differences over a field of of functions in several real variables, one can use the A. Einstein's approach to define the concept of *strength* of such a system as follows. Let

$$A_i(f_1, \ldots, f_s) = 0 \qquad (i = 1, \ldots, p) \qquad (2)$$

be a system of equations in finite differences with respect to s unknown grid functions $f_1, \ldots, f_s$ in n real variables $x_1, \ldots, x_n$ with coefficients in some functional field K. We also assume that the difference grid, whose nodes form the domain of considered functions, has equal cells of dimension $h_1 \times \cdots \times h_n$ ($h_1, \ldots, h_n \in \mathbf{R}$) and fills the whole space $\mathbf{R}^n$. As an example, one can consider a field K consisting of a zero function and fractions of the form u/v where u and v are grid functions defined almost everywhere (i. e., there are only finitely many nodes where such a function is not defined) and vanishing at a finite number of nodes.

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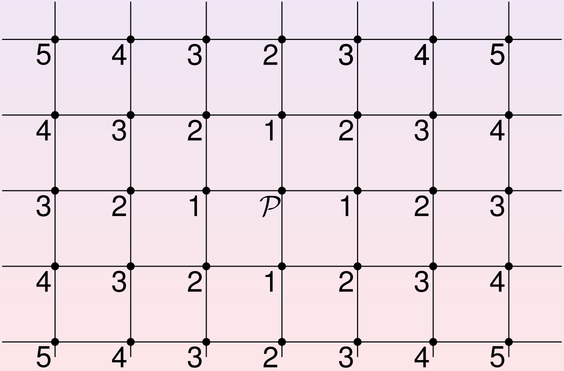
Let us fix some node $\mathcal{P}$ say that *a node $\mathcal{Q}$ has order i* (with respect to $\mathcal{P}$) if the shortest path from $\mathcal{P}$ to $\mathcal{Q}$ along the edges of the grid consists of *i* steps (by a step we mean a path from a node of the grid to a neighbor node along the edge between these two nodes). Say, the orders of the nodes in the two-dimensional case are as follows (a number near a node shows the order of this node).



Let us consider the values of the unknown grid functions $f_1, \ldots, f_s$ at the nodes whose order does not exceed r $(r \in \mathbf{N})$. If $f_1, \ldots, f_s$ should not satisfy any system of equations (or any other condition), their values at nodes of any order can be chosen arbitrarily. Because of the system in finite differences (and equations obtained from the equations of the system by transformations of the form $f_j(x_1, \ldots, x_s) \mapsto f_j(x_1 + k_1 h_1, \ldots, x_s + k_n h_n$ with $k_1, \ldots, k_n \in \mathbf{Z}$, $1 \leq j \leq s$), the number of independent values of the functions $f_1, \ldots, f_s$ at the nodes of order $\leq r$ decreases. This number, which is a function of r, is considered as a ”measure of strength” of the system in finite differences (in the sense of A. Einstein). We denote it by S_r.

With the above conventions, suppose that the transformations α_j of the field of coefficients K defined by

$$\alpha_j f(x_1, \dots, x_n) = f(x_1, \dots, x_{j-1}, x_j + h_j, \dots, x_n)$$

$(1 \leq j \leq n)$ are automorphisms of this field. Then K can be considered as an inversive difference field with the basic set $\sigma = \{\alpha_1, \dots, \alpha_n\}$. The replacement of the unknown functions f_i by difference indeterminates y_i $(i = 1, \dots, s)$ leads to a system of algebraic difference equations of the form (2). If this system is prime (e.g., we deal with a system of linear difference equations), then its difference dimension polynomial $\psi(t)$ expresses the strength S_r. Thus, this polynomial can be naturally viewed as the measure of the A. Einstein's strength of a given system of equations in finite differences.

Methods of computation of Δ- and Δ^*- dimension polynomials of a system of algebraic partial differential or difference equations developed in the 1980s were based either on building of a characteristic set of the considered above associated Δ- (or Δ^*-) ideal P in $K\{y_1, \dots, y_s\}$ (respectively, in $K\{y_1, \dots, y_s\}^*$) or on constructing a free resolution of the module of Kähler differentials associated with the extension $K\rangle\eta_1, \dots, \eta_s\langle$ (or $K\langle\eta_1, \dots, \eta_s\rangle^*$). The corresponding computations can be found, for example, in [Kondrateva, M.V.; Levin, A. B.; Mikhalev, A. V.; Pankratev, E. V. Differential and Difference Dimension Polynomials. *Kluwer Acad. Publ.*, 1998].

The main drawback of these approaches is the lack of efficient algorithms for constructing characteristic sets and serious restriction on the systems to which one can apply the method of free resolutions.

For example, the only case when we can use the method of free resolutions is the case of systems of inversive difference equations where the inversive difference operators are linear and symmetric, that is, whenever an equation involves a Δ^*-operator $\omega = a_1\delta_1^{k_{11}}\dots\delta_m^{k_{1m}} + \dots + a_r\delta_1^{k_{r1}}\dots\delta_m^{k_{rm}}$ ($a_i \in K$), which contains a term $a\delta_1^{l_1}\dots\delta_m^{l_m}$ ($a \in K,\ a \neq 0$), then it also contains all terms of the form $b\delta_1^{\pm l_1}\dots\delta_m^{\pm l_m}$ with nonzero coefficients $b \in K$ and all 2^m distinct combinations of signs before $l_1, \dots, l_m$.

In what follows we consider Gröbner basis method that yields algorithms of computation of dimension polynomials (and therefore, the strength of a system of algebraic partial differential or difference equations), which does not have some of these restrictions.

The following considerations lead to an essential generalization of Theorems 1 and 2 on differential and difference dimension polynomials. We will consider the differential case and show that there is a multivariate numerical polynomial associated with any partition of the basic set of derivation operators. These multivariate differential dimension polynomials represent the ”generalized” strength of a system of algebraic differential equations, which is defined in the same way as the Einstein's concept of strength if one imposes separate restrictions on the orders of derivations with respect to each group of basic derivation operators.

Let K be a differential field ($Char\, K = 0$) whose basic set Δ is a union of p disjoint finite sets ($p \geq 1$): $\Delta = \Delta_1 \bigcup \cdots \bigcup \Delta_p$, where $\Delta_i = \{\delta_{i1}, \dots, \delta_{im_i}\}$ ($i = 1, \dots, p$). Thus, we fix a partition of the set Δ.

For any $\theta = \delta_{11}^{k_{11}} \dots \delta_{1m_1}^{k_{1m_1}} \delta_{21}^{k_{21}} \dots \delta_{pm_p}^{k_{pm_p}} \in \Theta$, we define the order of the element θ with respect to Δ_i as follows: $ord_i\, \theta = \sum_{j=1}^{m_i} k_{ij}$ ($i = 1, \dots, p$).

Furthermore, for any $r_1, \dots, r_p \in \mathbf{N}$, we set $\Theta(r_1, \dots, r_p) = \{\theta \in \Theta | ord_i\, \theta \leq r_i \text{ for } i = 1, \dots, p\}$.

Theorem 3 (L., 2007)

Let $L = K\langle \eta_1, \dots, \eta_n \rangle$ be a Δ-field extension generated by a set $\eta = \{\eta_1, \dots, \eta_n\}$. Then there exists a polynomial $\Phi_\eta(t_1, \dots, t_p)$ in p variables $t_1, \dots, t_p$ with rational coefficients such that

(i) $\Phi_\eta(r_1, \dots, r_p) = tr.deg_K K(\bigcup_{j=1}^{n} \Theta(r_1, \dots, r_p)\eta_j)$ for all sufficiently large $(r_1, \dots, r_p) \in \mathbf{N}^p$ (i. e., there exist $s_1, \dots, s_p \in \mathbf{N}$ such that the last equality holds for all elements $(r_1, \dots, r_p) \in \mathbf{N}^p$ with $r_1 \geq s_1, \dots, r_p \geq s_p$);

(ii) $deg_{t_i} \Phi_\eta \leq m_i$ $(1 \leq i \leq p)$, so that $deg\, \Phi_\eta \leq m$ and the polynomial $\Phi_\eta(t_1, \dots, t_p)$ can be represented as

$$\Phi_\eta = \sum_{i_1=0}^{m_1} \dots \sum_{i_p=0}^{m_p} a_{i_1 \dots i_p} \binom{t_1 + i_1}{i_1} \dots \binom{t_p + i_p}{i_p}$$

where $a_{i_1 \dots i_p} \in \mathbf{Z}$ for all $i_1, \dots, i_p$.

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• $\Phi_\eta(t_1, \dots, t_p)$ is called the *differential dimension polynomial* of the extension L/K associated with the set of differential generators η (and the given partition of the basic set Δ).

• For any permutation $(j_1, \dots, j_p)$ of the set $\{1, \dots, p\}$, we define the lexicographic order $<_{j_1,\dots,j_p}$ on $\mathbf{N}^p$ as follows: $(r_1, \dots, r_p) <_{j_1,\dots,j_p} (s_1, \dots, s_p)$ if and only if either $r_{j_1} < s_{j_1}$ or there exists $k \in \mathbf{N}$, $1 \le k \le p-1$, such that $r_{j_\nu} = s_{j_\nu}$ for $\nu = 1, \dots, k$ and $r_{j_{k+1}} < s_{j_{k+1}}$.

If $\Sigma \subseteq \mathbf{N}^p$, then Σ' denotes the set $\{e \in \Sigma | e$ is a maximal element of Σ with respect to one of the $p!$ lexicographic orders $<_{j_1,\dots,j_p}\}$. For example, if $\Sigma = \{(3,0,2), (2,1,1), (0,1,4), (1,0,3), (1,1,6), (3,1,0), (1,2,0)\} \subseteq \mathbf{N}^3$, then $\Sigma' = \{(3,0,2), (3,1,0), (1,1,6), (1,2,0)\}$.

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Theorem 4

Let K be a differential field whose basic set of derivations Δ is a union of p disjoint finite sets ($p \geq 1$): $\Delta = \Delta_1 \bigcup \cdots \bigcup \Delta_p$, where $\Delta_i = \{\delta_{i1}, \dots, \delta_{im_i}\}$ ($i = 1, \dots, p$). Let $L = K\langle \eta_1, \dots, \eta_n \rangle$ and
$\Phi_\eta = \sum_{i_1=0}^{m_1} \dots \sum_{i_p=0}^{m_p} a_{i_1 \dots i_p} \binom{t_1 + i_1}{i_1} \dots \binom{t_p + i_p}{i_p}$ the corresponding dimension polynomial. Let

$E_\eta = \{(i_1, \dots, i_p) \in \mathbf{N}^p | 0 \leq i_k \leq m_k$ for $k = 1, \dots, p$ and $a_{i_1 \dots i_p} \neq 0\}$.

Then $d = deg\, \Phi_\eta$, $a_{m_1 \dots m_p}$, elements $(j_1, \dots, j_p) \in E'_\eta$, the corresponding coefficients $a_{j_1 \dots j_p}$, and the coefficients of the terms of total degree d do not depend on η. Furthermore, $a_{m_1, \dots, m_p} = \Delta\text{-}tr.deg_K L$.

The proof of Theorems 3 and 4, as well as a method of computation of multivariate dimension polynomials, is based on the corresponding results for differential, difference, and inversive difference modules.

Considering the differential case we treat the module of differentials $\Omega_K(L)$ associated with the differential field extension $L = K\langle \eta_1, \dots, \eta_s \rangle$ of K as a Δ-L-module generated by the differentials $d\eta_1, \dots, d\eta_s$ ($d\eta_i(D) = D(\eta_i)$ for any K-derivation $D : L \to L$, and $d(\delta_j(\eta_i)) = \delta_j(d\eta_i)$ for all $i = 1, \dots, s$; $j = 1, \dots, m$).

In other words, $\Omega_K(L)$ is a finitely generated module over the ring of Δ-operators $\mathcal{D} = \{\sum_{\theta \in \Theta} c_\theta \theta \,|\, c_\theta = 0$ for almost all $\theta \in \Theta\}$.

$\mathcal{D}$ has a natural structure of a K-vector space and multiplication determined by the rule $\delta a = a\delta + \delta(a)$ for any $\delta \in \Delta,\ a \in K$.

As before, let us consider a partition $\Delta = \Delta_1 \bigcup \cdots \bigcup \Delta_p$, where $\Delta_i = \{\delta_{i1}, \dots, \delta_{im_i}\}$ ($i = 1, \dots, p$).

If $\theta = \delta_{11}^{k_{11}} \dots \delta_{1m_1}^{k_{1m_1}} \delta_{21}^{k_{21}} \dots \delta_{pm_p}^{k_{pm_p}} \in \Theta$, then we define the order of the element θ with respect to Δ_i as follows: $ord_i\, \theta = \sum_{j=1}^{m_i} k_{ij}$ ($i = 1, \dots, p$).

For any $r_1, \dots, r_p \in \mathbf{N}$, let $\Theta(r_1, \dots, r_p) = \{\theta \in \Theta | ord_i\, \theta \le r_i$ for $i = 1, \dots, p\}$ and let $\mathcal{D}_{r_1,\dots,r_p}$ denote the vector K-subspace of $\mathcal{D}$ generated by the $\Theta(r_1, \dots, r_p)$.

Setting $\mathcal{D}_{r_1,\dots,r_p} = 0$ for any $(r_1, \dots, r_p) \in \mathbf{Z}^p \setminus \mathbf{N}^p$, we obtain a family $\{\mathcal{D}_{r_1,\dots,r_p} | (r_1, \dots, r_p) \in \mathbf{Z}^p\}$ of vector K-subspaces of $\mathcal{D}$ which is called the *standard p-dimensional filtration* of the ring $\mathcal{D}$.

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Clearly, $\mathcal{D}_{r_1,\dots,r_p} \subseteq \mathcal{D}_{s_1,\dots,s_p}$ if $(r_1,\dots,r_p)$ is less than $(s_1,\dots,s_p)$ w.r.t. the product order on $\mathbf{Z}^p$, and for any $(r_1,\dots,r_p)$, $(i_1,\dots,i_p) \in \mathbf{N}^p$ one has $\mathcal{D}_{i_1,\dots,i_p}\mathcal{D}_{r_1,\dots,r_p} = \mathcal{D}_{r_1+i_1,\dots,r_p+i_p}$.

Let M be a vector Δ-K-module, that is, a left $\mathcal{D}$-module. A family $\{M_{r_1,\dots,r_p} | (r_1,\dots,r_p) \in \mathbf{Z}^p\}$ is said to be a p-dimensional filtration of M if the following four conditions hold:

(i) $M_{r_1,\dots,r_p} \subseteq M_{s_1,\dots,s_p}$ for any p-tuples $(r_1,\dots,r_p)$, $(s_1,\dots,s_p) \in \mathbf{Z}^p$ such that $(r_1,\dots,r_p) \leq_P (s_1,\dots,s_p)$.

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(ii) $\bigcup_{(r_1,\dots,r_p)\in \mathbf{Z}^p} M_{r_1,\dots,r_p} = M$.

(iii) There exists a p-tuple $(r_1^{(0)},\dots,r_p^{(0)}) \in \mathbf{Z}^p$ such that $M_{r_1,\dots,r_p} = 0$ if $r_i < r_i^{(0)}$ for at least one index i $(1 \leq i \leq p)$.

(iv) $\mathcal{D}_{r_1,\dots,r_p} M_{s_1,\dots,s_p} \subseteq M_{r_1+s_1,\dots,r_p+s_p}$ for any $(r_1,\dots,r_p),(s_1,\dots,s_p) \in \mathbf{Z}^p$.

If every vector K-space $M_{r_1,\dots,r_p}$ is finite-dimensional and there exists an element $(h_1,\dots,h_p) \in \mathbf{Z}^p$ such that $\mathcal{D}_{r_1,\dots,r_p} M_{h_1,\dots,h_p} = M_{r_1+h_1,\dots,r_p+h_p}$ for any $(r_1,\dots,r_p) \in \mathbf{N}^p$, the p-dimensional filtration $\{M_{r_1,\dots,r_p} | (r_1,\dots,r_p) \in \mathbf{Z}^p\}$ is called **excellent**.

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Theorem 5

Let K be a differential field with a basic set $\Delta = \{\delta_1, \dots, \delta_m\}$ and let $\mathcal{D}$ be the ring of differential operators over K equipped with the standard p-dimensional filtration corresponding to a partition $\Delta = \Delta_1 \bigcup \cdots \bigcup \Delta_p$. Furthermore, let $m_i = Card\,\Delta_i\ (i = 1, \dots, p)$ and let $\{M_{r_1 \dots r_p} | (r_1, \dots, r_p) \in \mathbf{Z}^p\}$ be an excellent p-dimensional filtration of a vector σ-K-space M. Then there exists a polynomial $\phi(t_1, \dots, t_p) \in \mathbf{Q}[t_1, \dots, t_p]$ such that

(i) $\phi(r_1, \dots, r_p) = dim_K M_{r_1 \dots r_p}$ for all sufficiently large $(r_1, \dots, r_p) \in \mathbf{Z}^p$;

(ii) $deg_{t_i} \phi \le m_i$ $(1 \le i \le p)$, so that $deg\,\phi \le m$ and the polynomial $\phi(t_1, \dots, t_p)$ can be represented as

$$\phi(t_1, \dots, t_p) = \sum_{i_1=0}^{m_1} \dots \sum_{i_p=0}^{m_p} a_{i_1 \dots i_p} \binom{t_1 + i_1}{i_1} \dots \binom{t_p + i_p}{i_p}$$

where $a_{i_1 \dots i_p} \in \mathbf{Z}$ for all $i_1, \dots, i_p$.

If $L = K\langle\eta_1, \dots, \eta_s\rangle$, then $\Omega_K(L)$ has a natural excellent p-dimension filtration $\{(\Omega_K(L))_{r_1 \dots r_p}) \mid r_1, \dots, r_p \in \mathbf{N}\}$, where $\Omega_K(L))_{r_1 \dots r_p}$ is a vector L-space generated by the set $\{d\eta \mid \eta \in K(\{\theta\eta_i \mid \theta \in \Theta(r_1, \dots, r_p),\, 1 \leq j \leq s\})\}$, and $dim_L \Omega_K(L)_{r_1 \dots r_p} = \phi_\eta(r_1, \dots, r_p)$ for all sufficiently large $r_1, \dots, r_p \in \mathbf{Z}$.

Thus, the proof of Theorem 3 and the computation of the dimension polynomial $\phi_\eta(t_1, \dots, t_p)$ can be reduced to the proof of the existence and computation of the corresponding multivariate dimension polynomial of a finitely generated differential module. In what follows we consider a method of such computation based on some generalization of the Gröbner basis technique.

Let K be a differential (Δ-) field and $\Delta = \Delta_1 \bigcup \dots \bigcup \Delta_p$, where $\Delta_i = \{\delta_{i1}, \dots, \delta_{im_i}\}$ ($i = 1, \dots, p$). As before, if $\theta = \delta_{11}^{k_{11}} \dots \delta_{1m_1}^{k_{1m_1}} \delta_{21}^{k_{21}} \dots \delta_{pm_p}^{k_{pm_p}} \in \Theta$, we set $ord_i\, \theta = \sum_{j=1}^{m_i} k_{ij}$ ($i = 1, \dots, p$).
Let us consider p orders $<_1, \dots, <_p$ on Θ such that $\theta = \delta_{11}^{k_{11}} \dots \delta_{1m_1}^{k_{1m_1}} \delta_{21}^{k_{21}} \dots \delta_{pm_p}^{k_{pm_p}} <_i \theta' = \delta_{11}^{l_{11}} \dots \delta_{1m_1}^{l_{1m_1}} \delta_{21}^{l_{21}} \dots \delta_{pm_p}^{l_{pm_p}}$ if and only if the vector $(ord_i\theta, ord\, \theta, ord_1\theta, \dots, ord_{i-1}\theta, ord_{i+1}\theta, \dots, ord_p\theta, k_{i1}, \dots, k_{im_i}, k_{11}, \dots, k_{pm_p})$ is less than $(ord_i\theta', ord\, \theta', ord_1\theta', \dots, ord_{i-1}\theta', ord_{i+1}\theta', \dots, ord_p\theta', l_{i1}, \dots, l_{im_i}, l_{11}, \dots, l_{pm_p})$ with respect to the lexicographic order on $\mathbf{N}^{m+p+1}$.

Clearly, Θ is well-ordered with respect to each $<_i$.

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Let E be a free left $\mathcal{D}$-module with free generators $e_1, \dots, e_n$. Elements θe_j ($\theta \in \Theta$, $1 \leq j \leq n$) are called *terms* while the elements of the semigroup Θ are called *monomials*. The set of all terms is denoted by Θe; it generates E as a vector space over the field K.
The order of a term θe_j with respect to Δ_i ($1 \leq i \leq p$) is defined as $ord_i \theta$.
A term $u' = \theta' e_j$ is said to be a *multiple* of $u = \theta e_k$ (we write $u|u'$) if $j = k$ and θ' is a multiple of θ in Θ.
The least common multiple of two terms $u = \theta_1 e_i$ and $v = \theta_2 e_j$ is defined as follows:

$$lcm(u, v) = \begin{cases} 0, & \text{if} \quad i \neq j, \\ lcm(\theta_1, \theta_2) e_i & \text{if} \quad i = j. \end{cases}$$

We consider p orderings of Θe corresponding to the introduced orderings of Θ (we denote them by the same symbols): $\theta e_j <_i \theta' e_k$ if and only if $\theta <_i \theta'$ in Θ or $\theta = \theta'$ and $j < k$.

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Since the set Θe is a basis of the vector K-space E, every element $f \in E$ has a unique (up to the order of the terms in the sum) representation in the form

$$f = a_1\theta_1 e_{i_1} + \cdots + a_l\theta_l e_{i_l} \qquad (3)$$

where $\theta_1 e_{i_1}, \dots, \theta_l e_{i_l}$ are distinct elements of Θe, $a_1, \dots, a_l$ are non-zero elements of K, and $1 \leq i_1, \dots, i_l \leq n$.
Let f be an element of the $\mathcal{D}$-module E written in the form (3) and let $\theta_\nu e_{i_\nu}$ ($1 \leq \nu \leq p$) be the greatest term of the set $\{\theta_1 e_{i_1}, \dots, \theta_l e_{i_l}\}$ with respect to the order $<_j$ ($1 \leq j \leq p$). Then the term $\theta_\nu e_{i_\nu}$ is called the ***j*-leader** of the element f; it is denoted by $u_f^{(j)}$.
(It is possible that $u_f^{(j)} = u_f^{(j')}$ for $j \neq j'$.)
The number $ord_i u_f^{(i)}$ is called the ith order of f and denoted by $ord_i f$ ($i = 1, \dots, p$). The coefficient of $u_f^{(j)}$ in f is said to be the j-leading coefficient of f; it is denoted by $lc_j(f)$.

Let $f, g \in E$ and let $k, i_1, \dots, i_r$ be distinct elements of the set $\{1, \dots, p\}$. Then f is said to be $(<_k, <_{i_1}, \cdots <_{i_r})$-**reduced** with respect to g if f does not contain any multiple $\theta u_g^{(k)}$ $(\theta \in \Theta)$ such that $ord_{i_\nu}(\theta u_g^{(i_\nu)}) \leq ord_{i_\nu} u_f^{(i_\nu)}$ $(\nu = 1, \dots, r)$.
An element $f \in E$ is said to be $(<_k, <_{i_1}, \cdots <_{i_r})$-reduced with respect to a set $G \subseteq E$, if f is $(<_k, <_{i_1}, \cdots <_{i_r})$-reduced with respect to every element of G.

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Let us consider $p-1$ new symbols $z_1, \dots, z_{p-1}$ and the free commutative semigroup Γ of all power products $\gamma = \delta_1^{k_1} \dots \delta_m^{k_m} z_1^{l_1} \dots z_{p-1}^{l_{p-1}}$ with non-negative integer exponents. Let $\Gamma e = \{\gamma e_j | \gamma \in \Gamma, 1 \leq j \leq n\} = \Gamma \times \{e_1, \dots, e_n\}$. For any element $f \in E$, let $d_i(f) = ord_i u_f^{(i)} - ord_i u_f^{(1)}$ $(2 \leq i \leq p)$ and let $\rho : E \to \Gamma e$ be defined by $\rho(f) = z_1^{d_2(f)} \dots z_{p-1}^{d_{p(f)}} u_f^{(1)}$.
Let N be a $\mathcal{D}$-submodule of E. A finite set $G = \{g_1, \dots, g_r\} \subseteq N$ is called a **Gröbner basis of N with respect to the orders** $<_1, \dots, <_p$ if for any $f \in N$, there exists $g_i \in G$ such that $\rho(g_i) | \rho(f)$ in Γe.

It is clear that every Gröbner basis of N with respect to the orders $<_1, \dots, <_p$ is a Gröbner basis of N in the usual sense. Therefore, every Gröbner basis of N with respect to the orders $<_1, \dots, <_p$ generates N as a left D-module.

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A set $\{g_1, \ldots, g_r\} \subseteq E$ is said to be a *Groebner basis with respect to the orders* $<_1, \ldots, <_p$ if G is a Gröbner basis of $N = \sum_{i=1}^{r} \mathcal{D} g_i$ with respect to $<_1, \ldots, <_p$.

Given $f, g, h \in E$, with $g \neq 0$, we say that the element f $(<_k, <_{i_1}, \cdots <_{i_l})$-**reduces** to h **modulo** g in one step and write $f \xrightarrow[<_k, <_{i_1}, \cdots <_{i_l}]{g} h$ if and only if f contains some term w with a coefficient a such that $u_g^{(k)} | w$,

$$h = f - a(\frac{w}{u_g^{(k)}}(lc_k(g)))^{-1}\frac{w}{u_g^{(k)}} g$$

and $ord_{i_\nu} \frac{w}{u_g^{(k)}} u_g^{(i_\nu)} \leq ord_{i_\nu} u_f^{(i_\nu)}$ $(1 \leq \nu \leq l)$.

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Let $f, h \in E$ and let $G = \{g_1, \ldots, g_r\}$ be a finite set of non-zero elements of E. We say that f $(<_k, <_{i_1}, \cdots <_{i_l})$-**reduces** to h **modulo** G and write $f \xrightarrow[<_k;<_{i_1},\cdots<_{i_l}]{G} h$ if and only if there exists a sequence of elements $g^{(1)}, g^{(2)}, \ldots g^{(q)} \in G$ and a sequence of elements $h_1, \ldots, h_{q-1} \in E$ such that

$$f \xrightarrow[<_k,<_{i_1},\cdots<_{i_l}]{g^{(1)}} h_1 \xrightarrow[<_k,<_{i_1},\cdots<_{i_l}]{g^{(2)}} \ldots \xrightarrow[<_k,<_{i_1},\cdots<_{i_l}]{g^{(q-1)}} h_{q-1} \xrightarrow[<_k,<_{i_1},\cdots<_{i_l}]{g^{(q)}} h.$$

Theorem 6.

With the above notation, let $G = \{g_1, \ldots, g_r\} \subseteq E$ be a Gröbner basis with respect to the orders $<_1, \ldots, <_p$ on Te. Then there exist elements $g \in E$ and $Q_1, \ldots, Q_r \in \mathcal{D}$ such that $f - g = \sum_{i=1}^{r} Q_i g_i$ and g is reduced with respect to the set G.

The process of reduction is described by the following algorithm (that can be used for the reduction with respect to any finite set of elements of the free $\mathcal{D}$-module E).

Algorithm 1 $(f, r, g_1, \dots, g_r;\, g)$
Input: $f \in E$, $r \in \mathbf{N}$, $G = \{g_1, \dots, g_r\} \subseteq E$ where all $g_i \neq 0$
Output: Element $g \in E$ and elements $Q_1, \dots, Q_r \in \mathcal{D}$ such that $g = f - (Q_1 g_1 + \dots + Q_r g_r)$ and g is reduced with respect to G
Begin: $Q_1 := 0, \dots, Q_r := 0, g := f$
While there exist i, $1 \leq i \leq r$, and a term w, that appears in g with a nonzero coefficient $c(w)$, such that $u_{g_i}^{(1)} | w$ and $ord_j(\frac{w}{u_{g_i}^{(1)}} u_{g_i}^{(j)}) \leq ord_j u_g^{(j)}$ for $j = 2, \dots, p$ **do**
$z :=$ the greatest (w.r.t. $<_1$) such a term w.
$k := \min\{i \mid u_{g_i}^{(1)}$ is the greatest (w.r.t. $<_1$) 1-leader of an element $g_i \in G$ with $u_{g_i}^{(1)} | z$ and $ord_j(\frac{z}{u_{g_i}^{(1)}} u_{g_i}) \leq ord_j u_g^{(j)}$ $(j = 2, \dots, p)\}$.

$$Q_k := Q_k + c(z)\left(\frac{z}{u_{g_k}^{(1)}}(lc_1(g_k)\right)^{-1} \frac{z}{u_{g_k}^{(1)}}\, g_k$$

$$g := g - c(z)\left(\frac{z}{u_{g_k}^{(1)}}(lc_1(g_k)\right)^{-1} \frac{z}{u_{g_k}^{(1)}}\, g_k$$

End

Proposition

Let $G = \{g_1, \ldots, g_r\}$ be a Gröbner basis of a $\mathcal{D}$-submodule N of E with respect to the orders $<_1, \ldots, <_p$. Then

(i) *$f \in N$ if and only if $f \xrightarrow[<_1,<_2,\cdots<_p]{G} 0$.*

(ii) *If $f \in N$ and f is $(<_1, <_2, \cdots <_p)$-reduced with respect to G, then $f = 0$.*

Let f and g be two elements in the free $\mathcal{D}$-module E and let $k \in \{1, \ldots, p\}$. Then the element

$$S_k(f,g) = \left(\frac{lcm(u_f^{(k)}, u_g^{(k)})}{u_f^{(k)}} (lc_k(f)) \right)^{-1} \frac{lcm(u_f^{(k)}, u_g^{(k)})}{u_f^{(k)}} f -$$

$$\left(\frac{lcm(u_f^{(k)}, u_g^{(k)})}{u_g^{(k)}} (lc_k(g)) \right)^{-1} \frac{lcm(u_f^{(k)}, u_g^{(k)})}{u_g^{(k)}} g$$

is called the k**th S-polynomial** of f and g.

Proposition

Let $f, g_1, \ldots, g_r \in E$ $(r \geq 1)$ *and let* $f = \sum_{i=1}^{r} c_i \omega_i g_i$ *where* $\omega_i \in \Theta$, $c_i \in K$ $(1 \leq i \leq r)$. *Let* $k \in \{1, \ldots, p\}$ *and for any* $\nu, j \in \{1, \ldots, r\}$, *let* $u_{\nu j}^{(k)} = lcm(u_{g_\nu}^{(k)}, u_{g_j}^{(k)})$. *Furthermore, suppose that* $\omega_1 u_{g_1}^{(k)} = \cdots = \omega_r u_{g_r}^{(k)} = u$ *for some* $k \in \{1, \ldots, p\}$, $u_f^{(k)} <_k u$ *and there is a nonempty set* $I \subseteq \{1, \ldots, p\} \setminus \{k\}$ *such that* $\omega_i u_{g_i}^{(l)} \leq_l u_f^{(l)}$ *for all* $i \in \{1, \ldots, r\}$, $l \in I$. *Then there exist elements* $c_{\nu j} \in K$ $(1 \leq \nu \leq s, 1 \leq j \leq t)$ *such that*

$$f = \sum_{\nu=1}^{s} \sum_{j=1}^{t} c_{\nu j} \theta_{\nu j} S_k(g_\nu, g_j)$$

where $\theta_{\nu j} = \frac{u}{u_{\nu j}^{(k)}}$ *and* $\theta_{\nu j} u_{S_k(g_\nu, g_j)}^{(k)} <_k u$, $\theta_{\nu j} u_{S_k(g_\nu, g_j)}^{(l)} \leq_l u_f^{(l)}$ $(1 \leq \nu \leq s$, $1 \leq j \leq t$, $l \in I)$.

Theorem 7.

With the above notation, let $G = \{g_1, \ldots, g_r\}$ be a Gröbner basis of a $\mathcal{D}$-submodule N of E with respect to each of the following sequences of orders: $<_p$; $<_{p-1}, <_p$; ...; $<_{k+1}, \ldots, <_p$ $(1 \leq k \leq p-1)$. Furthermore, suppose that

$$S_k(g_i, g_j) \xrightarrow[<_k, <_{k+1}, \cdots <_p]{G} 0 \text{ for any } g_i, g_j \in G.$$

Then G is a Gröbner basis of N with respect to $<_k, <_{k+1}, \ldots, <_p$.

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Theorem 8.

Let $\mathcal{D}$ be the ring of Δ-operators over a Δ-field K, M a left $\mathcal{D}$-module generated by a finite set $\{f_1, \dots, f_n\}$, and E a free left $\mathcal{D}$-module with free generators $e_1, \dots, e_n$. Let $\pi : E \longrightarrow M$ be the natural $\mathcal{D}$-epimorphism ($\pi(e_i) = f_i$ for $i = 1, \dots, m$), $N = Ker\,\pi$, and $G = \{g_1, \dots, g_d\}$ a Gröbner basis of N with respect to $<_1, \dots, <_p$. Furthermore, for any $(r_1, \dots, r_p) \in \mathbf{Z}^p$, let $M_{r_1 \dots r_p} = \sum_{i=1}^{n} \mathcal{D}_{r_1 \dots r_p} f_i$ and let
$V_{r_1 \dots r_p} = \{u \in \Theta e | ord_i u \leq r_i$ for $i = 1, \dots, p$, and $u \neq \theta u_g^{(1)}$ for any $\theta \in \Theta$, $g \in G\}$,
$W_{r_1 \dots r_p} = \{u \in \Theta e \setminus V_{r_1 \dots r_p} | ord_i u \leq r_i$ for $i = 1, \dots, p$ and for every $\theta \in \Theta, g \in G$ such that $u = \theta u_g^{(1)}$, there exists $i \in \{2, \dots, p\}$ such that $ord_i \theta u_g^{(i)} > r_i\}$, and
$U_{r_1 \dots r_p} = V_{r_1 \dots r_p} \bigcup W_{r_1 \dots r_p}$.
Then for any $(r_1, \dots, r_p) \in \mathbf{N}^p$, the set $\pi(U_{r_1 \dots r_p})$ is a basis of the vector K-space $M_{r_1 \dots r_p}$.

Theorem 2.2.5 of [Kondrateva, M.V.; Levin, A. B.; Mikhalev, A. V.; Pankratev, E. V. Differential and Difference Dimension Polynomials. *Kluwer Acad. Publ.*, 1998], establishes the existence of a polynomial $f(t_1, \ldots, t_p) \in \mathbf{Q}[t_1, \ldots, t_p]$ such that $f(r_1, \ldots, r_p) = \mathit{Card}\ V_{r_1 \ldots r_p}$ for all sufficiently large $(r_1, \ldots, r_p) \in \mathbf{N}^p$. Using the combinatorial method of inclusion and exclusion, $\mathit{Card}\ V_{r_1 \ldots r_p}$ can be obtained as an alternative sum of the polynomials of the form

$$\phi_{j;k_1,\ldots,k_\mu}(t_1, \ldots, t_p) = \binom{t_1 + m_1 - b_{1j}}{m_1} \ldots \binom{t_{k_1-1} + m_{k_1-1} - b_{k_1-1,j}}{m_{k_1-1}}$$

$$\left[\binom{t_{k_1} + m_{k_1} - a_{k_1,j}}{m_{k_1}} - \binom{t_{k_1} + n_{k_1} - b_{k_1,j}}{n_{k_1}}\right] \binom{t_{k_1+1} + m_{k_1+1} - b_{k_1+1,j}}{m_{k_1+1}}$$
$$\ldots \left[\binom{t_{k_\mu} + m_{k_\mu} - a_{k_\mu,j}}{m_{k_\mu}} - \binom{t_{k_\mu} + m_{k_\mu} - b_{k_\mu,j}}{m_{k_\mu}}\right]$$

Thus, Theorem 8 provides a method of computation of the multivariate dimension polynomial associated with a system of Δ-equations and therefore the strength of such a system. In what follows we consider another method of computation of differential and difference dimension polynomials and present examples of computation of the strength polynomials for systems of PDEs and their difference approximations via different difference schemes. In particular, we will be able to compare different difference schemes from the point of view of their strength.

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Relative Reduction, Characterization of Relative Gröbner Bases and Computation of Multivariate Differential and Difference Dimension Polynomials

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Motivation

Relative Gröbner bases are well-established as one of the main tools for the algorithmic computation of bivariate (difference-differential) dimension polynomials.

However not much is known about the computational complexity of the algorithm provided by Meng Zhou and Franz Winkler for computing them.

At DEAM2 the first speaker pointed out a characterization of Gröbner bases with respect to several orderings as introduced by him in 2007 which can be extended to relative Gröbner bases and leads to some complexity considerations.

Milestones

1965 Buchberger introduces the concept of Gröbner bases

2007 The first speaker introduces his notion of Gröbner bases with respect to several orderings

2008 Zhou and Winkler introduce relative Gröbner bases for modules of difference-differential operators and give an algorithm for computing them

Outline

Preliminaries

Symmetry and Characterization of Relative Gröbner Bases

Complexity Considerations for Relative Gröbner Bases

Computation of Differential and Difference Dimension Polynomials

Admissible Orders

Let K be a field of characteristic 0 and let $\Delta = \{\delta_1, \ldots, \delta_m\}$ be a set of derivations or endomorphisms on K. Let $E = \{e_1, \ldots, e_q\}$ be a finite set. By $[\Delta]$ we denote the set of terms in the indeterminates $\delta_1, \ldots, \delta_m$.

Definition

Let $\prec$ be a total order on $[\Delta]E$ such that for all $k_1, k_2, l \in \mathbb{N}^n, e, \tilde{e} \in E$ we have

(i) $e \preceq \delta^{k_1} e$, and

(ii) $\delta^{k_1} e \prec \delta^{k_2} \tilde{e} \Longrightarrow \delta^{k_1+l} e \prec \delta^{k_2+l} \tilde{e}$.

Then $\prec$ is called an admissible order.

Relative Reduction

Definition

Let $f, g \in K[\Delta]E \setminus \{0\}$ and let $\prec, \prec'$ be admissible orders. If there exists a term $\lambda \in [\Delta]$ such that

(i) $\mathrm{lt}_{\prec}(\lambda g) = \mathrm{lt}_{\prec}(f)$, and

(ii) $\mathrm{lt}_{\prec'}(\lambda g) \preceq' \mathrm{lt}_{\prec'}(f)$

then we say f is $\prec$-reducible modulo g relative to $\prec'$.

Relative Reduction Algorithm

Algorithm 1 `Reduction`

IN: $0 \neq f \in K[\Delta]E$, finite $G \subseteq K[\Delta]E \setminus \{0\}$, and admissible orders $\prec, \prec'$,

OUT: $r \in K[\Delta]E$ such that f is $\prec$-reducible to r modulo G relative to $\prec'$ and there exist no $\lambda \in [\Delta], g \in G$ with

(i) $\mathrm{lt}_\prec(\lambda g) = \mathrm{lt}_\prec(r)$, and

(ii) $\mathrm{lt}_{\prec'}(\lambda g) \preceq' \mathrm{lt}_{\prec'}(f)$.

1: $r := f$

2: **while** there exist $g \in G$ and $\lambda \in K[\Delta]$ such that (i) and (ii) **do**

3: $\quad r := r - \lambda \frac{\mathrm{lc}_\prec(r)}{\mathrm{lc}_\prec(g)} g$

4: **end while**

5: **return** r

Relative Gröbner Bases

Definition
Let $M \subseteq K[\Delta]E$ be a submodule, $G \subseteq M \setminus \{0\}$ finite and let $\prec, \prec'$ be admissible orders such that every $0 \neq f \in M$ is $\prec$-reducible to 0 modulo G. relative to $\prec'$.

Then G is called $\prec$-Gröbner basis of M relative to $\prec'$.

If no confusion is possible we say that G is a relative Gröbner basis.

S-Polynomials

Definition

Let $\prec$ be an admissible order on $[\Delta]E$, $g_1, g_2 \in K[\Delta]E \setminus \{0\}$, $\lambda_1, \lambda_1 \in [\Delta], e_1, e_2 \in E$ such that $\mathrm{lt}_\prec(g_1) = \lambda_1 e_1$ and $\mathrm{lt}_\prec(g_2) = \lambda_2 e_2$. The least common multiple $\mathrm{lcm}(\mathrm{lt}_\prec(g_1), \mathrm{lt}_\prec(g_2))$ of $\mathrm{lt}_\prec(g_1)$ and $\mathrm{lt}_\prec(g_2)$ is defined by

$$\mathrm{lcm}(\mathrm{lt}_\prec(g_1), \mathrm{lt}_\prec(g_2)) := \begin{cases} \mathrm{lcm}(\lambda_1, \lambda_2) e_1 & \text{if } e_1 = e_2, \\ 0 & \text{if } e_1 \neq e_2. \end{cases}$$

Then the *S-polynomial* $S_\prec(g_1, g_2)$ of g_1 and g_2 is given by

$$S_\prec(g_1, g_2) := \frac{\mathrm{lcm}(\mathrm{lt}_\prec(g_1), \mathrm{lt}_\prec(g_2))}{\mathrm{lt}_\prec(g_1)} \frac{g_1}{\mathrm{lc}_\prec(g_1)} - \frac{\mathrm{lcm}(\mathrm{lt}_\prec(g_1), \mathrm{lt}_\prec(g_2))}{\mathrm{lt}_\prec(g_2)} \frac{g_2}{\mathrm{lc}_\prec(g_2)}.$$

S-Polynomial Criterion

Zhou and Winkler provide the following theorem from which they deduce an algorithm for computing relative Gröbner bases.

Theorem

Let $\prec, \prec'$ be admissible orders, $G = \{g_1, \ldots, g_r\} \subseteq K[\Delta]E \setminus \{0\}$ a finite $\prec'$-Gröbner basis of $M = \langle G \rangle$ such that for any $s, t \in \{1, \ldots, r\}$ the S-polynomial $S_\prec(g_s, g_t)$ is $\prec$-reducible to 0 modulo G relative to $\prec'$.Then G is a $\prec$-Gröbner basis of M relative to $\prec'$.

Buchberger's Algorithm

Algorithm 2 `Buchberger's_algorithm`

IN: $G \subseteq K[\Delta]E \setminus \{0\}$ a finite Gröbner basis w.r.t. $\prec'$, a total order $\prec$ on $[\Delta]$,

OUT: $\tilde{G} \subseteq K[\Delta]E \setminus \{0\}$ being a $\prec$-Gröbner basis of the left $K[\Delta]$-module generated by G relative to $\prec'$.

1: **while** there exist $g_1, g_2 \in \tilde{G}$ with $\mathtt{Reduction}(S_\prec(g_1, g_2), \tilde{G}, \prec, \prec') \neq 0$ **do**

2: $\quad \tilde{G} := \tilde{G} \cup \{\mathtt{Reduction}(S_\prec(g_1, g_2), \tilde{G}, \prec, \prec')\}$

3: **end while**

4: **return** $\tilde{G}$

Relative Gröbner Bases

The following lemma is due to Zhou and Winkler.

Lemma

Let $M \trianglelefteq K[\Delta]E$ be a submodule and $G \subseteq M$ a finite basis for M. Let $\prec$ and $\prec'$ be two admissible orders. Then the following are equivalent:

(i) *G is a $\prec$-Gröbner basis of M relative to $\prec'$,*

(ii) *$0 \neq f \in M$ iff f is $\prec$-reducible to 0 modulo G relative to $\prec'$,*

(iii) *$f \in M$ is not $\prec$-reducible modulo G relative to $\prec'$ iff $f = 0$.*

Symmetry of Relative Gröbner Bases

Let G be a $\prec$-Gröbner basis of M relative to $\prec'$ and let $0 \neq f_0 \in M$. Then by the previous lemma f_0 is $\prec$-reducible to 0 modulo G relative to $\prec'$, i.e., there exist $s \in \mathbb{N}$ and $f_1, \ldots, f_{s-1} \in K[\Delta]E \setminus \{0\},\ f_s = 0$ such that for all $i = 1, \ldots, s$ the polynomial f_{i-1} is $\prec$-reducible to f_i modulo G relative to $\prec'$ in one reduction step and there exists at least one $i_0 \in \{1, \ldots, s-1\}$ and $j \in \{1, \ldots, r\}$ with

(i) $\operatorname{lt}_{\prec'}(f_0) = \operatorname{lt}_{\prec'}(f_{i_0}) = \operatorname{lt}_{\prec'}\left(\frac{\operatorname{lt}_{\prec}(f_{i_0})}{\operatorname{lt}_{\prec}(g_j)} g_j\right)$, and

(ii) $\operatorname{lt}_{\prec}\left(\frac{\operatorname{lt}_{\prec}(f_{i_0})}{\operatorname{lt}_{\prec}(g_j)} g_j\right) = \operatorname{lt}_{\prec}(f_{i_0}) \preceq \operatorname{lt}_{\prec}(f_0)$.

Hence, f_0 is $\prec'$-reducible modulo G relative to $\prec$ and we conclude that G is a $\prec'$-Gröbner basis of M relative to $\prec$.

Relative Gröbner Bases and Sum Representations

Lemma

Let $M \trianglelefteq K[\Delta]E$ be a submodule and let $G = \{g_1, \ldots, g_r\} \subseteq M \setminus \{0\}$ be finite. Let $\prec$ and $\prec'$ be two admissible term orders. The following are equivalent:

(i) *G is a $\prec$-Gröbner basis of M relative to $\prec'$.*

(ii) *For every $0 \neq f \in M$ there exist $h_1, \ldots, h_r \in K[\Delta]$ such that*

  (a) $f = \sum_{i=1}^{r} h_i g_i$,

  (b) *for $i = 1, \ldots, r$ with $h_i \neq 0$ we have $\mathrm{lt}_{\prec}(h_i g_i) \preceq \mathrm{lt}_{\prec}(f)$ and $\mathrm{lt}_{\prec'}(h_i g_i) \preceq' \mathrm{lt}_{\prec'}(f)$.*

Change of Term Orders

The following lemma extends a well-known result from Gröbner bases to relative Gröbner bases.

Lemma

Let $\prec_1, \prec'_1, \prec_2, \prec'_2$ be admissible orders and let $G = \{g_1, \ldots, g_r\} \subseteq K[\Delta]E$ be such that for all $i \in \{1, \ldots, r\}$ we have

$$\begin{aligned} \mathrm{lt}_{\prec_1}(g_i) &= \mathrm{lt}_{\prec_2}(g_i), \\ \mathrm{lt}_{\prec'_1}(g_i) &= \mathrm{lt}_{\prec'_2}(g_i). \end{aligned}$$

TFAE

(i) *G is a $\prec_1$-Gröbner basis relative to $\prec'_1$,*

(ii) *G is a $\prec_2$-Gröbner basis relative to $\prec'_2$.*

Towards a Characterization of Relative Gröbner Bases

Consider the case $E = \{1\}$. It is well known that for the admissible order $\prec'$ there exist $n \in \{1, \ldots, m\}$ and $U \in \mathbb{R}^{m \times n}$ such that

$$\begin{array}{rcl} \alpha : ([\Delta], \prec') & \to & (\mathbb{R}^n, <_{\text{lex}}) \\ \delta^k & \mapsto & kU \end{array}$$

is an injective homomorphism. Note that for $\lambda, \mu \in [\Delta]$ we have

$$\alpha(\lambda\mu) = \alpha(\lambda) + \alpha(\mu).$$

Towards a Characterization of Relative Gröbner Bases

Let us consider a new symbol z and let

$$\begin{aligned} [\Delta, z]_U &:= \{\delta^k z^{kU} \mid k \in \mathbb{N}^m\}, \\ \overline{[\Delta, z]_U} &:= \{\delta^k z^l \mid k \in \mathbb{N}^m,\ l \in \mathbb{Z}^m U,\ 0 \leq_{\text{lex}} l - kU\}. \end{aligned}$$

Towards a Characterization of Relative Gröbner Bases

For $l_1, l_2 \in \mathbb{Z}^n U, k \in \mathbb{N}^n$ define $z^{l_1} z^{l_2} = z^{l_1+l_2}$ and $z^{l_1}\delta^k = \delta^k z^{l_1}$. Then $[\Delta, z]_U$ and $\overline{[\Delta, z]_U}$ can be considered as multiplicative monoids.

Define $\rho : K[\Delta] \to \overline{[\Delta, z]_U}$ by

$$\rho(f) := \mathrm{lt}_{<}(f) z^{\alpha(\mathrm{lt}_{<'}(f))}.$$

Characterization of Relative Gröbner Bases for Ideals

Lemma

Let $G = \{g_1, \ldots, g_r\} \subseteq K[\Delta]$ be finite, $I := {}_{K[\Delta]}\langle G \rangle$ and let $\prec, \prec'$ be two admissible orders on $[\Delta]$. TFAE

(i) *G is a $\prec$-Gröbner basis of I relative to $\prec'$,*

(ii) $\rho(I) \subseteq \overline{[\Delta, z]_U}\rho(G)$,

(iii) $\overline{[\Delta, z]_U}\rho(I) = \overline{[\Delta, z]_U}\rho(G)$,

(iv) ${}_{K\overline{[\Delta, z]_U}}\langle \rho(I) \rangle = {}_{K\overline{[\Delta, z]_U}}\langle \rho(G) \rangle$.

Complexity Considerations

It is not obvious that for every ideal $I \subseteq K[\Delta]$ there exists a finite basis of $_{\overline{K[\Delta,z]_U}}\langle \rho(I) \rangle$. Let $\Delta = \{\delta_1, \delta_2, \delta_3\}$ consist of trivial derivations on K.

Consider the ideal

$$I := \langle\ \delta_1^3\delta_2^2 + \delta_1^4\delta_2\ ,\ \delta_1^3\delta_2\delta_3 + \delta_2^2\delta_3^2\ \rangle$$

and the two admissible orders

$$\prec := \mathrm{lex}(\delta_3 > \delta_1 > \delta_2)\ , \qquad \prec' := \mathrm{grevlex}(\delta_3, \delta_2, \delta_1).$$

Using any major CAS it is easy to see that

$$\{f_0 := \delta_1^3\delta_2^2 + \delta_1^4\delta_2\ ,\ f_1 := \delta_2^3\delta_3^2 + \delta_1\delta_2^2\delta_3^2\ ,\ f_2 := \delta_1^3\delta_2\delta_3 + \delta_2^2\delta_3^2\}$$

is a $\prec'$-Gröbner basis of I.

Complexity Considerations

For $i \in \mathbb{N}$ let

$$g_i := \delta_1^{3+4i}\delta_2\delta_3 + \delta_2^{2+4i}\delta_3^2$$

and note that $g_0 = f_2$.

For every $i \in \mathbb{N}$ it turns out that $S_{\prec}(f_0, g_i)$ is $\prec$-reducible to g_{i+1} modulo $f_0, f_1, g_0, \ldots, g_i$ relative to $\prec'$ and that g_{i+1} is not $\prec$-reducible modulo $f_0, f_1, g_0, \ldots, g_i$ relative to $\prec'$.

Hence the algorithm for computing relative Gröbner bases will not terminate for this example.

Existence of Relative Gröbner Bases

In fact, one can show that there exists no $p \in I$ such that infinitely many g_i are $\prec$-reducible modulo p relative to $\prec'$, i.e., the ideal I cannot possess a $\prec$-Gröbner basis relative to $\prec'$.

Dimension Polynomials and Gröbner Bases w.r.t. Several Orderings

Theorem (Levin 2007)

Let M be a difference K-vector space generated (as a left $K[\Delta]$-module) by elements $m_1, \dots, m_q$, F a free $K[\Delta]$-module with set of free generators $E = \{e_1, \dots, e_q\}$, $\pi : F \to M$ the difference epimorphism $\forall_{1 \le i \le q} e_i \mapsto m_i$ and $N := \ker(\pi)$. Let $G \subseteq K[\Delta]E$ be a Gröbner basis of N with respect to $\prec_1, \dots, \prec_p$ as defined by Alexander this morning. For $r_1, \dots, r_p \in \mathbb{Z}$ let

$$
\begin{aligned}
M_{r_1,\dots,r_p} &:= \{\lambda m \in [\Delta]\{m_1, \dots, m_q\} \mid \operatorname{ord}_i \lambda \le r_i, (i = 1, \dots, p)\} \text{ and} \\
U_{r_1,\dots,r_p} &:= \{\lambda e \in [\Delta]E \mid \operatorname{ord}_i \lambda \le r_i \ (i = 1, \dots, p), \text{ for all } \mu \in [\Delta], g \in G \\
&\quad \text{ such that } \operatorname{lt}_{\prec_1}(\mu g) = \lambda e \text{ there exists } j \in \{2, \dots, p\} \text{ with} \\
&\quad \operatorname{ord}_j \operatorname{lt}_{\prec_j}(\mu g) > r_j\}.
\end{aligned}
$$

Then for any $r_1, \dots, r_p \in \mathbb{N}$ the set $\pi(U_{r_1,\dots,r_p})$ is a basis for the K-vector space $M_{r_1,\dots,r_p}$

Example: Diffusion Equation in 1-Space

The diffusion equation in 1-space for a constant collective diffusion coefficient a and unknown function $u(x, t)$ describing the density of the diffusing material at given position x and time t is given by

$$\frac{\partial u(x,t)}{\partial t} = a \frac{\partial^2 u(x,t)}{\partial x^2}. \tag{1}$$

Example: Differential Dimension Polynomial for Diffusion Equation in 1-space

Let K be a differential field with basic set $\Delta = \{\delta_x = \frac{\partial}{\partial x}, \delta_t = \frac{\partial}{\partial t}\}$ containing a and let M be a differential K-vector space generated as $K[\Delta]$-module by one generator m satisfying the defining equation

$$\delta_t m = a\delta_x^2 m.$$

Then M is isomorphic to the factor module of a free $K[\Delta]$ module with free generator e by its submodule N generated by

$$G := \{\delta_t e - a\delta_x^2 e\}.$$

Example cont.: Differential Dimension Polynomial for Diffusion Equation in 1-space

Since G consists of only one element there are no S-polynomials. Therefore G is already a Gröbner basis of N for any admissible order on $[\Delta]e$. Let the admissible order $\prec$ on $[\Delta]e$ be given by

$$\delta_x^{k_x}\delta_t^{k_t}e \prec \delta_x^{l_x}\delta_t^{l_t}e :\Longleftrightarrow (k_x + k_t, k_x, k_t) <_{\text{lex}} (l_x + l_t, l_x, l_t).$$

Example cont.: Differential Dimension Polynomial for Diffusion Equation in 1-space

Then for all $2 \leq r \in \mathbb{N}$ we have

$$\begin{aligned} U_r &= \{\delta_x^{k_x}\delta_t^{k_t} e \mid k_x + k_t \leq r,\ \delta_x^{k_x}\delta_t^{k_t} e \text{ is irreducible modulo } \delta_t e - a\delta_x^2\} \\ &= \{e, \delta_t e, \ldots, \delta_t^r e, \delta_x e, \delta_x\delta_t e, \ldots, \delta_x\delta_t^{r-1} e\}, \end{aligned}$$

and therefore $|U_r| = 2r + 1$. Hence, the differential dimension polynomial associated with the diffusion equation in one spatial dimension for a constant collective diffusion coefficient is given by $\phi(r) = 2r + 1$.

Example: Difference Dimension Polynomial for Forward Difference Scheme Associated with Diffusion Equation

In order to obtain a forward difference scheme for the diffusion equation (1) every occurence of $\frac{\partial u(x,t)}{\partial x}$ and $\frac{\partial u(x,t)}{\partial t}$ is replaced by $u(x+1,t) - u(x,t)$ and $u(x,t+1) - u(x,t)$, respectively.

We obtain

$$u(x,t+1) - u(x,t) = a(u(x+2,t) - 2u(x+1,t) + u(x,t)). \qquad (2)$$

Example cont.: Difference Dimension Polynomial for Forward Difference Scheme for Diffusion Equation

Let K be an inversive difference field with basic set $\Delta = \{\delta_x : x \mapsto x+1, \delta_t : t \mapsto t+1\}$ containing a and let M be an inversive difference K-vector space generated as a left $K[\Delta^*]$-module by one generator m satisfying the defining equation

$$\delta_t m - m = a(\delta_x^2 m - 2\delta_x m + m).$$

Then M is isomorphic to the factor module of a free $K[\Delta^*]$-module with free generator e by its submodule N generated by

$$G := \{\delta_t e - a\delta_x^2 e + 2a\delta_x e - (1+a)e\}.$$

Example cont.: Difference Dimension Polynomial for Forward Difference Scheme for Diffusion Equation

In order to use the algorithms provided above consider K a difference field with basic set

$$\Sigma = \{\alpha_x : x \mapsto x+1, \alpha_t : t \mapsto t+1, \beta_x : x \mapsto x-1, \beta_t : t \mapsto t-1\}.$$

Let $\tilde{G} := \{g_1 := a\alpha_x^2 e - 2a\alpha_x e + (1+a)e - \alpha_t e, g_2 := \alpha_x\beta_x e - e,$ $g_3 := \alpha_t\beta_t e - e\}$ and $I =_{K[\Sigma]} \langle \tilde{G} \rangle$. Then $K[\Sigma]e/I$ is isomorphic to $K[\Delta^*]e/N$ via the isomorphism

$$\alpha_x^{a_x}\alpha_t^{a_t}\beta_x^{b_x}\beta_t^{b_t} e \mapsto \delta_x^{a_x-b_x}\delta_t^{a_t-b_t} e.$$

Example cont.: Difference Dimension Polynomial for Forward Difference Scheme for Diffusion Equation

We fix an admissible order $\prec$ on $[\Sigma]e$ by

$$\alpha_x^{a_x}\alpha_t^{a_t}\beta_x^{b_x}\beta_t^{b_t}e \prec \alpha_x^{c_x}\alpha_t^{c_t}\beta_x^{d_x}\beta_t^{d_t}e :\Longleftrightarrow$$
$$(a_x + a_t + b_x + b_t, a_x, a_t, b_x, b_t) <_{\text{lex}} (c_x + c_t + d_x + d_t, c_x, c_t, d_x, d_t)$$

and compute a Gröbner basis of I with respect to $\prec$ using the algorithm provided above.

Example cont.: Difference Dimension Polynomial for Forward Difference Scheme for Diffusion Equation

A Gröbner basis for I is given by

$$
\begin{aligned}
\{\, g_1 &= a\alpha_x^2 e - 2a\alpha_x e + (1+a)e - \alpha_t e, \\
g_2 &= \alpha_x\beta_x e - e, \\
g_3 &= \alpha_t\beta_t e - e, \\
g_4 &= -\alpha_t\beta_x e + (1+a)\beta_x e + a\alpha_x e - 2ae, \\
g_5 &= \beta_x e - a\alpha_x\beta_t e - (1+a)\beta_x\beta_t e + 2a\beta_t e, \\
g_6 &= a\beta_t e - \beta_x^2 e + (1+a)\beta_x^2\beta_t e - 2a\beta_x\beta_t e \,\}.
\end{aligned}
$$

Example cont.: Difference Dimension Polynomial for Forward Difference Scheme for Diffusion Equation

For all $r \in \mathbb{N}$ sufficiently large we obtain

$$\begin{aligned} &|\{\lambda e \in [\Sigma]E \mid \text{ord}\, \lambda \le r, \text{ there exist no} \\ &\mu \in [\Sigma], g \in G \text{ such that } \text{lt}(\mu g) = \lambda e\}| \quad = \quad 5r. \end{aligned}$$

Hence, for all r sufficiently large the difference dimension polynomial ϕ associated with the forward difference scheme (2) is given by

$$\phi(r) = 5r.$$

Example cont.: Difference Dimension Polynomial for Space-Symmetric Difference Scheme for Diffusion Equation

A space-symmetric difference scheme for the diffusion equation (1) is obtained by replacing every occurence of $\frac{\partial u(x,t)}{\partial t}$ and $\frac{\partial^2 u(x,t)}{\partial x^2}$ by $u(x, t+1) - u(x, t)$ and $u(x+1, t) - 2u(x, t) + u(x-1, t)$, respectively.

We obtain

$$u(x, t+1) - u(x, t) = a(u(x+1, t) - 2u(x, t) + u(x-1, t)). \qquad (3)$$

The associated difference dimension polynomial is given by

$$\phi(r) = 4r.$$

Example cont.: Bivariate Differential Dimension Polynomial for Diffusion Equation in 1-space

Let K be a differential field with basic set $\Delta = \{\delta_x = \frac{\partial}{\partial x}, \delta_t = \frac{\partial}{\partial t}\}$ containing a and let M be a differential K-vector space generated as $K[\Delta]$-module by one generator m satisfying the defining equation

$$\delta_t m = a\delta_x^2 m.$$

Then M is isomorphic to the factor module of a free $K[\Delta]$ module with free generator e by its submodule N generated by

$$G := \{\delta_t e - a\delta_x^2 e\}.$$

Example cont.: Bivariate Differential Dimension Polynomial for Diffusion Equation in 1-space

Since G consists of only one element there are no S-polynomials. Therefore G is already a Gröbner basis of N w.r.t any two admissible orders on $[\Delta]e$. Let the admissible orders $\prec, \prec'$ be given by

$$\begin{aligned} \delta_x^{k_x}\delta_t^{k_t}e \prec \delta_x^{l_x}\delta_t^{l_t}e \quad &:\Longleftrightarrow \quad (k_x+k_t, k_x, k_t) <_{\text{lex}} (l_x+l_t, l_x, l_t), \\ \delta_x^{k_x}\delta_t^{k_t}e \prec' \delta_x^{l_x}\delta_t^{l_t}e \quad &:\Longleftrightarrow \quad (k_x+k_t, k_t, k_x) <_{\text{lex}} (l_x+l_t, l_t, l_x). \end{aligned}$$

Example cont.: Bivariate Differential Dimension Polynomial for Diffusion Equation in 1-space

Then for all $r_1, r_2 \in \mathbb{N}$ sufficiently large we have

$$\begin{aligned} U_{r_1,r_2} &= \{\delta_x^{k_x}\delta_t^{k_t}e \mid k_x \leq r_1,\ k_t \leq r_2,\ \delta_x^{k_x}\delta_t^{k_t}e \text{ is not } \prec\text{-} \\ &\quad \text{reducible modulo } \delta_t e - a\delta_x^2 \text{ relative to } \prec'\} \\ &= \{e, \delta_t e, \ldots, \delta_t^{r_2}e, \delta_x e, \ldots, \delta_x\delta_t^{r_2}e, \\ &\quad \delta_x^2\delta_t^{r_2}e, \ldots, \delta_x^{r_1}\delta_t^{r_2}e\}, \end{aligned}$$

and therefore $|U_{r_1,r_2}| = 2(r_1+1) + r_1 - 1 = r_1 + 2r_2 + 1$. Hence, the bivariate differential dimension polynomial associated with the diffusion equation in one spatial dimension for a constant collective diffusion coefficient is given by $\phi(r_1, r_2) = r_1 + 2r_2 + 1$.

Example cont.: Bivariate Difference Dimension Polynomial for Forward Scheme (2)

Let K be an inversive difference field with basic set $\Delta = \{\delta_x : x \mapsto x+1, \delta_t : t \mapsto t+1\}$ containing a and let M be an inversive difference K-vector space generated as a left $K[\Delta^*]$-module by one generator m satisfying the defining equation

$$\delta_t m - m = a(\delta_x^2 m - 2\delta_x m + m).$$

Then M is isomorphic to the factor module of a free $K[\Delta^*]$-module with free generator e by its submodule N generated by

$$G := \{\delta_t e - a\delta_x^2 e + a2\delta_x e - (1+a)e\}.$$

Example cont.: Bivariate Difference Dimension Polynomial for Forward Scheme (2)

In order to use the algorithms provided above consider K a difference field with basic set

$$\Sigma = \{\alpha_x : x \mapsto x+1, \alpha_t : t \mapsto t+1, \beta_x : x \mapsto x-1, \beta_t : t \mapsto t-1\}.$$

Let $\tilde{G} := \{g_1 := a\alpha_x^2 e - 2a\alpha_x e + (1+a)e - \alpha_t e, g_2 := \alpha_x\beta_x e - e,$ $g_3 := \alpha_t\beta_t e - e\}$ and $I =_{K[\Sigma]} \langle \tilde{G} \rangle$. Then $K[\Sigma]e/I$ is isomorphic to $K[\Delta^*]e/N$ via the isomorphism

$$\alpha_x^{a_x}\alpha_t^{a_t}\beta_x^{b_x}\beta_t^{b_t} e \mapsto \delta_x^{a_x-b_x}\delta_t^{a_t-b_t} e.$$

Example cont.: Bivariate Difference Dimension Polynomial for Forward Scheme (2)

We fix two admissible orders $\prec_x, \prec_t$ on $[\Sigma]e$ by

$$\begin{aligned}
&\alpha_x^{a_x}\alpha_t^{a_t}\beta_x^{b_x}\beta_t^{b_t}e \prec_x \alpha_x^{c_x}\alpha_t^{c_t}\beta_x^{d_x}\beta_t^{d_t}e :\Longleftrightarrow \\
&\qquad (a_x+b_x, a_x+a_t+b_x+b_t, a_t+b_t, a_x, b_x, a_t, b_t) \\
&\qquad <_{\text{lex}} (c_x+d_x, c_x+c_t+d_x+d_t, c_t+d_t, c_x, d_x, c_t, d_t) \\
&\alpha_x^{a_x}\alpha_t^{a_t}\beta_x^{b_x}\beta_t^{b_t}e \prec_t \alpha_x^{c_x}\alpha_t^{c_t}\beta_x^{d_x}\beta_t^{d_t}e :\Longleftrightarrow \\
&\qquad (a_t+b_t, a_x+a_t+b_x+b_t, a_x+b_x, a_t, b_t, a_x, b_x) \\
&\qquad <_{\text{lex}} (c_t+d_t, c_x+c_t+d_x+d_t, c_x+d_x, c_t, d_t, c_x, d_x)
\end{aligned}$$

and compute a Gröbner basis of I with respect to $\prec_x, \prec_t$ using the algorithm provided above.

Example cont.: Bivariate Difference Dimension Polynomial for Forward Scheme (2)

A Gröbner basis for I with respect to $\prec_x, \prec_t$ is given by

$$
\begin{aligned}
\{\, g_1 &= a\alpha_x^2 e - 2a\alpha_x e + (1+a)e - \alpha_t e, \\
g_2 &= \alpha_x\beta_x e - e, \\
g_3 &= \alpha_t\beta_t e - e, \\
g_4 &= -\alpha_t\beta_x e + (1+a)\beta_x e + a\alpha_x e - 2ae, \\
g_5 &= \beta_x e - a\alpha_x\beta_t e - (1+a)\beta_x\beta_t e + 2a\beta_t e, \\
g_6 &= a\beta_t e - \beta_x^2 e + (1+a)\beta_x^2\beta_t e - 2a\beta_x\beta_t e \,\}.
\end{aligned}
$$

Example cont.: Bivariate Difference Dimension Polynomial for Forward Scheme (2)

For all $r_1, r_2 \in \mathbb{N}$ sufficiently large we obtain

$$|\{\lambda e \in [\Sigma]E \mid \operatorname{ord}_x \lambda \leq r_1, \operatorname{ord}_t \lambda \leq r_2, \text{ for all } \mu \in [\Sigma], g \in G \text{ such that } \operatorname{lt}_{<_x}(\mu g) = \lambda e \text{ we have } \operatorname{ord}_t \operatorname{lt}_{<_t}(\mu g) > r_2\}| = 2r_1 + 4r_2 + 1.$$

Hence, for all r_1, r_2 sufficiently large the bivariate difference dimension polynomial ϕ associated with the forward difference scheme (2) is given by

$$\phi(r_1, r_2) = 2r_1 + 4r_2 + 1.$$

Example cont.: Bivariate Difference Dimension Polynomial for Space Symmetric Scheme (3)

If we consider the space symmetric difference scheme (3) for the diffusion equation in 1-space then the associated bivariate difference dimension polynomial is again given by $\phi(r_1, r_2) = 2r_1 + 4r_2 + 1$.

Example: Maxwell equations in 3-space

Let $E = (E_1, E_2, E_3), D = (D_1, D_2, D_3), H = (H_1, H_2, H_3), B = (B_1, B_2, B_3), J_f = (J_1, J_2, J_3)$ and ρ_f be functions in (x, y, z, t) denoting electric field strength, electric displacement vector, magnetic field strength, magnetic displacement vector, free current density and free charge density, respectively. With $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$ Maxwell's equations in 3-space are given by

$$\nabla \cdot D = \rho_f, \qquad \nabla \cdot B = 0, \qquad \nabla \times E + \frac{\partial B}{\partial t} = 0, \quad \text{and} \quad \nabla \times H = J_f + \frac{\partial D}{\partial t}.$$

Example cont.: Maxwell equations in 3-space

Assuming $J_f = 0$ and $\rho_f = 0$ Maxwell's equations can be considered as a set of homogeneous linear differential equations giving rise to a differential $K[\delta_x, \delta_y, \delta_z, \delta_t]$-module M with generators $e_1, e_2, e_3, d_1, d_2, d_3, h_1, h_2, h_3 b_1, b_2, b_3$ satisfying

$$
\begin{aligned}
\delta_x d_1 + \delta_y d_2 + \delta_z d_3 &= 0 = \delta_x b_1 + \delta_y b_2 + \delta_z b_3, \\
\delta_y e_3 - \delta_z e_2 + \delta_t b_1 &= 0 = \delta_y h_3 - \delta_z h_2 - \delta_t d_1, \\
\delta_z e_1 - \delta_x e_3 + \delta_t b_2 &= 0 = \delta_z h_1 - \delta_x h_3 - \delta_t d_2, \\
\delta_x e_2 - \delta_y e_1 + \delta_t b_3 &= 0 = \delta_x h_2 - \delta_y h_1 - \delta_t d_3.
\end{aligned}
$$

Example cont.: Maxwell equations in 3-space

Then M is isomorphic to the factor module of a free $K[\delta_x, \delta_y, \delta_z, \delta_t]$ module with free generators $p_1, \ldots, p_{12}$ by its submodule N generated by

$$\begin{aligned} G \;=\; & \{\delta_x p_7 + \delta_y p_8 + \delta_z p_9, \delta_x p_{10} + \delta_y p_{11} + \delta_z p_{12}, \delta_y p_3 - \delta_z p_2 + \delta_t p_{10}, \\ & \delta_y p_6 - \delta_z p_5 - \delta_t p_7, \delta_z p_1 - \delta_x p_3 + \delta_t p_{11}, \delta_z p_4 - \delta_x p_6 - \delta_t p_8, \\ & \delta_x p_2 - \delta_y p_1 + \delta_t p_{12}, \delta_x p_5 - \delta_y p_4 - \delta_t p_9\}. \end{aligned}$$

Example cont.: Maxwell equations in 3-space

We opt for the partition $\{\delta_x, \delta_y, \delta_z\}, \{\delta_t\}$ and define admissible orders $\prec_1, \prec_2$ by

$$\begin{aligned}
&\delta_x^{a_x}\delta_y^{a_y}\delta_z^{a_z}\delta_t^{a_t}e_{j_1} \prec_1 \delta_x^{b_x}\delta_y^{b_y}\delta_z^{b_z}\delta_t^{b_t}e_{j_2} :\Longleftrightarrow \\
&\qquad (a_x + a_y + a_z, a_x + a_y + a_z + a_t, a_t, j_1, a_x, a_y, a_z, a_t) \\
&\qquad <_{\text{lex}} (b_x + b_y + b_z, b_x + b_y + b_z + b_t, b_t, j_2, b_x, b_y, b_z, b_t), \\
&\delta_x^{a_x}\delta_y^{a_y}\delta_z^{a_z}\delta_t^{a_t}e_{j_1} \prec_2 \delta_x^{b_x}\delta_y^{b_y}\delta_z^{b_z}\delta_t^{b_t}e_{j_2} :\Longleftrightarrow \\
&\qquad (a_t, a_x + a_y + a_z + a_t, a_x + a_y + a_z, j_1, a_t, a_x, a_y, a_z) \\
&\qquad <_{\text{lex}} (b_t, b_x + b_y + b_z + b_t, b_x + b_y + b_z, j_2, b_t, b_x, b_y, b_z).
\end{aligned}$$

Example cont.: Maxwell equations in 3-space

Then G is a Gröbner basis w.r.t. $<_1, <_2$ and the differential dimension polynomial associated with Maxwell's equations for vanishing free current density and free charge density is given by

$$\phi(r_1, r_2) = r_1^3 r_2 + \frac{5}{3} r_1^3 + 8 r_1^2 r_2 + 11 r_1^2 + 19 r_1 r_2 + \frac{64}{3} r_1 + 12 r_2 + 12.$$

Example cont.: Forward Difference scheme for Maxwell equations in 3-space

In G we substitute every occurence of δ_i by $\delta_i - 1$ for $i \in \{x, y, z, t\}$ in order to obtain a forward difference scheme associated with Maxwell's equations for vanishing free current density and free charge density and consider K as an inversive difference field with $\delta_x, \delta_y, \delta_z$ and δ_t being shift operators.

Example cont.: Forward Difference scheme for Maxwell equations in 3-space

One verifies in finite time that a Gröbner basis of the associated $K[\alpha_x, \alpha_y, \alpha_z, \alpha_t, \beta_x, \beta_y, \beta_z, \beta_t]$ submodule of the free module with free generators $p_1, \ldots, p_{12}$ is given by

$$
\begin{aligned}
\tilde{G} \quad = \quad & \{-\alpha_t\beta_x\beta_y p_{12} - \beta_x\beta_y p_1 + \beta_x\beta_y p_2 + \beta_x\beta_y p_{12} + \beta_x p_1 - \beta_y p_2, \\
& \alpha_t\beta_x\beta_z p_{11} - \beta_x\beta_z p_1 + \beta_x\beta_z p_3 - \beta_x\beta_z p_{11} + \beta_x p_1 - \beta_z p_3, \\
& \alpha_t\beta_y\beta_z p_{10} + \beta_y\beta_z p_2 - \beta_y\beta_z p_3 - \beta_y\beta_z p_{10} - \beta_y p_2 + \beta_z p_3, \\
& -\alpha_z\beta_x p_1 - \alpha_t\beta_x p_{11} + \beta_x p_1 - \beta_x p_3 + \beta_x p_{11} + p_3, \\
& \alpha_z\beta_y p_2 - \alpha_t\beta_y p_{10} - \beta_y p_2 + \beta_y p_3 + \beta_y p_{10} - p_3, \\
& \alpha_t\beta_x\beta_y p_9 - \beta_x\beta_y p_4 + \beta_x\beta_y p_5 - \beta_x\beta_y p_9 + \beta_x p_4 - \beta_y p_5, \\
& -\alpha_t\beta_x\beta_z p_8 - \beta_x\beta_z p_4 + \beta_x\beta_z p_6 + \beta_x\beta_z p_8 + \beta_x p_4 - \beta_z p_6, \\
& -\alpha_t\beta_y\beta_z p_7 + \beta_y\beta_z p_5 - \beta_y\beta_z p_6 + \beta_y\beta_z p_7 - \beta_y p_5 + \beta_z p_6, \\
& -\alpha_z\beta_x p_4 + \alpha_t\beta_x p_8 + \beta_x p_4 - \beta_x p_6 - \beta_x p_8 + p_6, \\
& \alpha_z\beta_y p_5 + \alpha_t\beta_y p_7 - \beta_y p_5 + \beta_y p_6 - \beta_y p_7 - p_6,
\end{aligned}
$$

Example cont.: Forward Difference scheme for Maxwell equations in 3-space

$$-\alpha_y\beta_x p_4 - \alpha_t\beta_x p_9 + \beta_x p_4 - \beta_x p_5 + \beta_x p_9 + p_5,$$
$$-\alpha_y\beta_x p_1 + \alpha_t\beta_x p_{12} + \beta_x p_1 - \beta_x p_2 - \beta_x p_{12} + p_2,$$
$$\alpha_t\beta_t p_2 - \alpha_t\beta_t p_3 - \alpha_t\beta_t p_{10} - p_2 + p_3 + p_{10},$$
$$-\alpha_t\beta_t p_1 + \alpha_t\beta_t p_3 - \alpha_t\beta_t p_{11} + p_1 - p_3 + p_{11},$$
$$\alpha_t\beta_t p_1 - \alpha_t\beta_t p_2 - \alpha_t\beta_t p_{12} - p_1 + p_2 + p_{12},$$
$$\alpha_t\beta_t p_5 - \alpha_t\beta_t p_6 + \alpha_t\beta_t p_7 - p_5 + p_6 - p_7,$$
$$-\alpha_t\beta_t p_4 + \alpha_t\beta_t p_6 + \alpha_t\beta_t p_8 + p_4 - p_6 - p_8,$$
$$\alpha_t\beta_t p_4 - \alpha_t\beta_t p_5 + \alpha_t\beta_t p_9 - p_4 + p_5 - p_9,$$
$$\alpha_y\beta_x\beta_t p_1 - \alpha_z\beta_x\beta_t p_1 - \alpha_x\beta_x p_{10} - \alpha_y\beta_x p_{11} + \alpha_y\beta_t p_3 - \alpha_z\beta_x p_{12}$$
$$+\alpha_z\beta_z p_{10} - \alpha_z\beta_t p_2 + \beta_x\beta_t p_2 - \beta_x\beta_t p_3 + \beta_x\beta_t p_{11} + \beta_x\beta_t p_{12} + \beta_x p_{10}$$
$$-\beta_t p_{10},$$

Example cont.: Forward Difference scheme for Maxwell equations in 3-space

$$-\alpha_x\beta_x\beta_t p_3 + \alpha_z\beta_y\beta_t p_2 + \beta_x\beta_y\beta_t p_1 - \beta_x\beta_y\beta_t p_2 - \beta_x\beta_y\beta_t p_{12}$$
$$+\beta_x\beta_y p_{12} - \beta_x\beta_t p_1 + \beta_y\beta_t p_3 + \beta_y\beta_t p_{10} - \beta_y p_{10},$$
$$\alpha_y\beta_x\beta_t p_4 - \alpha_z\beta_x\beta_t p_4 + \alpha_x\beta_x p_7 + \alpha_y\beta_x p_8 + \alpha_y\beta_t p_6 + \alpha_z\beta_x p_9 - \alpha_z\beta_z p_7$$
$$-\alpha_z\beta_t p_5 + \beta_x\beta_t p_5 - \beta_x\beta_t p_6 - \beta_x\beta_t p_8 - \beta_x\beta_t p_9 - \beta_x p_7 + \beta_t p_7,$$
$$-\alpha_x\beta_x\beta_t p_6 + \alpha_z\beta_y\beta_t p_5 + \beta_x\beta_y\beta_t p_4 - \beta_x\beta_y\beta_t p_5 + \beta_x\beta_y\beta_t p_9$$
$$-\beta_x\beta_y p_9 - \beta_x\beta_t p_4 + \beta_y\beta_t p_6 - \beta_y\beta_t p_7 + \beta_y p_7,$$
$$\alpha_x p_7 + \alpha_y p_8 + \alpha_z p_9 - p_7 - p_8 - p_9,$$
$$\alpha_x p_{10} + \alpha_y p_{11} + \alpha_z p_{12} - p_{10} - p_{11} - p_{12},$$
$$\alpha_y p_3 - \alpha_z p_2 + \alpha_t p_{10} + p_2 - p_3 - p_{10},$$
$$-\alpha_x p_3 + \alpha_z p_1 + \alpha_t p_{11} - p_1 + p_3 - p_{11},$$
$$\alpha_x p_2 - \alpha_y p_1 + \alpha_t p_{12} + p_1 - p_2 - p_{12},$$
$$\alpha_y p_6 - \alpha_z p_5 - \alpha_t p_7 + p_5 - p_6 + p_7,$$

Example cont.: Forward Difference scheme for Maxwell equations in 3-space

$$-\alpha_x p_6 + \alpha_z p_4 - \alpha_t p_8 - p_4 + p_6 + p_8,$$
$$\alpha_x p_5 - \alpha_y p_4 - \alpha_t p_9 + p_4 - p_5 + p_9,$$
$$\alpha_x \beta_x p_1 - p_1,$$
$$\alpha_y \beta_y p_1 - p_1,$$
$$\alpha_z \beta_z p_1 - p_1,$$
$$\alpha_t \beta_t p_1 - p_1,$$
$$\alpha_x \beta_x p_2 - p_2,$$
$$\alpha_y \beta_y p_2 - p_2,$$
$$\alpha_z \beta_z p_2 - p_2,$$
$$\alpha_t \beta_t p_2 - p_2,$$
$$\alpha_x \beta_x p_3 - p_3,$$
$$\alpha_y \beta_y p_3 - p_3,$$

Example cont.: Forward Difference scheme for Maxwell equations in 3-space

$$\alpha_z\beta_z p_3 - p_3,$$
$$\alpha_t\beta_t p_3 - p_3,$$
$$\alpha_x\beta_x p_4 - p_4,$$
$$\alpha_y\beta_y p_4 - p_4,$$
$$\alpha_z\beta_z p_4 - p_4,$$
$$\alpha_t\beta_t p_4 - p_4,$$
$$\alpha_x\beta_x p_5 - p_5,$$
$$\alpha_y\beta_y p_5 - p_5,$$
$$\alpha_z\beta_z p_5 - p_5,$$
$$\alpha_t\beta_t p_5 - p_5,$$
$$\alpha_x\beta_x p_6 - p_6,$$
$$\alpha_y\beta_y p_6 - p_6,$$

Example cont.: Forward Difference scheme for Maxwell equations in 3-space

$$\alpha_z\beta_z p_6 - p_6,$$
$$\alpha_t\beta_t p_6 - p_6,$$
$$\alpha_x\beta_x p_7 - p_7,$$
$$\alpha_y\beta_y p_7 - p_7,$$
$$\alpha_z\beta_z p_7 - p_7,$$
$$\alpha_x\beta_x p_8 - p_8,$$
$$\alpha_y\beta_y p_8 - p_8,$$
$$\alpha_z\beta_z p_8 - p_8,$$
$$\alpha_x\beta_x p_9 - p_9,$$
$$\alpha_y\beta_y p_9 - p_9,$$
$$\alpha_x\beta_x p_{10} - p_{10},$$
$$\alpha_y\beta_y p_{10} - p_{10},$$

Example cont.: Forward Difference scheme for Maxwell equations in 3-space

$$\alpha_z\beta_z p_{10} - p_{10},$$
$$\alpha_x\beta_x p_{11} - p_{11},$$
$$\alpha_y\beta_y p_{11} - p_{11},$$
$$\alpha_z\beta_z p_{11} - p_{11},$$
$$\alpha_x\beta_x p_{12} - p_{12},$$
$$\alpha_y\beta_y p_{12} - p_{12},$$
$$\alpha_x\beta_z p_7 + \alpha_y\beta_z p_8 - \beta_z p_7 - \beta_z p_8 - \beta_z p_9 + p_9,$$
$$\alpha_x\beta_z p_{10} + \alpha_y\beta_z p_{11} - \beta_z p_{10} - \beta_z p_{11} - \beta_z p_{12} + p_{12},$$
$$\alpha_y\beta_t p_3 - \alpha_z\beta_t p_2 + \beta_t p_2 - \beta_t p_3 - \beta_t p_{10} + p_{10},$$
$$-\alpha_x\beta_t p_3 + \alpha_z\beta_t p_1 - \beta_t p_1 + \beta_t p_3 - \beta_t p_{11} + p_{11},$$
$$\alpha_x\beta_t p_2 - \alpha_y\beta_t p_1 + \beta_t p_1 - \beta_t p_2 - \beta_t p_{12} + p_{12},$$
$$\alpha_y\beta_t p_6 - \alpha_z\beta_t p_5 + \beta_t p_5 - \beta_t p_6 + \beta_t p_7 - p_7,$$

Example cont.: Forward Difference scheme for Maxwell equations in 3-space

$$-\alpha_x\beta_t p_6 + \alpha_z\beta_t p_4 - \beta_t p_4 + \beta_t p_6 + \beta_t p_8 - p_8,$$
$$\alpha_x\beta_t p_5 - \alpha_y\beta_t p_4 + \beta_t p_4 - \beta_t p_5 + \beta_t p_9 - p_9,$$
$$\alpha_x\beta_x\beta_y p_{10} + \alpha_y\beta_x\beta_y p_{11} - \alpha_z\beta_x\beta_z p_{11} - \alpha_z\beta_y\beta_z p_{10} + \beta_x\beta_y\beta_z p_{10}$$
$$+\beta_x\beta_y\beta_z p_{11} + \beta_x\beta_y\beta_z p_{12} - \beta_x\beta_y p_{12} - \beta_x\beta_z p_{11} - \beta_y\beta_z p_{10},$$
$$-\alpha_z\beta_x\beta_t p_1 + \beta_x\beta_t p_1 - \beta_x\beta_t p_3 + \beta_x\beta_t p_{11} - \beta_x p_{11} + \beta_t p_3,$$
$$-\beta_x\beta_z\beta_t p_1 + \beta_x\beta_z\beta_t p_3 - \beta_x\beta_z\beta_t p_{11} + \beta_x\beta_z p_{11} + \beta_x\beta_t p_1 - \beta_z\beta_t p_3,$$
$$\alpha_z\beta_y\beta_t p_2 - \beta_y\beta_t p_2 + \beta_y\beta_t p_3 + \beta_y\beta_t p_{10} - \beta_y p_{10} - \beta_t p_3,$$
$$\beta_y\beta_z\beta_t p_2 - \beta_y\beta_z\beta_t p_3 - \beta_y\beta_z\beta_t p_{10} + \beta_y\beta_z p_{10} - \beta_y\beta_t p_2 + \beta_z\beta_t p_3,$$
$$\alpha_x\beta_x\beta_y p_7 + \alpha_y\beta_x\beta_y p_8 - \alpha_z\beta_x\beta_z p_8 - \alpha_z\beta_y\beta_z p_7 + \beta_x\beta_y\beta_z p_7 + \beta_x\beta_y\beta_z p_8$$
$$+\beta_x\beta_y\beta_z p_9 - \beta_x\beta_y p_9 - \beta_x\beta_z p_8 - \beta_y\beta_z p_7,$$
$$-\alpha_z\beta_x\beta_t p_4 + \beta_x\beta_t p_4 - \beta_x\beta_t p_6 - \beta_x\beta_t p_8 + \beta_x p_8 + \beta_t p_6,$$
$$-\beta_x\beta_z\beta_t p_4 + \beta_x\beta_z\beta_t p_6 + \beta_x\beta_z\beta_t p_8 - \beta_x\beta_z p_8 + \beta_x\beta_t p_4 - \beta_z\beta_t p_6,$$

Example cont.: Forward Difference scheme for Maxwell equations in 3-space

$$\alpha_z\beta_y\beta_t p_5 - \beta_y\beta_t p_5 + \beta_y\beta_t p_6 - \beta_y\beta_t p_7 + \beta_y p_7 - \beta_t p_6,$$
$$\beta_y\beta_z\beta_t p_5 - \beta_y\beta_z\beta_t p_6 + \beta_y\beta_z\beta_t p_7 - \beta_y\beta_z p_7 - \beta_y\beta_t p_5 + \beta_z\beta_t p_6,$$
$$-\alpha_x\beta_y\beta_z p_7 + \beta_y\beta_z p_7 + \beta_y\beta_z p_8 + \beta_y\beta_z p_9 - \beta_y p_9 - \beta_z p_8,$$
$$-\alpha_x\beta_y\beta_z p_{10} + \beta_y\beta_z p_{10} + \beta_y\beta_z p_{11} + \beta_y\beta_z p_{12} - \beta_y p_{12} - \beta_z p_{11}\}.$$

Then the bivariate difference dimension polynomial associated with the forward difference scheme for Maxwell's equations with vanishing free current density and free charge density is given by

$$\phi(r_1, r_2) = 16r_1^3 r_2 + \frac{40}{3}r_1^3 + 52r_1^2 r_2 + 27r_1^2 + 60r_1 r_2 + \frac{89}{3}r_1 + 24r_2 + 12.$$

Example cont.: Symmetric Difference scheme for Maxwell equations in 3-space

Instead of substituting every occurence of δ_i by $\delta_i - 1$ we substitute by $\frac{\delta_i - \delta_i^{-1}}{2}$ for $i \in \{x, y, z, t\}$ in order to obtain a symmetric difference scheme associated with Maxwell's equations for vanishing free current density and free charge density. The associated bivariate difference dimension polynomial is then given by

$$\phi(r_1, r_2) = 16r_1^3r_2 + \frac{56}{3}r_1^3 + 56r_1^2r_2 + 28r_1^2 + 64r_1r_2 + \frac{64}{3}r_1 + 12r_2 + 22.$$

Conclusion

In the first part of the talk we have presented several recent results on relative Gröbner bases including an example in which they do not exist.

In the second part of the talk we gave several examples of dimension polynomials of systems of differential equations and associated difference schemes.

Thank you for your attention