Differential and Difference Chow Form, Sparse Resultant, and Toric Variety

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Outline

- Background
- Sparse Differential Resultant
- Differential Chow Form
- Difference Binomial and Toric Variety
Sparse Differential Resultant for Laurent Differential Polynomials
Sylvester Resultant

Two polynomials:

\[ f = a_l x^l + a_{l-1} x^{l-1} + \cdots + a_1 x + a_0 \]
\[ g = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0. \]

\[
\text{Res}(f, g) = \begin{vmatrix}
  a_l & a_{l-1} & a_{l-2} & \cdots & a_0 \\
  a_l & a_{l-1} & a_{l-2} & \cdots & a_0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  a_l & a_{l-1} & a_{l-2} & \cdots & a_0 \\
  b_m & b_{m-1} & b_{m-2} & \cdots & b_0 \\
  b_m & b_{m-1} & b_{m-2} & \cdots & b_0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  b_m & b_{m-1} & b_{m-2} & \cdots & b_0 \\
\end{vmatrix}.
\]

Property: \( \text{Res}(f, g) = 0 \iff f(x) = g(x) = 0 \) has common solutions

A Brief History of Resultant

**Algebraic Resultant**

- Sylvester (1883) resultant for two polynomials ($n = 1$)
- Macaulay (1902) multivariate resultant
- Gelfand & Sturmfels (1994) sparse resultant
A Brief History of Resultant

**Algebraic Resultant**
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**Differential Resultant**
- Ritt (1932): Differential resultant for \( n = 1 \).
- Ferro (1997): Diff-Res as Macaulay resultant. **Not complete.**

No rigorous definition for differential multi-variate resultant
No study of differential sparse resultant
Sparse Differential Polynomials

- **Sparse Differential Polynomials**: with fixed monomials

Most differential polynomials in practice are sparse

\[
f = \sum_{i+j \leq 5} * y^i (y')^j
\]

**Dense Diff Polynomials**

\[
f = \sum_{i+j \leq 5} * y^i (y')^j
\]

**Sparse Diff Polynomials**

\[
f = * + * y^4 + * y'^5 + * y^2 y'^2
\]
Ordinary differential field: \((\mathcal{F}, \delta)\), e.g. \((\mathbb{Q}(x), \frac{d}{dx})\)

Diff Indeterminates: \(\mathbb{Y} = \{y_1, \ldots, y_n\}\).

Notation: \(y_i^{(k)} = \delta^k y_i\).

Laurent Diff Monomial: \(M = \prod_{k=1}^{n} \prod_{l=0}^{o} (y_k^{(l)})^{d_{kl}}\) with \(d_{kl} \in \mathbb{Z}\);

Laurent Diff Poly: \(f = \sum_{k=1}^{m} a_k M_k\), \(M_k\) Laurent diff monomials.

Support of \(f\): \(\mathcal{A} = \{M_1, \ldots, M_m\}\).

Laurent Diff Poly Ring: \(\mathcal{F}\{\mathbb{Y}^\pm\}\).

Example. Laurent Differential Polynomial

\[P = y_1 + y_1'y_2 \quad \Leftrightarrow \quad P = 1 + y_1^{-1}y_1'y_2\]
Intersection Theorem is not true in diff case:

\[ \dim(V \cap W) \geq \dim(V) + \dim(W) - n \]
**Intersection Theorem** is not true in diff case:

\[
\dim(V \cap W) \geq \dim(V) + \dim(W) - n
\]

**Theorem**

\(\mathcal{I} \subset \mathcal{F}\{Y\} : \) a prime diff ideal with dimension \(d > 0\) and order \(h\).

\(f : \) a generic diff poly of order \(s\) with \(u_f\) the set of its coefficients.

Then \(\mathcal{I}_1 = [\mathcal{I}, f]\) is a prime diff ideal in \(\mathcal{F}\langle u_f\rangle\{Y\}\) with dimension \(d - 1\) and order \(h + s\).
Intersection Theorem is not true in diff case:

\[
\dim(V \cap W) \geq \dim(V) + \dim(W) - n
\]

**Theorem**

\(\mathcal{I} \subset \mathcal{F}\{Y\} : \) a prime diff ideal with dimension \(d > 0\) and order \(h\).

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Then \(\mathcal{I}_1 = [\mathcal{I}, f]\) is a prime diff ideal in \(\mathcal{F}\langle \mathbf{u}_f \rangle\{Y\}\) with dimension \(d - 1\) and order \(h + s\).

**Dimension Conjecture** (Ritt, 1950): \(\dim[f_1, \ldots, f_r] \geq n - r\).
Differential Dimension Conjecture in Generic Case

**Intersection Theorem** is not true in diff case:

\[
\dim(V \cap W) \geq \dim(V) + \dim(W) - n
\]

**Theorem**

\(\mathcal{I} \subset \mathcal{F}\{\mathbb{Y}\} : \) a prime diff ideal with dimension \(d > 0\) and order \(h\).

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Then \(\mathcal{I}_1 = [\mathcal{I}, f]\) is a prime diff ideal in \(\mathcal{F}\langle u_f \rangle\{\mathbb{Y}\}\) with dimension \(d - 1\) and order \(h + s\).

**Dimension Conjecture** (Ritt, 1950): \(\dim[f_1, \ldots, f_r] \geq n - r\).

**Theorem (Generic Dimension Theorem)**

\(f_1, \ldots, f_r (r \leq n) : \) generic diff polynomials. Then

\([f_1, \ldots, f_r] : \) a prime diff ideal of dimension \(n - r\) and order \(\sum_i \text{ord}(f_i)\).
Sparse Differential Resultant

- **Generic Sparse Differential Polynomials:**
  \[ \mathcal{A}_i = \{ M_{i0}, M_{i1}, \ldots, M_{il_i} \} \quad (i = 0, \ldots, n) \]: Monomial sets
  \[ P_i = \sum_{j=0}^{l_i} u_{ij} M_{ij} \] and \[ u_i = \{ u_{i1}, \ldots, u_{il_i} \} \].

  \[ [P_0, P_1, \ldots, P_n] \subset \mathbb{Q}\{u_0, u_1, \ldots, u_n, Y, Y^{-1}\} \]
Sparse Differential Resultant

**Generic Sparse Differential Polynomials:**
\[ A_i = \{M_{i0}, M_{i1}, \ldots, M_{il_i}\} (i = 0, \ldots, n): \text{Monomial sets} \]
\[ P_i = \sum_{j=0}^{l_i} u_{ij} M_{ij} \quad \text{and} \quad u_i = \{u_{i1}, \ldots, u_{il_i}\}. \]
\[ [P_0, P_1, \ldots, P_n] \subset \mathbb{Q}\{u_0, u_1, \ldots, u_n, \gamma, \gamma^{-1}\} \]

**Sparse Differential Resultant Exists, if the eliminant ideal:**
\[ [P_0, \ldots, P_n] \cap \mathbb{Q}\{u_0, u_1, \ldots, u_n\} = \text{sat}(R(u_0, \ldots, u_n)) \]
\[ \text{is of codimension 1} \]

**Definition**

**R:** *Sparse Differential Resultant* of \( P_0, \ldots, P_n \) or \( A_0, \ldots, A_n \).
Sparse Differential Resultant

**Generic Sparse Differential Polynomials:**
\[ A_i = \{ M_{i0}, M_{i1}, \ldots, M_{il_i} \} \ (i = 0, \ldots, n) \] are Monomial sets.
\[ P_i = \sum_{j=0}^{l_i} u_{ij} M_{ij} \] and \[ u_i = \{ u_{i1}, \ldots, u_{il_i} \}. \]
\[ [P_0, P_1, \ldots, P_n] \subset \mathbb{Q}\{u_0, u_1, \ldots, u_n, Y, Y^{-1}\} \]

**Sparse Differential Resultant Exists, if the eliminant ideal:**
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is of codimension 1

\[ \Leftrightarrow \text{P}_i \text{ are Laurent differentially essential:} \]
There exist \( k_i \ (i = 0, \ldots, n) \) with \( 1 \leq k_i \leq l_i \) such that
\[ \text{d.tr.deg} \mathbb{Q}\langle \frac{M_{0k_0}}{M_{00}}, \frac{M_{1k_1}}{M_{10}}, \ldots, \frac{M_{nk_n}}{M_{n0}} \rangle / \mathbb{Q} = n. \]

**Definition**

**R:** Sparse Differential Resultant of \( P_0, \ldots, P_n \) or \( A_0, \ldots, A_n \).
Example

\( n = 2, \)

\[ \mathbb{P}_i = u_{i0} y_1'' + u_{i1} y_1''' + u_{i2} y_2''' \quad (i = 0, 1, 2). \]

d.\text{tr.}\text{deg} \, Q \langle \frac{y_1''''}{y_1''}, \frac{y_2''''}{y_1''} \rangle / Q = 2 \implies \mathbb{P}_i \text{ form a diff essential system.} \]

The sparse differential resultant is

\[ R = \begin{vmatrix} u_{00} & u_{01} & u_{02} \\ u_{10} & u_{11} & u_{12} \\ u_{20} & u_{21} & u_{22} \end{vmatrix}. \]
Criterion for Existence of Sparse Resultant

\[ P_i = \sum_{j=0}^{l_i} u_{ij} M_{ij} \ (i = 0, \ldots, n). \]

- \( M_{ij} / M_{i0} = \prod_{k=1}^{n} \prod_{l=0}^{s_i} (y_k^{(l)}) d_{ijkl} \).
  - \( d_{ijk} = \sum_{l=0}^{s_i} d_{ijkl} x_k^l \in \mathbb{Q}[x_k]. \)

**Symbolic Support Vector of** \( M_{ij} / M_{i0}: \) \( \beta_{ij} = (d_{ij1}, \ldots, d_{ijn}) \)
Criterion for Existence of Sparse Resultant

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Symbolic Support Vector of \( M_{ij}/M_{i0} \): \( \beta_{ij} = (d_{ij1}, \ldots, d_{ijn}) \)

Symbolic Support Vector of \( P_i \): \( \beta_i = \sum_{j=0}^{l_i} u_{ij} \beta_{ij} = (d_{i1}, \ldots, d_{in}) \).

Symbolic Support Matrix of \( P_0, \ldots, P_n \):

\[
M_P = \begin{pmatrix}
\beta_0 \\
\beta_1 \\
\vdots \\
\beta_n
\end{pmatrix} = \begin{pmatrix}
d_{01} & d_{02} & \ldots & d_{0n} \\
d_{11} & d_{12} & \ldots & d_{1n} \\
\vdots & \vdots & \ddots & \vdots \\
d_{n1} & d_{n2} & \ldots & d_{nn}
\end{pmatrix}
\]
Criterion for Existence of Sparse Resultant

\[ P_i = \sum_{j=0}^{l_i} u_{ij} M_{ij} \quad (i = 0, \ldots, n). \]

- \[ M_{ij} / M_{i0} = \prod_{k=1}^{n} \prod_{l=0}^{s_i} (y_k^{(l)}) d_{ijkl}. \]
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**Symbolic Support Vector of** \( M_{ij} / M_{i0} \): \( \beta_{ij} = (d_{ij1}, \ldots, d_{ijn}) \)

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- **Symbolic Support Matrix of** \( P_0, \ldots, P_n \):
  \[
  M_P = \begin{pmatrix}
    \beta_0 \\
    \beta_1 \\
    \vdots \\
    \beta_n
  \end{pmatrix}
  = \begin{pmatrix}
    d_{01} & d_{02} & \ldots & d_{0n} \\
    d_{11} & d_{12} & \ldots & d_{1n} \\
    \vdots & \vdots & \ddots & \vdots \\
    d_{n1} & d_{n2} & \ldots & d_{nn}
  \end{pmatrix}
  \]

**Theorem (Like Linear Algebra!)**

\[ \text{Sparse resultant exists for } P_i \iff \text{rk}(M_P) = n. \]
Properties of Sparse Differential Resultant
Lemma

\((P_i, u_i)\) specializes to \((\overline{P}_i, v_i)\) by setting \(u_i = v_i \in F\).

If \(\overline{P}_0 = \cdots = \overline{P}_n = 0\) has a non-poly solution,
then \(R(v_0, \ldots, v_n) = 0\).
Lemma

\((P_i, u_i)\) specializes to \((\overline{P}_i, v_i)\) by setting \(u_i = v_i \in \mathcal{F}\).

If \(\overline{P}_0 = \cdots = \overline{P}_n = 0\) has a non-poly solution,
then \(R(v_0, \ldots, v_n) = 0\).

Example (Why Non-Polynomial solution?)

\(\mathcal{F} = \mathbb{Q}(x)\), differential operator: \(\frac{\partial}{\partial x}\)

\(P_i = u_{i0}y''_1 + u_{i1}y'''_1 + u_{i2}y'''_2\) \((i = 0, 1, 2)\).

The sparse differential resultant \(R = \begin{vmatrix} u_{00} & u_{01} & u_{02} \\ u_{10} & u_{11} & u_{12} \\ u_{20} & u_{21} & u_{22} \end{vmatrix} \neq 0\).

Let \(a_1 = x + 1, a_2 = x^2 + x + 1\).
Then \(a_1'' = a_2''' = 0\). \((a_1, a_2)\): a solution of \(P_0 = P_1 = P_2 = 0\).
Conditions for Existence of Non-poly Solutions

\[ A_i = (M_{i0}, \ldots, M_{il_i}) : \text{Differential Monomials} \]
Conditions for Existence of Non-poly Solutions

- $A_i = (\mathcal{M}_{i0}, \ldots, \mathcal{M}_{il_i})$: Differential Monomials
- $\mathcal{L}(A_i) = \{ F_i = \sum_{j=0}^{l_i} c_{ij} M_{ij} \}$: all diff polys with support $A_i$. 

$Z_0(A_0, \ldots, A_n)$: set of $F_i$ having a common non-poly solution.

$Z_0(A_0, \ldots, A_n)$: Kolchin diff closure of $Z_0(A_0, \ldots, A_n)$.

Theorem

$Z_0(A_0, \ldots, A_n) = V(\text{sat}(\text{Res } A_0, \ldots, A_n))$.

On a Kolchin open set of $V(\text{sat}(\text{Res } A_0, \ldots, A_n))$, $F_0 = \cdots = F_n = 0$ have non-poly solutions $\iff \text{Res } F_0, \ldots, F_n = 0$. 

Xiao-Shan Gao (AMSS, CAS)
Conditions for Existence of Non-poly Solutions

- $A_i = (M_{i0}, \ldots, M_{il_i})$: Differential Monomials
- $\mathcal{L}(A_i) = \{ F_i = \sum_{j=0}^{l_i} c_i M_{ij} \}$: all diff polys with support $A_i$.
- $\mathcal{Z}_0(A_0, \ldots, A_n)$: set of $F_i$ having a common non-poly solution.
- $\overline{\mathcal{Z}_0(A_0, \ldots, A_n)}$: Kolchin diff closure of $\mathcal{Z}_0(A_0, \ldots, A_n)$. 

\[ Z_0(A_0, \ldots, A_n) = V(\text{sat}(\text{Res} A_0, \ldots, A_n)). \]

On a Kolchin open set of $V(\text{sat}(\text{Res} A_0, \ldots, A_n))$, $F_0 = \cdots = F_n = 0$ have non-poly solutions $\iff$ $\text{Res} F_0, \ldots, F_n = 0$. 

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Conditions for Existence of Non-poly Solutions

- $A_i = (M_{i0}, \ldots, M_{il_i})$: Differential Monomials
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- $Z_0(A_0, \ldots, A_n)$: set of $F_i$ having a common non-poly solution.
- $\overline{Z_0(A_0, \ldots, A_n)}$: Kolchin diff closure of $Z_0(A_0, \ldots, A_n)$.

**Theorem**

\[ \overline{Z_0(A_0, \ldots, A_n)} = \mathbb{V}(\text{sat}(\text{Res}_{A_0, \ldots, A_n})). \]

On a Kolchin open set of $\mathbb{V}(\text{sat}(\text{Res}_{A_0, \ldots, A_n}))$,

$F_0 = \cdots = F_n = 0$ have non-poly solutions $\iff \text{Res}_{F_0, \ldots, F_n} = 0$. 
$\mathcal{G} = \{g_1, \ldots, g_n\}$: differential polynomials.

**Jacobi Number:** $\text{Jac}(\mathcal{G}) = \max_{\sigma} \sum_{i=1}^{n} \text{ord}(g_i, y_{\sigma}(i))$,
where $\sigma$ is a permutation of $\{1, \ldots, n\}$.
\[ \mathcal{G} = \{g_1, \ldots, g_n\} \text{: differential polynomials.} \]

**Jacobi Number:** \( \text{Jac}(\mathcal{G}) = \max_\sigma \sum_{i=1}^{n} \text{ord}(g_i, y_\sigma(i)) \),

where \( \sigma \) is a permutation of \( \{1, \ldots, n\} \).

\[ \delta \text{Res}(u_0, \ldots, u_n) \text{ is differentially homogeneous in each } u_i \text{ and is of order } h_i = s - s_i \text{ in } u_i \ (i = 0, \ldots, n) \text{ where } s = \sum_{i=0}^{n} s_i. \]

\[ \text{S-} \delta \text{Res}(u_0, \ldots, u_n) \text{ is differentially homogeneous in each } u_i \text{ and is of order } h_i \leq J_i = \text{Jac}(\mathcal{P}^*_i) \text{ in } u_i, \text{ where } \mathcal{P}^*_i = \{\mathcal{P}_0, \ldots, \mathcal{P}_n\} \setminus \{\mathcal{P}_i\}. \]
Poisson-Type Product Formula

- **Algebraic Resultant:** \( \text{Res}(A(x), B(x)) = c \prod_{\eta, B(\eta)=0} A(\eta). \)
Algebraic Resultant: \( \text{Res}(A(x), B(x)) = c \prod_{\eta, B(\eta) = 0} A(\eta) \).

Differential Resultant:
\[
\delta \text{Res}(u_0, \ldots, u_n) = A(u_0, \ldots, u_n) \prod_{\tau = 1}^{t_0} \mathbb{P}_0(\eta_{\tau 1}, \ldots, \eta_{\tau n})^{(h_0)}.
\]
And \( (\eta_{\tau 1}, \ldots, \eta_{\tau n}) \) are generic points of \([\mathbb{P}_1, \ldots, \mathbb{P}_n]\).
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- **Sparse Differential Resultant:**
  \[
  S-\delta \text{Res}(u_0, \ldots, u_n) = A \prod_{\tau=1}^{t_0} (u_{00} + \sum_{k=1}^{l_0} u_{0k} \xi_{\tau k})^{(h_0)}.
  \]
Poisson-Type Product Formula

- **Algebraic Resultant:** \( \text{Res}(A(x), B(x)) = c \prod_{\eta, B(\eta) = 0} A(\eta) \).

- **Differential Resultant:**
  \[ \delta \text{Res}(u_0, \ldots, u_n) = A(u_0, \ldots, u_n) \prod_{T=1}^{t_0} \mathbb{P}_0(\eta_{T1}, \ldots, \eta_{Tn})(h_0) \]
  And \((\eta_{T1}, \ldots, \eta_{Tn})\) are generic points of \([\mathbb{P}_1, \ldots, \mathbb{P}_n]\).

- **Sparse Differential Resultant:**
  \[ S-\delta \text{Res}(u_0, \ldots, u_n) = A \prod_{T=1}^{t_0} (u_{00} + \sum_{k=1}^{l_0} u_{0k} \xi_{Tk})(h_0) \]

  When 1) Any \(n\) of the \(A_i\) diff independent and
  2) \(e_j \in \text{Span}_\mathbb{Z}\{\alpha_{ij} - \alpha_{i0}\}\),

  the result can be strengthened:

  \[ S-\delta \text{Res}(u_0, \ldots, u_n) = A \prod_{T=1}^{t_0} \left( \mathbb{P}_0(\eta_{T1}, \ldots, \eta_{Tn}) \right)(h_0) \]

  And \(\eta_T = (\eta_{T1}, \ldots, \eta_{Tn})\) are generic points of \([\mathbb{P}_0^{N1}, \ldots, \mathbb{P}_n^{N1}] : m\).
Sylvester Resultant: \( \text{Res}(A(x), B(x)) = c \prod_{\eta, B(\eta) = 0} A(\eta) \).
Poisson-Type Product Formula

- **Sylvester Resultant:** \( \text{Res}(A(x), B(x)) = c \prod_{\eta, B(\eta) = 0} A(\eta). \)

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And \((\eta_{\tau 1}, \ldots, \eta_{\tau n})\) are generic points of \([P_1, \ldots, P_n]\).

**Sylvester Resultant:**
\[
\text{Res}(A(x), B(x)) = A(x)T(x) + B(x)W(x),
\]
where \(\text{deg}(T) < \text{deg}(B), \text{deg}(W) < \text{deg}(A)\).
Poisson-Type Product Formula

- **Sylvester Resultant:** \( \text{Res}(A(x), B(x)) = c \prod_{\eta, B(\eta) = 0} A(\eta) \).

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  \]
  where \(\deg(T) < \deg(B), \deg(W) < \deg(A)\).

- **Differential Resultant:**
  \[
  \delta \text{Res}(u_0, \ldots, u_n) = \sum_{i=0}^{n} \sum_{j=0}^{s-s_i} h_{ij}P_i^{(j)}
  \]
  where \(s_i = \text{ord}(P_i)\) and \(s = s_0 + \cdots + s_n\), and
  \[
  \deg(G_{ij}P_i^{(j)}) \leq (m + 1)\deg(R) \leq (m + 1)^{ns+n+2}.
  \]
**Laurent Diff Essential System**: $P_i, \text{ord}(P_i) = s_i$ and $\text{deg}(P_i) = m_i$.

$R$: the sparse resultant of $P_0, \ldots, P_n$. 

Theorem (Degree Bounds)

1. $\deg(R) \leq \prod_{i=0}^{n} (m_i + 1) h_i + 1 \leq (m + 1) n s + n + 1,$ where $m = \max_i\{m_i\}.$

2. $\deg(G_{ij} P_j) \leq (m + 1) \deg(R) \leq (m + 1) n s + n + 2.$
Laurent Diff Essential System: $P_i$, $\text{ord}(P_i) = s_i$ and $\text{deg}(P_i) = m_i$.

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Laurent Diff Essential System: \( P_i, \text{ord}(P_i) = s_i \) and \( \text{deg}(P_i) = m_i \).

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**Theorem (Degree Bounds)**

1. \( \text{deg}(R) \leq \prod_{i=0}^{n} (m_i + 1)^{h_i+1} \leq (m + 1)^{ns+n+1}, \) where \( m = \max_i \{ m_i \} \).

2. \( R = \sum_{i=0}^{n} \sum_{j=0}^{s-s_i} h_{ij} P_i^{(j)} \) 

\( \text{deg}(G_{ij} P_i^{(j)}) \leq (m + 1) \text{deg}(R) \leq (m + 1)^{ns+n+2} \).
Theorem

\( \mathbb{P}_i (i = 0, \ldots, n) \): generic diff polynomials in \( \mathbb{Y} \) with order \( s_i \), coefficient set \( u_i \), and \( s = \sum_{i=0}^{n} s_i \). Then

\[ \deg(R, u_i) \leq \sum_{k=0}^{s-s_i} \mathcal{M}((Q_{jl})_{j \neq i, 0 \leq l \leq s-s_j}, Q_{i0}, \ldots, Q_{i,k-1}, Q_{i,k+1}, \ldots, Q_{i,s-s_i}). \]

\( Q_{jl} \): Newton polytope of \( \mathbb{P}_j^{(l)} \) as a polynomial in \( y_1^{[s]}, \ldots, y_n^{[s]} \).

\( \mathcal{M}(Q_1, \ldots, Q_n) \): Mixed volume of \( Q_1, \ldots, Q_n \).
BKK Degree Bound for Differential Resultant

**Theorem**

\[ P_i (i = 0, \ldots, n) \text{: generic diff polynomials in } Y \text{ with order } s_i, \text{ coefficient set } u_i, \text{ and } s = \sum_{i=0}^{n} s_i. \text{ Then } \]

\[ \deg(R, u_i) \leq \sum_{k=0}^{s-s_i} M((Q_{jl})_{j \neq i, 0 \leq l \leq s-s_j}, Q_{i0}, \ldots, Q_{i,k-1}, Q_{i,k+1}, \ldots, Q_{i,s-s_i}). \]

\( Q_{jl} \): Newton polytope of \( P_j^{(l)} \) as a polynomial in \( y_1^{[s]}, \ldots, y_n^{[s]} \).

\( M(Q_1, \ldots, Q_n) \): Mixed volume of \( Q_1, \ldots, Q_n \).

**Example**

\[ P_0 = u_{00} + u_{01}y + u_{02}y' + u_{03}y^2 + u_{04}yy' + u_{05}(y')^2 \]

\[ P_1 = u_{10} + u_{11}y + u_{12}y' + u_{13}y^2 + u_{14}yy' + u_{15}(y')^2 \]

Bézout-type degree bound: \( \deg(R) \leq (2 + 1)^4 = 81. \)

BKK-type degree bound: \( \deg(R) \leq 20. \)
Outline of the Algorithm. Knowing order and degree bounds, we compute sparse diff resultant by solving linear equations. Precisely,
Outline of the Algorithm. Knowing order and degree bounds, we compute sparse diff resultant by solving linear equations. Precisely,

1. Search for \( R(u_0, \ldots, u_n) \) with order \( h_i = 0, \ldots, s - s_i \) and with degree from \( D = 1, \ldots, \prod_{i=0}^{n} (m_i + 1)^{h_i+1} \).

2. With fixed \( h_i \) and \( D \), computing coefficients of \( R \) and \( G_{ik} \) by solving linear equations raising from

\[
R(u_0, \ldots, u_n) = \sum_{i=0}^{n} \sum_{k=0}^{h_i} h_{ik} P_i^{(k)}.
\]
An Algorithm for Sparse Differential Resultant

Outline of the Algorithm. Knowing order and degree bounds, we compute sparse diff resultant by solving linear equations. Precisely,

1. Search for $R(u_0, \ldots, u_n)$ with order $h_i = 0, \ldots, s - s_i$ and with degree from $D = 1, \ldots, \prod_{i=0}^n (m_i + 1)^{h_i + 1}$.

2. With fixed $h_i$ and $D$, computing coefficients of $R$ and $G_{ik}$ by solving linear equations raising from

$$R(u_0, \ldots, u_n) = \sum_{i=0}^n \sum_{k=0}^{h_i} h_{ik} \mathbb{P}_i^{(k)}.$$

Theorem (Computing Complexity)

$$O(m^{O(nls^2)}) \text{ } \mathbb{Q}-\text{arithmetic operations.}$$

$n$: number of variables; $s$: order of system; $l$: size of sparse system
**Comparison with differential sparse resultant:**

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Differential Chow Form
Example: Plücker Coordinates

Using coordinates to represent algebraic variety
Using coordinates to represent algebraic variety

Lines in \( \mathbb{P}(3) \):

- **Line** \( L \) := \[
\begin{align*}
    a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3 &= 0 \\
    b_0x_0 + b_1x_1 + b_2x_2 + b_3x_3 &= 0
\end{align*}
\]

\( \iff \) (one to one correspondence)

**Plücker Coordinates**: \( p_{ij} = \left| \begin{array}{cc} a_i & a_j \\ b_i & b_j \end{array} \right| , \ i, j = 0, 1, 2, 3 \)
Example: Plücker Coordinates

Using coordinates to represent algebraic variety

Lines in $\mathbb{P}(3)$:

- Line $\mathbf{L} := \begin{cases} a_0 x_0 + a_1 x_1 + a_2 x_2 + a_3 x_3 = 0 \\ b_0 x_0 + b_1 x_1 + b_2 x_2 + b_3 x_3 = 0 \end{cases}$

$\iff$ (one to one correspondence)

**Plücker Coordinates**: $p_{ij} = \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix}, i, j = 0, 1, 2, 3$

- Plücker coordinate $C = (p^{01}, p^{02}, p^{03}, p^{23}, p^{31}, p^{12}) \in \mathbb{P}(5)$

$C$ represents a line in $\mathbb{P}(3)$

$\iff$

$C$ is on hypersurface $p^{23} p^{01} + p^{31} p^{02} + p^{12} p^{03} = 0$. 
Using coordinates to represent algebraic variety:

- Lines in $\mathbb{P}(3)$: **Plücker Coordinates**
  \[
  \{\text{Line } L \subset \mathbb{P}(3)\} \leftrightarrow \text{hypersurface } p^{23}p^{01} + p^{31}p^{02} + p^{12}p^{03} = 0.
  \]
Background: Chow Form

Using coordinates to represent algebraic variety:

- **Lines in \( \mathbb{P}(3) \):** Plücker Coordinates

  \[ \{ \text{Line } \mathbf{L} \subset \mathbb{P}(3) \} \leftrightarrow \text{hypersurface } p^{23} p^{01} + p^{31} p^{02} + p^{12} p^{03} = 0. \]

- **Subspace of \( d \)-dim in \( \mathbb{P}(n) \):** Grassmann Coordinates

  \[ \{ S_d \subset \mathbb{P}(n) \} \leftrightarrow \text{Grassmann Variety} \]
Background: Chow Form

Using coordinates to represent algebraic variety:

- Lines in $\mathbb{P}(3)$: **Plücker Coordinates**
  \[ \{ \text{Line } L \subset \mathbb{P}(3) \} \leftrightarrow \text{hypersurface } p^{23}p^{01} + p^{31}p^{02} + p^{12}p^{03} = 0. \]

- Subspace of $d$-dim in $\mathbb{P}(n)$: **Grassmann Coordinates**
  \[ \{ S_d \subset \mathbb{P}(n) \} \leftrightarrow \text{Grassmann Variety} \]

- Algebraic Variety in $\mathbb{P}(n)$: **Chow Coordinates**
  \[ \{(r,d)\text{ cycles}\} \leftrightarrow \text{Chow Variety} \]
Using coordinates to represent algebraic variety:

- Lines in $\mathbb{P}(3)$: **Plücker Coordinates**
  \[
  \{\text{Line } L \subset \mathbb{P}(3)\} \longleftrightarrow \text{hypersurface } p^{23}p^{01} + p^{31}p^{02} + p^{12}p^{03} = 0.
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- Subspace of $d$-dim in $\mathbb{P}(n)$: **Grassmann Coordinates**
  \[
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  \]

- Algebraic Variety in $\mathbb{P}(n)$: **Chow Coordinates**
  \[
  \{(r,d)\text{– cycles}\} \longleftrightarrow \text{Chow Variety}
  \]

- **Differential Analog?**
Definition of Differential Chow Form

\( \mathcal{I} \subset \mathcal{F}\{Y\} \): prime differential ideal of dimension \( d \).

**\( d + 1 \) Generic Differential Primes:**

\[
\mathbb{P}_i = u_{i0} + u_{i1}y_1 + \cdots + u_{in}y_n \quad (i = 0, \ldots, d).
\]

\( u_i = (u_{i0}, \ldots, u_{in}) \): coefficient set of \( \mathbb{P}_i \)
Definition of Differential Chow Form

\( \mathcal{I} \subset \mathcal{F}\{Y\} \): prime differential ideal of dimension \( d \).

d + 1 \textbf{Generic Differential Primes:}

\[ P_i = u_{i0} + u_{i1}y_1 + \cdots + u_{in}y_n \quad (i = 0, \ldots, d) \]

\( u_i = (u_{i0}, \ldots, u_{in}) \): coefficient set of \( P_i \)

**Theorem**

By intersecting \( \mathcal{I} \) with the \( d + 1 \) primes, the \textbf{eliminant ideal}

\[ [\mathcal{I}, P_0, \ldots, P_d] \cap \mathcal{F}\{u_0, u_1, \ldots, u_d\} = \text{sat}(\mathcal{F}(u_0, u_1, \ldots, u_d)) \]

is a \textbf{prime ideal of co-dimension one}.

**Differential Chow form** of \( \mathcal{I} \) or \( \mathcal{V}(\mathcal{I}) \):

\[ \mathcal{F}(u_0, u_1, \ldots, u_d) = f(u; u_{00}, \ldots, u_{d0}) \]
Chow form of $\mathcal{I}$: $F(u_0, u_1, \ldots, u_d) = f(u; u_{00}, u_{10}, \ldots, u_{d0})$

Property of Chow form.
- $F(\ldots, u_\sigma, \ldots, u_\rho, \ldots) = (-1)^{r_\sigma \rho} F(\ldots, u_\rho, \ldots, u_\sigma, \ldots)$.
- $\text{ord}(F, u_{00}) \neq 0$, $\text{ord}(F, u_{00}) = \text{ord}(F, u_{ij})$ if $u_{ij}$ occurs in $F$. 

Order of Chow form: $\text{ord}(F) = \text{ord}(f, u_{00})$. 

Theorem (Order of Chow Form) $\text{ord}(F) = \text{ord}(\mathcal{I})$. 

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Chow form of $\mathcal{I}$: $F(u_0, u_1, \ldots, u_d) = f(u; u_{00}, u_{10}, \ldots, u_{d0})$

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Property of Chow form.
- $F(\ldots, u_{\sigma}, \ldots, u_{\rho}, \ldots) = (-1)^{r_{\sigma \rho}} F(\ldots, u_{\rho}, \ldots, u_{\sigma}, \ldots)$.
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Order of Chow form: $\text{ord}(F) = \text{ord}(f, u_{00})$.

Theorem (Order of Chow Form)

$\text{ord}(F) = \text{ord}(\mathcal{I})$. 
Differentially homogenous diff poly of degree $m$:

$$p(ty_0, ty_1 \ldots, ty_n) = t^m p(y_0, y_1, \ldots, y_n)$$

**Theorem**

$F(u_0, u_1, \ldots, u_d)$: differential Chow form of $V$.

Then $F(u_0, u_1, \ldots, u_d)$ is differentially homogenous of degree $r$ in each set $u_i$ and $F$ is of total degree $(d + 1)r$.

**Definition (Differential degree)**

$r$ as above is defined to be the **differential degree** of $\mathcal{I}$, which is an invariant of $\mathcal{I}$ under invertible linear transformations.
Factorization of Differential Chow Form

$V$: a diff irreducible variety of dimension $d$ and order $h$.

$F(u_0, u_1, \ldots, u_d)$: the differential Chow form of $V$.

Theorem ($F(u_0, u_1, \ldots, u_d)$ can be uniquely factored)

$$F(u_0, u_1, \ldots, u_d) = A(u_0, u_1, \ldots, u_d) \prod_{\tau=1}^{g} (u_{00} + \sum_{\rho=1}^{n} u_{0\rho} \xi_{\tau\rho})^{(h)}$$

$$= A(u_0, u_1, \ldots, u_d) \prod_{\tau=1}^{g} \mathbb{P}_0(\xi_{\tau 1}, \ldots, \xi_{\tau n})^{(h)}$$

where $g = \deg(F, u_{00}^{(h)})$ and $\xi_{\tau\rho}$ are in an extension field of $F$.

And the points $(\xi_{\tau 1}, \ldots, \xi_{\tau n}) (\tau = 1, \ldots, g)$ are generic points of the variety $V$. 
Leading Differential Degree

Differential primes:
\[ P_i := u_{i0} + u_{i1} y_1 + \cdots + u_{in} y_n \quad (i = 1, \ldots, d), \]

Algebraic primes:
\[ a_{P_0} := u_{00} + u_{01} y_1 + \cdots + u_{0n} y_n, \]
\[ a_{P_0}^{(s)} := u_{00}^{(s)} + \sum_{j=1}^{n} \sum_{k=0}^{s} \binom{s}{k} u_{0j}^{(k)} y_j^{(s-k)} \quad (s = 1, 2, \ldots) \]

Theorem
\((\xi_{\tau 1}, \ldots, \xi_{\tau n}) \quad (\tau = 1, \ldots, g)\) are the only elements of \( V \) which lie on \( P_1, \ldots, P_d \) as well as on \( a_{P_0}, a_{P_0}', \ldots, a_{P_0}^{(h-1)} \).

Definition
Number \( g \) is defined to be the leading diff degree of \( V \) or \( I \).
A diff variety $V$ has **index** $(n, d, h, g, m)$ if $V \subset \mathcal{E}^n$ has **invariants**: dim $d$, order $h$, leading diff degree $g$, and diff degree $m$. 

**Chow Form of $V$**: $F \sum_i s_i V_i$, $V_i$ irreducible of index $(d, h, g, m)$.

**Index of $V$**: $(\sum_i s_i g_i, \sum_i s_i m_i)$. 

**Definition**: A diff variety $V$ is a Chow Variety if $(\bar{a}_i) \in V \iff \bar{F}$ with coef $(\bar{a}_i)$: Chow form with index $(n, d, h, g, m)$. 

$V$: diff cycle of index $(n, d, h, g, m)$.

**Chow Coordinate of $V$**: $(\bar{a}_i)$

In affine case, Chow Variety is a constructible set.
A diff variety $V$ has **index** $(n, d, h, g, m)$ if $V \subset \mathcal{E}^n$ has **invariants**: dim $d$, order $h$, leading diff degree $g$, and diff degree $m$.

**Diff Cycle**: $V = \sum_i s_i V_i$, $V_i$ irreducible of index $(d, h, g, m)$
- Chow Form of $V : \prod_i F_i^{s_i}$, $F_i$ Chow form of $V_i$
- Index of $V : (d, h, \sum_i s_i g_i, \sum_i s_i m_i)$
A diff variety $V$ has **index** $(n, d, h, g, m)$ if $V \subset \mathcal{E}^n$ has **invariants**: dim $d$, order $h$, leading diff degree $g$, and diff degree $m$.

**Diff Cycle**: $V = \sum_i s_i V_i$, $V_i$ irreducible of index $(d, h, g, m)$

- Chow Form of $V$: $\prod_i F_i^{s_i}$, $F_i$ Chow form of $V_i$
- Index of $V$: $(d, h, \sum_i s_ig_i, \sum_i s_im_i)$

**Definition**

A diff variety $\mathcal{V}$ is a **Chow Variety** if $(\bar{a}_i) \in \mathcal{V}$

$\iff$ $\bar{F}$ with coef $(\bar{a}_i)$: Chow form with index $(n, d, h, g, m)$.

$\iff$ $V$: diff cycle of index $(n, d, h, g, m)$. 
A diff variety $V$ has **index** $(n, d, h, g, m)$ if $V \subset \mathcal{E}^n$ has **invariants**: $\dim d$, order $h$, leading diff degree $g$, and diff degree $m$.

**Diff Cycle**: $V = \sum_i s_i V_i$, $V_i$ irreducible of index $(d, h, g, m)$
- Chow Form of $V : \prod_i F_i^{s_i}$, $F_i$ Chow form of $V_i$
- Index of $V : (d, h, \sum_i s_i g_i, \sum_i s_i m_i)$

**Definition**

A diff variety $\mathcal{V}$ is a **Chow Variety** if $(\bar{a}_i) \in \mathcal{V}$
$\Leftrightarrow \bar{F}$ with coef $(\bar{a}_i)$: Chow form with index $(n, d, h, g, m)$.
$\Leftrightarrow V$: diff cycle of index $(n, d, h, g, m)$.

**Chow Coordinate** of $V$: $(\bar{a}_i)$

In affine case, Chow Variety is a constructible set.
Theorem (Gao-Li-Yuan, 2013)

In the case \( g = 1 \), the differential Chow variety exists.

**Difficulty for the general case:** Eliminating both differential and algebraic variables.
Theorem (Gao-Li-Yuan, 2013)

In the case $g = 1$, the differential Chow variety exists.

**Difficulty for the general case:** Eliminating both differential and algebraic variables.

Theorem (Freitag-Li-Scanlon, 2015)

The differential Chow variety exists.

**Key Ideas:** Use prolongation admissible varieties and prolongation sequences to reduce the construction to the algebraic case. Definability in $ACF$ and $DCF_0$ is also used.
Differential Toric Variety
Differential Toric Variety

- $\mathcal{A} = \{M_0, \ldots, M_l\}$: a set of diff monomials

Consider the map $\phi_A : (E \wedge)^n \rightarrow P(l) \eta(M_0(\eta), M_1(\eta), \ldots, M_l(\eta))$

**Definition**
The image of $\phi_A$ is called the differential toric variety w.r.t. $A$, denoted by $X_A$.

$X_A$ is an irreducible projective diff variety.

**Theorem**
$Res A$: Sparse differential resultant of $P_i = \sum_j u_{ij} M_j, i = 0, \ldots, n$ is the differential Chow form of $X_A$. 

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\( \mathcal{A} = \{M_0, \ldots, M_l\} \): a set of diff monomials

Consider the map

\[
\phi_\mathcal{A} : (\mathcal{E}^\wedge)^n \rightarrow \mathbb{P}(I)
\]

\[
\eta \mapsto (M_0(\eta), M_1(\eta), \ldots, M_l(\eta))
\]

Definition

The image of \( \phi_\mathcal{A} \) is called the differential toric variety w.r.t. \( \mathcal{A} \), denoted by \( X_\mathcal{A} \).

\( X_\mathcal{A} \) is an irreducible projective diff variety.

Theorem

Res\( \mathcal{A} \): Sparse differential resultant of \( P_i = \sum u_{ij} M_j \), \( i = 0, \ldots, n \) is the differential Chow form of \( X_\mathcal{A} \).
Differential Toric Variety

- $\mathcal{A} = \{M_0, \ldots, M_l\}$: a set of diff monomials

Consider the map

$$\phi_{\mathcal{A}} : (\mathcal{E}^\wedge)^n \rightarrow \mathbf{P}(l)$$

$$\eta \mapsto (M_0(\eta), M_1(\eta), \ldots, M_l(\eta))$$

Definition

The image of $\phi_{\mathcal{A}}$ is called the **differential toric variety w.r.t.** $\mathcal{A}$, denoted by $X_{\mathcal{A}}$. $X_{\mathcal{A}}$ is an irreducible projective diff variety.
Differential Toric Variety

- \( \mathcal{A} = \{M_0, \ldots, M_l\} \): a set of diff monomials

Consider the map

\[ \phi_{\mathcal{A}} : (\mathcal{E}^\wedge)^n \rightarrow \mathbb{P}(l) \]

\[ \eta \mapsto (M_0(\eta), M_1(\eta), \ldots, M_l(\eta)) \]

**Definition**

The image of \( \phi_{\mathcal{A}} \) is called the **differential toric variety w.r.t.** \( \mathcal{A} \), denoted by \( X_{\mathcal{A}} \). \( X_{\mathcal{A}} \) is an irreducible projective diff variety.

**Theorem**

\( \text{Res}_{\mathcal{A}} : \text{Sparse differential resultant} \) of \( P_i = \sum_j u_{ij} M_j, i = 0, \ldots, n \) is the differential Chow form of \( X_{\mathcal{A}} \).
Example

Let $n = 1$ and a set of monomials $A = \{y_1, y'_1, y_1^2\}$.

**Toric Variety**: all possible values of a set of monomials $A$

$$X_A = \{(y_1, y'_1, y_1^2) \mid y_1 \in \mathcal{E}\}$$

Defining equations of the toric variety

$$X_A = \text{Zero}(\text{sat}(z_1z_2 - (z_0z'_2 - z'_0z_2))).$$

The sparse differential resultant $\text{Res}_A$ is equal to the differential Chow form of $X_A$. 
Example

Let $n = 1$ and a set of monomials $\mathcal{A} = \{y_1, y'_1, y''_1\}$.

**Toric Variety**: all possible values of a set of monomials $\mathcal{A}$

$$X_\mathcal{A} = \{(y_1, y'_1, y''_1) \mid y_1 \in \mathcal{E}\}$$

Defining equations of the toric variety

$$X_\mathcal{A} = \text{Zero}(\text{sat}(z_1z_2 - (z_0z'_2 - z'_0z_2))).$$

The sparse differential resultant $\text{Res}_\mathcal{A}$ is equal to the differential Chow form of $X_\mathcal{A}$.

The defining ideal of a diff toric variety is not **binomial**!
Binomial $\sigma$-ideal and Toric $\sigma$-variety
In this talk, **difference field** \((\mathcal{F}, \sigma)\): \(\sigma: \mathcal{F} \to \mathcal{F}\) is a field automorphism. \(\mathcal{F}\) is also assumed to be algebraically closed.

Example: \(\mathbb{Q}(x) : \sigma(x) = x + 1\).
In this talk, **difference field** \((\mathcal{F}, \sigma)\): \(\sigma : \mathcal{F} \to \mathcal{F}\) is a field automorphism. \(\mathcal{F}\) is also assumed to be algebraically closed.

Example: \(\mathbb{Q}(x) : \sigma(x) = x + 1\).

**\(\sigma\)-Exponent:** For \(p = \sum_{i=0}^{s} c_i x^i \in \mathbb{Z}[x]\), denote \(a^p = \prod_{i=0}^{s} (\sigma^i a)^{c_i}\).

Example: \(a^{3x^2-1} = (\sigma^2(a))^3 / a\)
In this talk, **difference field** \((\mathcal{F}, \sigma)\): \(\sigma : \mathcal{F} \Rightarrow \mathcal{F}\) is a field automorphism. \(\mathcal{F}\) is also assumed to be algebraically closed.

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**\(\sigma\)-Exponent:** For \(p = \sum_{i=0}^{s} c_i x^i \in \mathbb{Z}[x]\), denote \(a^p = \prod_{i=0}^{s} (\sigma^i a)^{c_i}\).

Example: \(a^{3x^2-1} = (\sigma^2(a))^3 / a\)

\(Y = \{y_1, \ldots, y_n\}\): \(\sigma\)-indeterminates

**\(\sigma\)-monomial with support \(f\):**
\[
Y^f = \prod_{i=1}^{n} y_i^{f_i}
\]
where \(f = (f_1, \ldots, f_n)^\top \in \mathbb{N}[x]^n\)

\(\mathcal{F}\{Y\}\): \(\sigma\)-polynomial ring
\( \mathbb{Z}[x] \) Lattice: \( \mathbb{Z}[x] \) module in \( \mathbb{Z}[x]^n \)

Two kinds of representations:
- **Generators:** \( L = \text{Span}_{\mathbb{Z}[x]}\{f_1, \ldots, f_s\} = (f_1, \ldots, f_s), f_i \in \mathbb{Z}[x]^n \)
- **Matrix representation:** \( F = [f_1, \ldots, f_s]_{n \times s} \)

**Rank of** \( L \): \( \text{rk}(F) \)
Binomial $\sigma$-ideal

$\sigma$-binomial: $f = aY^a + bY^b$, $a, b \in \mathbb{N}[x]^n$, $a, b \in \mathcal{F}$.

Normal Form: $f = aY^g(Y^{f^+} - cY^{f^-})$,

$f \in \mathbb{Z}[x]^n$ and $f = a - b = f^+ - f^-$ for $f^+, f^- \in \mathbb{N}[x]^n$. 
Binomial $\sigma$-ideal

$\sigma$-binomial: $f = aY^a + bY^b$, $a, b \in \mathbb{N}[x]^n$, $a, b \in \mathcal{F}$.

Normal Form: $f = aY^g(Y^{f^+} - cY^{f^-})$,

$f \in \mathbb{Z}[x]^n$ and $f = a - b = f^+ - f^-$ for $f^+, f^- \in \mathbb{N}[x]^n$.

Normal binomial $\sigma$-ideal $\mathcal{I}$:

- $\mathcal{I}$ is generated by $\sigma$-binomials
- $Mp \in \mathcal{I} \Rightarrow p \in \mathcal{I}, M: \sigma$-monomial
Binomial $\sigma$-ideal

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Normal binomial $\sigma$-ideal $\mathcal{I}$:

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Partial Character: A homomorphism from a $\mathbb{Z}[x]$ lattice $L_\rho$ to the multiplicative group $\mathcal{F}^*$ satisfying $\rho(xf) = \sigma(\rho(f))$.

Lemma

$\mathcal{I}$ is a normal binomial $\sigma$-ideal $\iff$

$\mathcal{I} = \mathcal{I}(\rho) = \{Y^{f^+} - \rho(f)Y^{f^-} | f \in L_\rho\}$ for a partial character $\rho$. 
Criteria for Normal LB$\sigma$-ideal

Definition

A $\mathbb{Z}[x]$ lattice $L$ in $\mathbb{Z}[x]^n$ is called

- **$\mathbb{Z}$-saturated** if, for $a \in \mathbb{Z}$ and $f \in \mathbb{Z}[x]^n$, $af \in L$ implies $f \in L$.
- **$x$-saturated** if, for $f \in \mathbb{Z}[x]^n$, $xf \in L$ implies $f \in L$.
- **$M$-saturated** if, for $f \in \mathbb{Z}[x]^n$ and $m \in \mathbb{N}$, $mf \in L \Rightarrow (x - o_m)f \in L$. 

\[ \text{Theorem} \]

Let $\rho$ be a partial character over $\mathbb{Z}[x]^n$.

- $L_\rho$ is $\mathbb{Z}$-saturated $\iff$ $I(\rho)$ is prime
- $L_\rho$ is $x$-saturated $\iff$ $I(\rho)$ is reflexive
- If $\langle I(\rho) \rangle : M \neq [1]$, then $L_\rho$ is $M$-saturated $\iff$ $I(\rho)$ is well-mixed
- If $\{I(\rho)\} : M \neq [1]$, then $L_\rho$ is $x$-$M$-saturated $\iff$ $I(\rho)$ is perfect
Criteria for Normal LB_{\sigma}-ideal

**Definition**

A \( \mathbb{Z}[x] \) lattice \( L \) in \( \mathbb{Z}[x]^n \) is called

- **\( \mathbb{Z} \)-saturated** if, for \( a \in \mathbb{Z} \) and \( f \in \mathbb{Z}[x]^n \), \( af \in L \) implies \( f \in L \).
- **\( x \)-saturated** if, for \( f \in \mathbb{Z}[x]^n \), \( xf \in L \) implies \( f \in L \).
- **\( M \)-saturated** if, for \( f \in \mathbb{Z}[x]^n \) and \( m \in \mathbb{N} \), \( mf \in L \) implies \( (x - o_m)f \in L \).

**Theorem**

Let \( \rho \) be a partial character over \( \mathbb{Z}[x]^n \).

- \( L_{\rho} \) is **\( \mathbb{Z} \)-saturated** \( \iff \) \( \mathcal{I}(\rho) \) is prime
- \( L_{\rho} \) is **\( x \)-saturated** \( \iff \) \( \mathcal{I}(\rho) \) is reflexive
- If \( \langle \mathcal{I}(\rho) \rangle : \mathbb{M} \neq [1] \), then \( L_{\rho} \) is **\( M \)-saturated** \( \iff \) \( \mathcal{I}(\rho) \) is well-mixed
- If \( \{ \mathcal{I}(\rho) \} : \mathbb{M} \neq [1] \), then \( L_{\rho} \) is **\( x \)-\( M \)-saturated** \( \iff \) \( \mathcal{I}(\rho) \) is perfect
For $\alpha = \{\alpha_1, \ldots, \alpha_n\}$, $\alpha_i \in \mathbb{Z}[x]^m$, $i = 1, \ldots, n$
Define a map $\phi_\alpha : (\mathbb{A}^*)^m \mapsto (\mathbb{A}^*)^n$:
$T = (t_1, \ldots, t_m) \mapsto T^\alpha = (T^{\alpha_1}, \ldots, T^{\alpha_n})$. 

Toric $\sigma$-variety
For \( \alpha = \{ \alpha_1, \ldots, \alpha_n \} \), \( \alpha_i \in \mathbb{Z}[x]^m \), \( i = 1, \ldots, n \)
Define a map \( \phi_\alpha : (\mathbb{A}^*)^m \mapsto (\mathbb{A}^*)^n : \)
\[
T = (t_1, \ldots, t_m) \mapsto T^\alpha = (T^{\alpha_1}, \ldots, T^{\alpha_n}).
\]

**Toric Variety** \( X_\alpha \): the Cohn closure of \( \phi_\alpha((\mathbb{C}^*)^m) \) in \( (\mathbb{A})^n \).

- Toric Variety: \( \sigma \)-variety parameterized by \( \sigma \)-monomials.
- \( X_\alpha \) is an irreducible \( \sigma \)-variety of dim \( \text{rk}(A) \), where
  \[
  A = [\alpha_1, \ldots, \alpha_n]_{m \times n}
  \]
Toric $\sigma$-variety

For $\varpi = \{\alpha_1, \ldots, \alpha_n\}$, $\alpha_i \in \mathbb{Z}[x]^m$, $i = 1, \ldots, n$
Define a map $\phi_{\varpi} : (\mathbb{A}^*)^m \hookrightarrow (\mathbb{A}^*)^n$
$\mathcal{T} = (t_1, \ldots, t_m) \mapsto \mathcal{T}_{\varpi} = (T^{\alpha_1}, \ldots, T^{\alpha_n})$.

Toric Variety $X_{\varpi}$: the Cohn closure of $\phi_{\varpi}((C^*)^m)$ in $(\mathbb{A})^n$.

- Toric Variety: $\sigma$-variety parameterized by $\sigma$-monomials.
- $X_{\varpi}$ is an irreducible $\sigma$-variety of dim $\text{rk}(A)$, where $A = [\alpha_1, \ldots, \alpha_n]_{m \times n}$

Example

The support: $\varpi = \{[1, 1]^\tau, [x, x]^\tau, [0, 1]^\tau\}$.
The $\sigma$-monomial: $(t_1 t_2, t_1^x t_2^x, t_2)$.
The map: $y_1 = t_1 t_2$, $y_2 = t_1^x t_2^x$, $y_3 = t_2$
Toric $\sigma$-variety: $X_{\varpi} : y_1^x - y_2 = 0$. Note that $y_3$ is free.
Toric $\sigma$-ideal

**Toric $\mathbb{Z}[x]$ Lattice** $L$: $pf \in L \Rightarrow f \in L$  ($p \in \mathbb{Z}[x]$ and $f \in \mathbb{Z}[x]^n$)

Example (Reflexive prime but not toric)

Let $L = ([1-x, x-1])$. Since $[1-x, x-1] = (x-1) \cdot [1, -1]$, $L$ is not $\mathbb{Z}[x]$ toric. The $\sigma$-ideal $I^+(L) = \langle y_1 x^i y_2 x^j - y_1 x^j y_2 x^i; 0 \leq i \leq j \in \mathbb{N} \rangle$ is reflexive prime but not toric.
Toric $\sigma$-ideal

**Toric $\mathbb{Z}[x]$ Lattice** $L$: $pf \in L \Rightarrow f \in L$ ($p \in \mathbb{Z}[x]$ and $f \in \mathbb{Z}[x]^n$)

**Toric $\sigma$-ideal**: $\mathcal{I}^+(L) = [y^f - y^{-f} | f \in L]$, where $L$ is a toric $\mathbb{Z}[x]$ lattice.
- $\mathcal{I}^+(L)$ is reflexive and prime $\sigma$-ideal of dimension $\text{rk}(L)$. 

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Toric $\sigma$-ideal

**Toric $\mathbb{Z}[x]$ Lattice** $L$: $pf \in L \Rightarrow f \in L$  ($p \in \mathbb{Z}[x]$ and $f \in \mathbb{Z}[x]^n$)

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- $\mathcal{I}^+(L)$ is reflexive and prime $\sigma$-ideal of dimension $\text{rk}(L)$.

**Theorem**

A $\sigma$-variety $V$ is toric iff $\mathbb{I}(V)$ is a toric $\sigma$-ideal.
Toric $\sigma$-ideal

**Toric $\mathbb{Z}[x]$ Lattice** \(L: pf \in L \Rightarrow f \in L\) (\(p \in \mathbb{Z}[x]\) and \(f \in \mathbb{Z}[x]^n\))

**Toric $\sigma$-ideal:** \(\mathcal{I}^+(L) = [\mathbb{Y} f^+ - \mathbb{Y} f^- | f \in L]\), where \(L\) is a toric \(\mathbb{Z}[x]\) lattice.

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**Theorem**

A $\sigma$-variety $V$ is toric iff $\mathcal{I}(V)$ is a toric $\sigma$-ideal.

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**Example (Reflexive prime but not toric)**

Let \(L = ([1 - x, x - 1]^T)\).

Since \([1 - x, x - 1] = (x - 1) \cdot [1, -1]\), \(L\) is not $\mathbb{Z}[x]$ toric.

The $\sigma$-ideal \(\mathcal{I}^+(L) = [y_1^{x^i} y_2^{x^j} - y_1^{x^j} y_2^{x^i}; 0 \leq i \leq j \in \mathbb{N}]\) is reflexive prime but not toric.
Conversion between $V = X_\alpha$ and $\mathcal{I}(V) = I^+(\rho_L)$

(1) **Implicitization:**

Given $X_\alpha (\alpha = (\alpha_1, \ldots, \alpha_n)) \Rightarrow \mathcal{I}(V) \subset \mathcal{F}\{Y\}$

$A = [\alpha_1, \ldots, \alpha_n]_{m \times n}$

$K_A = \text{ker}(A) = (f_1, \ldots f_s)$: a toric $\mathbb{Z}[x]$ lattice; Gröbner basis

$\mathcal{I}(X_\alpha) = I^+(K_A) = \text{sat}(Y_f^+ - Y_f^-, \ldots, Y_{fs}^+ - Y_{fs}^-)$
Conversion between $V = X_\alpha$ and $\mathbb{I}(V) = L_+^+(\rho_L)$

(1) Implicitization:
Given $X_\alpha (\alpha = (\alpha_1, \ldots, \alpha_n)) \Rightarrow \mathbb{I}(V) \subset \mathcal{F}\{Y\}$

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$\mathbb{I}(X_\alpha) = L_+^+(K_A) = \text{sat}(Yf_1^+ - Yf_1^-, \ldots, Yf_s^+ - Yf_s^-)$

(2) Parametrization:
Given $I = \text{sat}(Yf_1^+ - Yf_1^-, \ldots, Yf_s^+ - Yf_s^-) \Rightarrow X_\alpha = V(I)$

$L_\rho = (f_1, \ldots, f_s)$

$F = [f_1, \ldots, f_s]_{n \times s} \in \mathbb{Z}[x]^{n \times s}$

$K_F = \{X \in \mathbb{Z}[x]^n | F^tX = 0\}$ is a free $\mathbb{Z}[x]$ module.

$K_F$ has a basis $\{h_1, \ldots, h_{n-r}\}$

$H = [h_1, \ldots, h_{n-r}]_{n \times (n-r)}$

$\alpha = \{\alpha_1, \ldots, \alpha_n\} \in \mathbb{Z}[x]^{n-r}$ the rows of $H$.

If $L_\rho$ is a toric $\mathbb{Z}[x]$ lattice, then $X_\alpha = V(I)$
Affine $\mathbb{N}[x]$ module:

$\beta = \{\beta_1, \ldots, \beta_s\} \subseteq \mathbb{Z}[x]^m$

$M = \mathbb{N}[x](\beta) = \{\sum_{i=1}^{s} a_i \beta_i \mid a_i \in \mathbb{N}[x]\} \subseteq \mathbb{Z}[x]^m$.

Affine $\sigma$-algebra

$\mathcal{F}\{M\} = \{\sum_{\mathbf{f} \in M} a_{\mathbf{f}} T^\mathbf{f} \mid a_{\mathbf{f}} \in \mathcal{F}, a_\beta \neq 0 \text{ for finitely many } \beta\}$.
Coordinate Ring of Toric $\sigma$-variety

**Affine $\mathbb{N}[x]$ module:**

$\beta = \{\beta_1, \ldots, \beta_s\} \subset \mathbb{Z}[x]^m$

$M = \mathbb{N}[x](\beta) = \{\sum_{i=1}^{s} a_i \beta_i | a_i \in \mathbb{N}[x]\} \subset \mathbb{Z}[x]^m$.

**Affine $\sigma$-algebra**

$$\mathcal{F}\{M\} = \left\{ \sum_{\mathbf{f} \in M} a_\mathbf{f} T^\mathbf{f} | a_\mathbf{f} \in \mathcal{F}, a_\beta \neq 0 \text{ for finitely many } \beta \right\}.$$

**Theorem.** $X$ is a toric $\sigma$-variety

$\iff X \cong \text{Spec}^\sigma(\mathbb{Q}\{M\})$ for an affine $\mathbb{N}[x]$ module $M$.

$\iff$ the coordinate ring of $X$ is $\mathbb{Q}\{M\}$.
Toric $\sigma$-variety in terms of group action

The map $\phi_\alpha : (\mathbb{A}^*)^m \rightarrow (\mathbb{A}^*)^n$:

**Quasi $\sigma$-torus:** $T_\alpha = \phi_\alpha((\mathbb{A}^*)^m)$

In the algebraic case, $T_\alpha$ (the torus) is a variety: $T_\alpha = X_\alpha \cap (\mathbb{C}^*)^m$

This is not valid in the difference case.
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This is not valid in the difference case.

$\sigma$-torus: a $\sigma$-variety isomorphic to the Cohn $*$-closure of $T_\alpha$ in $(\mathbb{A}^*)^n$.

1) $T^*$ is a $\sigma$-variety which is open in $X_\alpha$.
2) A $\sigma$-torus is group under componentwise product.
The map $\phi_\alpha : (\mathbb{A}^*)^m \rightarrow (\mathbb{A}^*)^n$:

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**$\sigma$-torus:** a $\sigma$-variety isomorphic to the Cohn $*$-closure of $T_\alpha$ in $(\mathbb{A}^*)^n$.

1. $T^*$ is a $\sigma$-variety which is open in $X_\alpha$.
2. A $\sigma$-torus is group under componentwise product.

**Theorem (Toric $\sigma$-variety in terms of group action)**

A $\sigma$-variety $X$ is toric iff $X$ contains a $\sigma$-torus $T^*$ as an open subset and with a group action of $T^*$ on $X$ extending the natural group action of $T^*$ on itself.
Sparse differential/difference resultant is defined and properties similar to that of the Sylvester resultant are given. A single exponential algorithm to compute the sparse differential resultant is given.

Differential/difference Chow Form is defined and its basic properties are established.

Difference binomial ideals and difference toric varieties are introduced, which connects the difference Chow form and difference sparse resultant.
Thanks!


