Integro-differential Algebras of Combinatorial species

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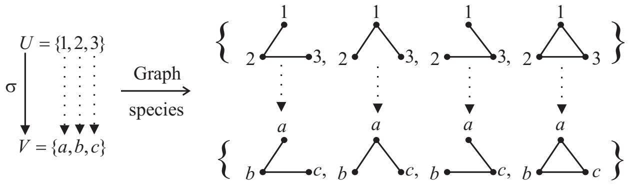
Outline

- The definition and some operators on (plane) species
- Integro-differential algebra
- Integro-differential algebra of plane species
- The definition and some operators on linear species
- Integro-differential algebra of linear species

An example

- Let U be a finite set. Denote by $\mathcal{G}[U]$ the set of all simple connected graphs $g = (\gamma, U)$ (i.e., undirected graphs without loops or multiple edges) on U, where γ is the edge set.
- Each bijection $\sigma : U \to V$ induces a function (bijection) $\mathcal{G}[\sigma] : \mathcal{G}[U] \to \mathcal{G}[V]$. Then $\mathcal{G}$ is called the graph species.
- For example, let $U = \{1, 2, 3\}$ and $V = \{a, b, c\}$.



The definition of species

- The concept of structure is fundamental, recurring in all branches of mathematics, as well as in computer science.
- Informally, a species is a class of finite structures on arbitrary finite sets which is closed under arbitrary “relabellings” along bijections.
- Formally, A (plane) species of structures is a rule F which
  1. produces, for each finite set U, a finite set $F[U]$,
  2. produces, for each bijection $\sigma : U \to V$, a function $F[\sigma] : F[U] \to F[V]$.

  The functions $F[\sigma]$ should further satisfy the following functorial properties:

  (i) for all bijections $\sigma : U \to V$ and $\tau : V \to W$,

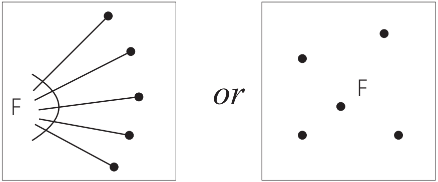
$$F[\sigma \circ \tau] = F[\sigma] \circ F[\tau]. \tag{1}$$

  (ii) for the identity map $\mathrm{id}_U : U \to U$,

$$F[\mathrm{id}_U] = \mathrm{id}_{F[U]}. \tag{2}$$

The definition of structure

- (A. Joyal 1981) In categorial terms, a species is a functor from the category of finite sets to the category of finite sets, where the morphisms are bijections.
- The following is a drawing of an F-structure, where the block dots represent the elements in the underlying set U and the circular arc labeled F represent the F-structure.



- An element $s \in F[U]$ is called an F-structure on U.
- The function $F[\sigma]$ is called the transport of F-structures along σ.

Some examples

- Let U be a finite set. In each of the following cases, the transport of structures $F[\sigma]$ is obvious.
- The empty species, denoted by 0, defined by $0[U] = \emptyset$ for all U.
- The species 1, characteristic of the empty set, defined by

$$1[U] := \begin{cases} \{U\} & \text{if } U = \emptyset, \\ \emptyset & \text{otherwise.} \end{cases}$$

- The species E_n with $n \geq 1$, characteristic of sets of cardinality n, defined by

$$E_n[U] := \begin{cases} \{U\} & \text{if } |U| = n, \\ \emptyset & \text{otherwise.} \end{cases}$$

  In particular, write $X := E_1$, characteristic of singletons.

Some examples

- The set species E, defined by $E[U] := \{U\}$. For each finite set U, there is a unique E-structure, namely the set U itself.
- The tree species α, defined by

$$\alpha[U] = \{\text{all trees on } U\}.$$

- The cyclic species C, defined by

$$C[U] = \{\text{all oriented cycles on } U\}.$$

- The linear order species L, defined by

$$L[U] = \{\text{all linear orders on } U\}.$$

- The permutation species $\mathcal{S}$, defined by

$$\mathcal{S}[U] = \{\text{all permutations on } U\}.$$

The combinatorial equality of species

- In categorical terms, an isomorphism of species is an invertible natural transformation of functors.
- Let F and G be two species of structures. An isomorphism of F to G is a family of bijections $\varphi_U : F[U] \to G[U]$ which satisfies the following naturality condition: For any bijection $\sigma : U \to V$ between two finite sets, the following diagram commutes:

$$\begin{array}{ccc} F[U] & \xrightarrow{\varphi_U} & G[U] \\ {\scriptstyle F[\sigma]}\downarrow & & \downarrow{\scriptstyle G[\sigma]} \\ F[V] & \xrightarrow{\varphi_V} & G[V] \end{array}$$

- Two isomorphic species essentially possess the ”same” combinatorial properties. Henceforth they will be considered as equal in the combinatorial algebra developed in the following.
- Thus we write $F = G$ to indicate that F and G are isomorphic.

The sum operator

- Let F and G be two species of structures. The species $F+G$, called the sum of F and G, is defined as follows:
- For any finite set U, one has

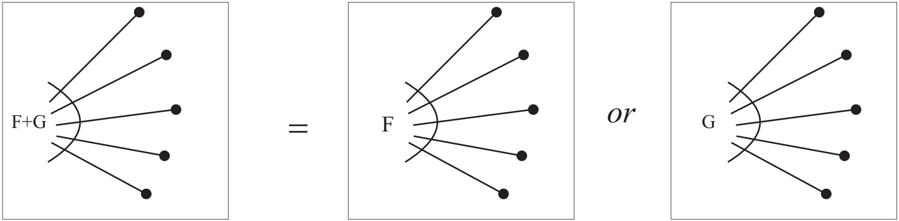
$$(F+G)[U] := F[U] + G[U], \quad \text{(disjoint union)}.$$

- The transport along a bijection $\sigma : U \to V$ is carried out by setting, for any $(F+G)$-structure s on U, one has

$$(F+G)[\sigma](s) := \begin{cases} F[\sigma](s) & \text{if } s \in F[U], \\ G[\sigma](s) & \text{if } s \in G[U]. \end{cases}$$

The illustration of the sum operator

- A $(F + G)$-structure can be represented as:



A remark of the sum

- It is possible that $F[U] \cap G[U] \neq \emptyset$. For example, let F be the graph species and G be the tree species.
- If $F[U] \cap G[U] \neq \emptyset$, then $F[U] + G[U]$ is still the disjoint union. To obtain this, we must at first form distinct copies of the sets $F[U]$ and $G[U]$. A standard way of distinguishing the F-structures from the G-structures is to replace the set $F[U]$ by the isomorphic set $F[U] \times \{1\}$ and $G[U]$ by $G[U] \times \{2\}$, and to set

$$(F + G)[U] := (F[U] \times \{1\}) \sqcup (G[U] \times \{2\}).$$

Some properties of the sum operator

- Recall $(F + G)[U] = F[U] + G[U]$.
- The operation of sum is commutative and associative, up to isomorphism, that is,

$$F + G = G + H \text{ and } (F + G) + H = F + (G + H).$$

- The empty species 0 is a neutral element for the operation of sum:

$$F + 0 = 0 + F = F$$

for all species F.

Extend the sum operator to summable familiy

- A family $(F_i)_{i\in I}$ of species is said to be summable if for each finite set U, $F_i[U] = \emptyset$ except for finitely many indices $i \in I$. The sum of a summable family $(F_i)_{i\in I}$ is denoted by $\sum_{i\in I} F_i$.
- For example, the family $(E_n)_{n\geq 0}$ is summable. Recall E_n is the species characteristic of sets of cardinality n with $n \geq 0$:

$$E_n[U] := \begin{cases} \{U\} & \text{if } |U| = n, \\ \emptyset & \text{otherwise.} \end{cases}$$

- $E = E_0 + E_1 + E_2 + \cdots$, where E is the set species: Let U be a finite set U and suppose $|U| = n$. Then we have $E[U] = \{U\}$ and

$$\begin{aligned} &(E_0 + E_1 + E_2 + \cdots)[U] \\ =&E_0[U] + E_1[U] + E_2[U] + \cdots + E_n[U] + \cdots = E_n[U] = \{U\}. \end{aligned}$$

Extend the sum operator to summable familiy

- In general, for any species F, we have

$$F = F_0 + F_1 + F_2 + \cdots ,$$

where F_n is the species F restricted to cardinality n, $n \geq 0$. More precisely,

$$F_n[U] = \begin{cases} F[U], & \text{if } |U| = n, \\ \emptyset, & \text{otherwise.} \end{cases}$$

The product operator

- Let F and G be two species of structures. The product of F and G, denote by $F \cdot G$ or FG, is defined as follows:
- For any finite set U

$$(FG)[U] := \sum_{U_1 + U_2 = U} F[U_1] \times G[U_2], \quad \text{(cartisian product of sets)}$$

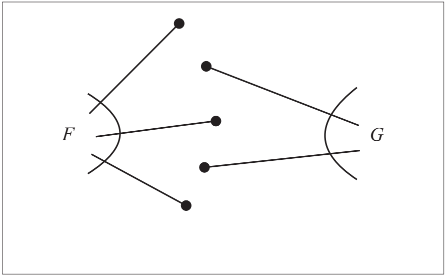
  where the disjoint sum being taken over all pairs (U_1, U_2) forming a decomposition of U.
- The transport $(FG)[\sigma]$ along a bijection $\sigma : U \to V$ is natural. More precisely, for each (FG)-structure $s = (f, g)$ on U,

$$(FG)[\sigma](s) := (F[\sigma_1](f), G[\sigma_2](g)), \tag{3}$$

  where $\sigma_1 := \sigma|_{U_1}$ and $\sigma_2 := \sigma|_{U_2}$ are the restriction of σ on U_1 and U_2, respectively.

The transport of the product operator

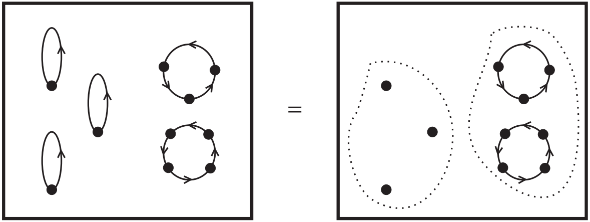
- In a pictorial fashion, any *FG*-structure can be represented as:



- Informally, an (*FG*)-structure is an ordered pair formed by an *F*-structure and a *G*-structure over complementary disjoint subsets.

An example of the product operator

- Let $\mathcal{S}$ be the permutation species and Der be the derangement species(the permutation without fixed points). Then $\mathcal{S} = E \cdot \mathrm{Der}$.



Some properties of the product operator

- The product of species is associative and commutative up to isomorphism:

$$F \cdot G = G \cdot F \text{ and } (F \cdot G) \cdot H = F \cdot (G \cdot H).$$

- The product admits the species 1 as neutral element, and the species 0 as absorbing element, i.e.,

$$1F = F1 = F \text{ and } 0F = F0 = 0.$$

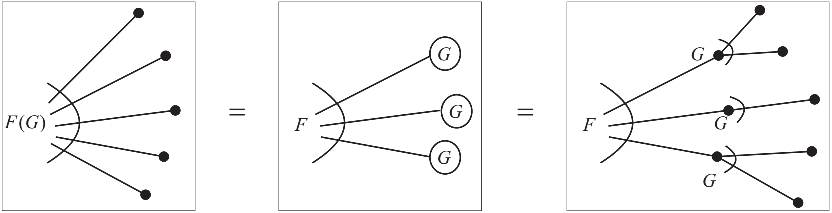
- The operator of product distributes over the operation of sum, that is,

$$F(G + H) = FG + FH.$$

- So the set of species is a semi-ring under the operations of sum and product.

The operator of substitution

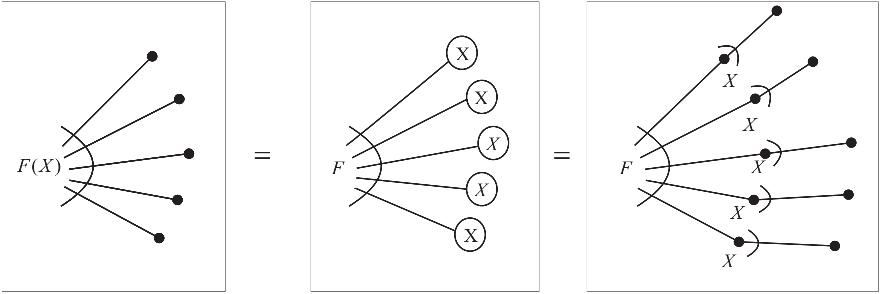
- The substitution of G in F, denoted by $F \circ G$ or $F(G)$, is a little bit complicated. Intuitively, an $F(G)$-structure is an F-assembly of disjoint G-structures:



An example of the operator of substitution

- Note that $F \circ X = F(X) = F$ for any species F. Recall $X := E_1$ is the species characteristic of singletons:

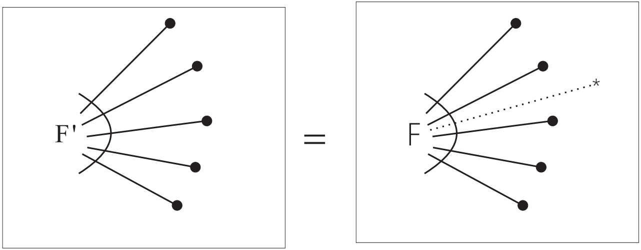
$$X[U] := \begin{cases} \{U\} & \text{if } |U| = 1, \\ \emptyset & \text{otherwise.} \end{cases}$$



The derivative operator

- Let $F = F(X)$ be a species of structures. The species F' (also denoted by dF or $\frac{d}{dX}F(X)$) called the derivative of F, is defined as follows:
- An F'-structure on U is an F-structure on $U^* := U \cup \{*\}$, where $*$ is a element chosen outside of U. In other words, for any finite set U , one sets

$$F'[U] := F[U^*], \qquad \text{where } U^* := U + \{*\}.$$



- The point $*$ is different from the points in the set U. We can image $*$ as a virtual point and elements in U as real points.

The derivative operator

- The transport along a bijection $\sigma : U \to V$ is carried out by setting, for any F'-structure s on U,

$$F'[\sigma](s) := F[\sigma^*](s),$$

where $\sigma^* : U + \{*\} \to V + \{*\}$ is the canonical extension of σ obtained by setting

$$\sigma^*(u) := \sigma(u) \text{ if } u \in U \text{ and } \sigma^*(*) := *.$$

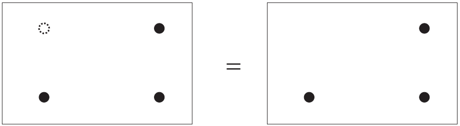
- Up to isomorphism, the derivation of species satisfies additivity and Leibniz rule, i.e.,

$$(F+G)' = F' + G' \quad \text{and} \quad (FG)' = F'G + FG'.$$

- The operator d is called the combinatorial differential operator.

Examples of the derivative operator

- We decompose the process of derivation $F'[U]$ to the following three steps:
  1. Add the point $*$ to the set U, that is, $U^* = U \cup \{*\}$;
  2. Form the structures $F[U^*]$. In this step, take the point $*$ as a real point temporarily;
  3. Take out the point $*$ from the structures in $F[U^*]$, because the point $*$ is indeed a virtual point.
- $(E_n)' = E_{n-1}$ for any $n \geq 1$. For example, $E_4' = E_3$:
  for any finite set U with $|U| \neq 3$, we have
  $E_4'[U] = E_4[U^*] = \emptyset = E_3[U]$, and for U with $|U| = 3$, we have



Examples of the derivative operator

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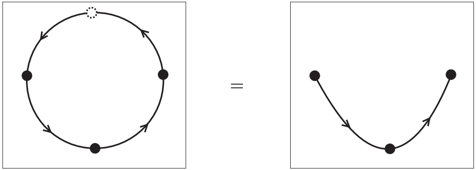
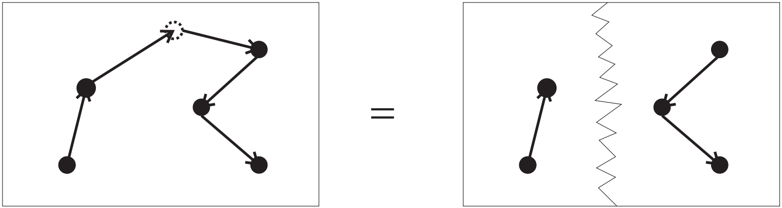
  1. Add the point $*$ to the set U, that is, $U^* = U \cup \{*\}$;
  2. Form the structures $F[U^*]$. In this step, take the point $*$ as a real point temporarily;
  3. Take out the point $*$ from the structures in $F[U^*]$, because the point $*$ is indeed a virtual point.
- $(C_n)' = L_{n-1} = X^{n-1}$ for any $n \geq 3$, where C_n is the n-cycle species (i.e. the cylce species C restrict to cardinality n) and L_{n-1} is the $(n-1)$-linear order species (i.e. the linear species L restrict to cardinality $n-1$).
- For example, $(C_4)' = L_3$: for any finite set U with $|U| \neq 3$, we have $C_4'[U] = C_4[U^*] = \emptyset = L_3[U]$, and for U with $|U| = 3$, we have



Examples of the derivative operator

- $L' = L^2$:



Combinatorial differential constant

- A combinatorial differential constant is a species $K = K(X)$ such that $K'(X) = 0$.
- There are plenty of "non constant" combinatorial differential constants. For example, $K = K(X) := 3 + X^5 - 5C_5$, as

$$K'(X) = (3 + X^5 - 5C_5)' = 0 + 5X^4 - 5X^4 = 0,$$

  where C_5 is the 5-cycle species.
- Another example of combinatorial differential constant is $K = K(X) := X^2 - 2E_2$, as

$$K' = (X^2 - 2E_2)' = 2X - 2E_1 = 2X - 2X = 0.$$

- Question: Characterize the combinatorial differential constant.

The virtual species

- To introduce the concept of the integral operator, we need the subtraction of species, that is, the concept of virtual species.
- The situation is analogous to the case of semi-ring $\mathbb{N}$ of natural numbers for which the subtraction is not everywhere defined.
- More precisely, a virtual species is an element of the quotient

$$\mathrm{Virt} := \mathrm{Spec} \times \mathrm{Spec}/\sim,$$

where $\sim$ is defined by

$$(F, G) \sim (H, K) \Leftrightarrow F + K = G + H.$$

Write

$$F - G := \text{ the class of of } (F, G) \text{ according to } \sim.$$

- We have the semi-ring embedding: Species $\hookrightarrow$ Virtual Species via $F \mapsto F - 0$.

The Joyal integral operator

- For a virtual species Φ, write $\Phi^{(n)} := d^n(\Phi)$ for the n-th derivation of Φ.
- **Theorem**(A. Joyal 1985) Every virtual species $\Phi = \Phi(X)$ has an integral $J_E(\Phi(X))$ such that $J_E(\Phi(0)) = 0$ defined by,

$$J_E(\Phi(X)) := J_E(\Phi) := E_1\Phi - E_2\Phi' + E_3\Phi^{(2)} + \cdots$$

- **Proof.** Want: $(J_E(\Phi))' = \Phi$. Differentiate term by term to obtain "telescopic" cancellation:

$$\begin{aligned}(J_E(\Phi))' &= (E_1\Phi)' - (E_2\Phi')' + (E_3\Phi'')' + \cdots \\ &= \Phi + E_1\Phi' - E_1\Phi' - E_2\Phi^{(2)} + E_2\Phi^{(2)} + E_3\Phi^{(3)} + \cdots \\ &= \Phi\end{aligned}$$

- Call J_E the Joyal integral operator.

Combinatorial differential tower

- To introduce other kind of integrals of virtual species, we need the concept of combinatorial differential tower.
- A combinatorial differential tower is a species $T = T(X)$ such that $T'(X) = T(X)$ and $T(0) = 1$.
- There are plenty of combinatorial differential towers. For example,

$$E(X) = 1 + X + E_2(X) + \cdots + E_n(X) + \cdots$$
$$e^X = 1 + \frac{X}{1!} + \frac{X^2}{2!} + \cdots + \frac{X^n}{n!} + \cdots$$

- Every combinatorial differential tower $T = T_0 + T_1 + \cdots + T_n + \cdots$ is characterized by $T_0 = 1$ and $T_n' = T_{n-1}$ for $n \geq 1$.

Variant of Joyal integral

- **Theorem** Let $T = T_0 + T_1 + \cdots + T_n + \cdots$ be any given combinatorial differential tower. Then every integral J of a virtual species $\Phi = \Phi(X)$ can be written in the form

$$J(\Phi) := J(\Phi(X)) := K(X) + J_T(\Phi(X)),$$

where $K(X)$ is a combinatorial differential constant and

$$\begin{aligned} &J_T(\Phi(X)) \\ :=&T_1(X)\Phi(X) - T_2(X)\Phi'(X) + \cdots + (-1)^{n-1}T_n(X)\Phi^{(n-1)}(X) + \cdots \end{aligned}$$

- If taking $T = E$, then J_T is precisely the Joyal integral.

Integro-differential algebra

- A (commutative) integro-differential algebra of weight λ is a differential **k**-algebra (R, D) of weight λ together with a linear operator $\Pi\colon R \to R$ satisfying the section axiom $D \circ \Pi = \mathrm{id}_R$ and the hybrid/differential Rota-Baxter axiom

$$\Pi(D(x)\Pi(y)) = x\Pi(y) - \Pi(xy) - \lambda\Pi(D(x)y), \quad x, y \in R.$$

- The hybrid Rota-Baxter axiom implies the Rota-Baxter axiom: Let $x = \Pi(\tilde{x})$. Then $D(x) = D(\Pi(\tilde{x}))) = \tilde{x}$ and the above Equation becomes

$$\begin{aligned} \Pi(\tilde{x}\Pi(y)) &= \Pi(\tilde{x})\Pi(y) - \Pi(\Pi(\tilde{x})y) - \lambda\Pi(\tilde{x}y) \\ \Leftrightarrow \Pi(\tilde{x})\Pi(y) &= \Pi(\tilde{x}\Pi(y)) + \Pi(\Pi(\tilde{x})y) + \lambda\Pi(\tilde{x}y) \end{aligned}$$

  which is the Rota-Baxter axiom.
- Every integro-differential algebra of weight λ is a Rota-Baxter algebra of weight λ.

Equivalent conditions

- Let (R, D) be a differential algebra of weight λ with a linear operator Π on R such that $D \circ \Pi = \mathrm{id}_R$. Denote $\mathcal{J} = \Pi \circ D$, called the initialization, and $\mathcal{E} = \mathrm{id}_R - \mathcal{J}$, called the evaluation. Then the following statements are equivalent:
  1. (R, D, Π) is an integro-differential algebra;
  2. $\mathcal{E}(xy) = \mathcal{E}(x)\mathcal{E}(y) \quad$ for all $x, y \in R$;
  3. $\mathcal{J}(x)\mathcal{J}(y) + \mathcal{J}(xy) = \mathcal{J}(x)y + x\mathcal{J}(y), \quad \forall x, y \in R$
  4. $\ker E = \mathrm{im}\mathcal{J}$ is an ideal;
  5. $\mathcal{J}(x\mathcal{J}(y)) = x\mathcal{J}(y) \quad$ for all $x, y \in R$;
  6. $\mathcal{J}(x\Pi(y)) = x\Pi(y) \quad$ for all $x, y \in R$;
  7. $x\Pi(y) = \Pi(D(x)\Pi(y)) + \Pi(xy) + \lambda\Pi(D(x)y) \quad$ for all $x, y \in R$;
  8. (R, D, Π) is a differential Rota-Baxter algebra and $\Pi(\mathcal{E}(x)y) = \mathcal{E}(x)\Pi(y) \quad$ for all $x, y \in R$;
  9. (R, D, Π) is a differential Rota-Baxter algebra and $\mathcal{J}(\mathcal{E}(x)\mathcal{J}(y)) = \mathcal{E}(x)\mathcal{J}(y) \quad$ for all $x, y \in R$.
- We will focus on 2: $\mathcal{E}(xy) = \mathcal{E}(x)\mathcal{E}(y)$. and 3: $\mathcal{J}(x)\mathcal{J}(y) + \mathcal{J}(xy) = \mathcal{J}(x)y + x\mathcal{J}(y), \quad \forall x, y \in R$.

Integro-differential algebra of virtual species

- Let $\Phi = \Phi(X)$ be a virtual species and J be an arbitrary integral on virtual species given by

$$J(\Phi) := J(\Phi(X)) := K(X) + J_T(\Phi(X)),$$

where $K(X)$ is a combinatorial differential constant and

$$J_T(\Phi(X)) \\ := T_1(X)\Phi(X) - T_2(X)\Phi'(X) + \cdots + (-1)^{n-1}T_n(X)\Phi^{(n-1)}(X) + \cdots$$

- **Theorem** (Gao-Guo-Rosenkranz-Zhang) The virtual species with the derivative d and integral J is a commutative integro-differential $\mathbb{Z}$-algebra of weight 0 if and only if $K(X) = 0$.
- **Corollay** (Gao-Guo-Rosenkranz-Zhang) The virtual species with the derivative d and the Joyal integral J_E is a commutative integro-differential $\mathbb{Z}$-algebra of weight 0.
- **Corollary** (Gao-Guo-Rosenkranz-Zhang) The set of species with the Joyal integral J_E is a Rota-Baxter $\mathbb{Z}$-algebra of weight 0.

Applications of Int-diff algebras

- The integro-differential algebra can be used to solve initial problems and boundary problems. For example:
- **Theorem** (Rosenkranz et. al.) Let (R, D, Π) be an ordinary integro-differential algebra. Given a regular differential operator $T = D^n + c_{n-1}D^{n-1} + \cdots + c_0 \in R[D]$ and a regular fundamental system $u_1, \cdots, u_n \in R$, the canonical initial value problem

$$\boxed{\begin{array}{l} Tu = f \\ u(0) = u'(0) = \cdots = u^{(n-1)}(0) = 0 \end{array}}$$

has the unique solution $u = Gf$, where

$$G = \sum_{i=1}^{n} u_i \Pi a^{-1} a_i$$

is the Green operator, $a = \det W(u_1, \cdots, u_n)$ and a_i is the determinant of the matrix W_i obtained form W by replacing the i-th column by the n-th unit vector.

Applications of Int-diff algebras

- Another application, having the integro-differential $\mathbb{Z}$-algebra S of virtual species, we can also form the integro-differential polynomials (say in one variable) with coefficients in S. This will give an exact algebraic meaning to nonlinear combinatorial differential equations, including initial conditions.
- For the concept of integro-differential polynomial, we refer to

References: M. Rosenkranz, G. Regensburger, L. Tec, and B. Buchberger, Symbolic analysis for boundary problems: From rewriting to parametrized Gröbner bases. In U. Langer and P. Paule (eds.), *Numerical and Symbolic Scientific Computing: Progress and Prospects*, Springer Vienna, 2012, 273–331.

The definition of linear species

- The linear species is defined on totally ordered set.
- A totally ordered set is a pair $\ell = (U, \leq)$, where U is a finite set and $\leq$ is a total order relation on U.
- The smallest element of a nonempty totally ordered set ℓ is denoted by m_ℓ, that is,
$$m_\ell := \min\{x \mid x \in \ell\}.$$
- Let $\ell = (U, \leq)$ be a totally ordered set. For subsets $U_1, \cdots, U_k$ of U such that $U_1 + \cdots + U_k = U$, we write
$$\ell_1 + \cdots + \ell_k = \ell$$
if $\ell_i := \ell|_{U_i}$ is the restriction of ℓ to U_i, for $i = 1, \cdots, k$.
- Between two ordered sets ℓ_1 and ℓ_2, an increasing function $f : \ell_1 \to \ell_2$ is function such that, for u and v in ℓ_1,
$$u <_{\ell_1} v \Rightarrow f(u) <_{\ell_2} f(v).$$

The definition of linear species

- A $\mathbb{L}$-species is a rule F which
  1. to each totally ordered set ℓ, associates a finite (weighted) set $F[\ell]$,
  2. to each increasing bijection $\sigma : \ell_1 \to \ell_2$, associates a function (morphism of weighted sets) $F[\sigma] : F[\ell_1] \to F[\ell_2]$; these functions $F[\sigma]$ must satisfy the following functoriality properties:

$$F[\mathrm{id}_\ell] = \mathrm{id}_{F[\ell]}, \quad F[\sigma \circ \tau] = F[\sigma] \circ F[\tau]. \tag{4}$$

- In category terms, an $\mathbb{L}$-species is a functor $\mathbb{L} \to \mathbb{E}$, where $\mathbb{L}$ denote the category of totally ordered sets and increasing bijections and $\mathbb{E}$, that of finite sets and bijections.

Some operators on linear species

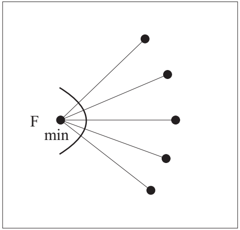
- Let F, G be two linear species and $\ell = (U, <)$ a totally ordered set. Define (In each case, the transport of structures is defined in the obvious way)
  1. the sum, $F + G$:
  $$(F + G)[\ell] := F[\ell] + G[\ell], \quad \text{(the disjoint union)};$$
  2. the product, FG:
  $$(FG)[\ell] := \sum_{\ell_1 + \ell_2 = \ell} F[\ell_1] \times G[\ell_2];$$
  3. the derivation, $\partial(F)$:
  $$\partial(F)[\ell] := F[1 +_{\mathcal{O}} \ell];$$
  where $1 +_{\mathcal{O}} \ell$ is the totally ordered set obtained by adding a new minimum element 1 to ℓ.

The integral operator

- 4. the integral, $\Pi(F)$:

$$\Pi(F)[\ell] := \begin{cases} \emptyset & \text{if } U = \emptyset, \\ F[\ell \setminus m_\ell] & \text{otherwise.} \end{cases}$$

The following figure illustrates graphically the $\Pi(F)$-structure, where $\min = \min \ell$.



- Note that the minimum element $\min \ell$ of the underlying total order set ℓ of a $\Pi(F)$-structure does not appear in the corresponding F-structure.

Integro-diff algebra of linear virtual species

- **Theorem** (Gao-Guo-Rosenkranz-Zhang) The set of virtual linear species with the derivative ∂ and integral Π is an integro-differential $\mathbb{Z}$-algebra of weight 0.

- **Corollary** (Gao-Guo-Rosenkranz-Zhang) The set of virtual linear species with the integral Π is a Rota-Baxter $\mathbb{Z}$-algebra of weight 0.

- References:

*Bergeron F., Labelle G. and Leroux P., Combinatorial species and tree-like structures, in: Encyclopedia of Mathematics and Its Applications, vol. 67, Cambridge U. Press (1998).

- **Thank You!**