Classification of Rota-Baxter Type Operators [1]

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[1] joint work with William Sit and Ronghua Zhang

Motivation:

- Throughout the history, mathematical objects are often understood through studying operators defined on them.
- Well-known examples include Galois theory where a field is studied by its automorphisms, and analysis and geometry were functions and manifolds are studied through their derivations and related vector fields.

Rota's Question:

- By the 1970s, several other operators had been found from studies in analysis, probability and combinatorics.

$$\begin{array}{rl} \text{Average operator} & P(x)P(y) = P(xP(y)), \\ \text{Inverse average operator} & P(x)P(y) = P(P(x)y), \\ \text{(Rota-)Baxter operator} & P(x)P(y) = P(xP(y) + P(x)y + \lambda xy) \\ & \text{where } \lambda \text{ is a fixed constant,} \\ \text{Reynolds operator} & P(x)P(y) = P(xP(y) + P(x)y - P(x)P(y) \end{array}$$

- Rota posed the question of finding all the identities that could be satisfied by a linear operator defined on associative algebras. He also suggested that there should not be many such operators other than these previously known ones.

Quotation from Rota

- ”In a series of papers, I have tried to show that other linear operators satisfying algebraic identities may be of equal importance in studying certain algebraic phenomena, and I have posed the problem of finding all possible algebraic identities that can be satisfied by a linear operator on an algebra. Simple computations show that the possibility are very few, and the problem of classifying all such identities is very probably completely solvable.[2] A partial (but fairly complete) list of such identities is the following. Besides endomorphisms and derivations, one has averaging operators, Reynolds operators and Baxter operators.”

[2] A notable step forward has been made in the unpublished (and unsubmitted) Harvard thesis of Alexander Doohovskoy.

Post Rota developments

- Little progress was made on finding all such operators. In the meantime, new operators have merged from physics and combinatorial studies, such as

Nijenhuis operator	$P(x)P(y) = P(xP(y) + P(x)y - P(xy)),$
Leroux's TD operator	$P(x)P(y) = P(xP(y) + P(x)y - xP(1)y).$

- These previously known operators continued to find remarkable applications in pure and applied mathematics. For the differential operator, we witnessed the establishing of differential algebra and difference algebra (and the DARTs). The Rota-Baxter algebra have found applications in classical Yang-Baxter equations, operads, combinatorics, and most prominently, the renormalization of quantum field theory through the Hopf algebra framework of Connes and Kreimer.
- **What do we mean by a linear operator satisfying an algebraic identity?**

A simplified analogy: PI algebras

- A $\mathbf{k}$-algebra R is called a PI algebra if it satisfies a polynomial identity: there is a fixed element $f(X)$ in a noncommutative polynomial algebra (that is, a free algebra) $\mathbf{k}\langle X\rangle$ over a set X such that $f(X)$ is sent to zero under any algebra homomorphism from $\mathbf{k}\langle X\rangle$ to R.
- So identities for a PI algebra are elements from the free algebra $\mathbf{k}\langle X\rangle$.
- Then identities for an operator on an algebra should be elements from free algebra with operators.

Operated algebras

- An **operated monoid** is a monoid U together with a map $\alpha : U \to U$. A morphism from an operated monoid (U, α) to an operated monoid (V, β) is a monoid homomorphism $f : U \to V$ such that $f \circ \alpha = \beta \circ f$.

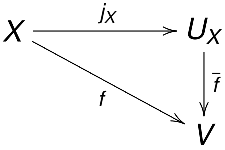
$$\begin{array}{ccc} U & \xrightarrow{\alpha} & U \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ V & \xrightarrow{\beta} & V \end{array}$$

  Let **k** be a commutative unitary ring. When monoid is replaced by semigroup, **k**-algebra or nonunitary **k**-algebra in the above definition, we obtain the concept of **operated semigroup**, **operated k-algebras** or **operated nonunitary k-algebra**.

- The set $\mathcal{F}$ of forests with the grafting map and the concatenation product is a operated semigroup. The **k**-module $\mathbf{k}\mathcal{F}$ generated by $\mathcal{F}$ is a operated nonunitary **k**-algebra.

Free operated algebras

- The adjoint functor of the forgetful functor from the category of operated monoids to the category of sets gives the free operated monoids in the usual way.
- More precisely, a **free operated monoid** on a set X is an operated monoid (U_X, α_X) together with a map $j_X : X \to U_X$ with the property that, for any operated monoid (V, β) together with a map $f : X \to V$, there is a unique morphism $\overline{f} : (U_X, \alpha_X) \to (V, \beta)$ of operated monoids such that $f = \overline{f} \circ j_X$.



- We similarly define the concept of free operated (nonunitary) **k**-algebras on a set X.

Construction of free operated algebras

- For a given set Y, let $S(Y)$ be the free semigroup generated by Y, let $M(Y)$ be the free monoid generated by Y and let $\lfloor Y \rfloor$ be the set $\{\lfloor y \rfloor \big| y \in Y\}$ which is just another copy of Y whose elements are denoted by $\lfloor y \rfloor$ for distinction.
- Let X be a set. We recursively define a direct system $\{i_{n,n+1} : \mathfrak{S}_n \to \mathfrak{S}_{n+1}\}$ of free semigroups $\{\mathfrak{S}_n\}_{n=0}^{\infty}$, and a direct system $\{\tilde{i}_{n,n+1} : \mathfrak{M}_n \to \mathfrak{M}_{n+1}\}$ of free monoids $\{\mathfrak{M}_n\}_{n=0}^{\infty}$, where the transition maps $i_{n,n+1}$ and $\tilde{i}_{n,n+1}$ are injective morphisms.
- We do this by first letting $\mathfrak{S}_0 = S(X)$ and $\mathfrak{M}_0 = M(X)$, and then define

$$\mathfrak{S}_1 = S(X \cup \lfloor \mathfrak{M}_0 \rfloor) = S(X \cup \lfloor M(X) \rfloor), \quad \mathfrak{M}_1 = M(X \cup \lfloor \mathfrak{M}_0 \rfloor)$$

with $i_{0,1}$ and $\tilde{i}_{0,1}$ being the natural injection

$$\begin{aligned} i_{0,1} : \quad & \mathfrak{S}_0 = S(X) \hookrightarrow \mathfrak{S}_1 = S(X \cup \lfloor \mathfrak{M}_0 \rfloor), \\ \tilde{i}_{0,1} : \quad & \mathfrak{M}_0 = M(X) \hookrightarrow \mathfrak{M}_1 = M(X \cup \lfloor \mathfrak{M}_0 \rfloor). \end{aligned}$$

- Inductively assume that $\mathfrak{S}_{n-1}$ and $\mathfrak{M}_{n-1}$ have been defined for $n \geq 2$, with the embeddings

$$i_{n-2,n-1} : \mathfrak{S}_{n-2} \hookrightarrow \mathfrak{S}_{n-1} \text{ and } \tilde{i}_{n-2,n-1} : \mathfrak{M}_{n-2} \to \mathfrak{M}_{n-1}.$$

- We define

$$\mathfrak{S}_n := S(X \cup \lfloor \mathfrak{M}_{n-1} \rfloor), \quad \mathfrak{M}_n := M(X \cup \lfloor \mathfrak{M}_{n-1} \rfloor) = \mathfrak{S}_n \cup \{\mathbf{1}\}. \tag{1}$$

- We also have the injections

$$\lfloor \mathfrak{M}_{n-2} \rfloor \hookrightarrow \lfloor \mathfrak{M}_{n-1} \rfloor \text{ and } X \cup \lfloor \mathfrak{M}_{n-2} \rfloor \hookrightarrow X \cup \lfloor \mathfrak{M}_{n-1} \rfloor,$$

yielding injective maps of free semigroups and free monoids

$$\begin{aligned} \mathfrak{S}_{n-1} &= S(X \cup \lfloor \mathfrak{M}_{n-2} \rfloor) \hookrightarrow S(X \cup \lfloor \mathfrak{M}_{n-1} \rfloor) = \mathfrak{S}_n, \\ \mathfrak{M}_{n-1} &= M(X \cup \lfloor \mathfrak{M}_{n-2} \rfloor) \hookrightarrow M(X \cup \lfloor \mathfrak{M}_{n-1} \rfloor) = \mathfrak{M}_n. \end{aligned}$$

- We finally define the semigroup

$$\mathfrak{S}(X) = \varinjlim \mathfrak{S}_n = \bigcup_{n \geq 0} \mathfrak{S}_n$$

and monoid

$$\mathfrak{M}(X) = \varinjlim \mathfrak{M}_n = \bigcup_{n \geq 0} \mathfrak{M}_n$$

with identity (the image of) **1.**

Construction of free operated algebras II

- Let $j_X : X \to \mathfrak{M}(X)$ be the natural embedding.
  1. The triple $(\mathfrak{M}(X), (\lfloor\ \rfloor), j_X)$ is the free operated monoid on X.
  2. The triple $(\mathbf{k}\,\mathfrak{M}(X), (\lfloor\ \rfloor), j_X)$ is the free operated unitary algebra on X.
- We also call $\mathfrak{M}(X)$ the set of **bracketed words** generated by X.
- Let $\mathfrak{M}'(X)$ be the set of **disjoint bracketed words** consisting of bracketed words with no pairs of brackets right next to each other, such as $\lfloor * \rfloor \lfloor * \rfloor$.
- $\mathfrak{M}'(X)$ is called the set of Rota-Baxter words since it forms a **k**-basis of the free Rota-Baxter **k**-algebra generated by X.
- Words in $X \cup \lfloor \mathfrak{M}(X) \rfloor$ are called **indecomposable.** Any $\mathbf{x} \in \mathfrak{M}(X) - \{1\}$ has a unique factorization $\mathbf{x} = \mathbf{x}_1 \cdots \mathbf{x}_b$ of indecomposable words, called the **standard decomposition.**

PI operated algebras

- For a commutative ring $\mathbf{k}$, denote $\mathbf{k}\{X\} = \mathbf{k}\mathfrak{M}(X)$ and $\mathbf{k}\{X\}' = \mathbf{k}\mathfrak{M}'(X)$.
- We generalize the concept of PI algebras to the context of operated algebras.
- Let $\phi(x_1, \cdots, x_k) \in \mathbf{k}\{x_1, \cdots, x_n\}$ be given. For a given operated algebra (R, P) and a set map $f : \{x_1, \cdots, x_k\} \to R$, we let $\bar{\phi}(f(x_1), \cdots, f(x_k))$ denote the image of $\phi(x_1, \cdots, x_k)$ under the unique operated algebra homomorphism $\bar{f} : \mathbf{k}\{x_1, \cdots, x_k\} \to R$ that extends f by the universal property of $\mathbf{k}\{x_1, \cdots, x_k\}$ as a free operated algebra on $\{x_1, \cdots, x_k\}$.
- Let (R, P) be an operated algebra. If

$$\bar{\phi}(u_1, \cdots, u_k) = 0, \quad \forall u_1, \cdots, u_k \in R,$$

then R is called a ϕ**-algebra** and P is called a ϕ**-operator.** A ϕ-algebra is called a **polynomial identity operated algebra** or a **PIO algebra** for short.

Examples

- When $\phi = [xy] - x[y] - [x]y$, then a ϕ-operator (resp. algebra) is a differential operator (resp. algebra).
- When $\phi = [x][y] - [x[y]] - [[x]y] - \lambda[xy]$, then a ϕ-operator (resp. ϕ-algebra) is a Rota-Baxter operator (resp. algebra) of weight λ.
- When ϕ is from the noncommutative polynomial algebra $\mathbf{k}\langle X\rangle$, then a ϕ-algebra is an algebra with polynomial identity. Here we regard an algebra as an operated algebra where the operator is taken to be the identity map.
- **Proposition** Let X be a given set. The free ϕ-algebra on X is given by the quotient operated algebra $\mathbf{k}\{X\}/I_\phi$ where I_ϕ is the operated ideal of $\mathbf{k}\{X\}$ generated by the set

$$\{\bar{\phi}(u_1, \cdots, u_k) \mid u_1, \cdots, u_k \in \mathbf{k}\{X\}\}.$$

Special kinds of PIO algebras

- What Rota-Baxter operator, average operator, Nijenhuis operator, etc. (will come to differential operator later) have in common is that they are of the form
$$[u][v] = [M(u, v)]$$
where $M(u, v)$ is an expression involving u, v and P, i.e. $M(u, v) \in \mathbf{k}\{u, v\}$.
- Also, $M(u, v)$ is formally associative:
$$M(M(u, v), w) = M(u, M(v, w))$$
modulo the relation $\phi_M := [u][v] - [M(u, v)]$.
- Further, free algebras in the corresponding categories (of Rota-Baxter algebras, of average algebras, ...) have a special basis. More precisely, The map
$$\mathbf{k}\{X\}' \to \mathbf{k}\{X\} \to \mathbf{k}\{X\}/I_M$$
is bijective. Thus a suitable multiplication on $\mathbf{k}\{X\}'$ makes it the free ϕ_M-algebra on X.
- As we will see, these properties are related.

Some concepts

- Let x, y be two symbols. A fixed element $M \in \mathbf{k}\{x, y\}'$ gives a map
$$\mathfrak{m} : \mathbf{k}\{X\} \times \mathbf{k}\{X\} \to \mathbf{k}\{X\} \tag{2}$$
by substitution.
- Let $M \in \mathbf{k}\{x, y\}'$ be given.
- Define $\phi := \phi_M := [x][y] - [M(x, y)]$ and denote $I_M = I_{\phi_M}$, namely I_M is the operated ideal of $\mathbf{k}\{X\}$ generated by the set
$$\{[a][b] - [M(a, b)] \mid a, b \in \mathbf{k}\{X\}\}.$$
- For a set X, denote $j_X : \mathbf{k}\{X\}' \to \mathbf{k}\{X\}$ for the inclusion map and $\pi_M : \mathbf{k}\{X\} \to \mathbf{k}\{X\}/I_M$ for the quotient map. Define ρ_M to be the composition
$$\rho_M : \mathbf{k}\{X\}' \xrightarrow{j_X} \mathbf{k}\{X\} \xrightarrow{\pi_X} \mathbf{k}\{X\}/I_M. \tag{3}$$
- Call M **bijective** if ρ_M is bijective for every set X.

- Define the composition of linear maps

$$\bar{\rho}_M : \mathbf{k}\{X\}' \to \mathbf{k}\{X\}'\mathbf{k}\{X\}' \to \frac{\mathbf{k}\{X\}'\mathbf{k}\{X\}'}{I_M \cap \mathbf{k}\{X\}'\mathbf{k}\{X\}'} \cong \frac{\mathbf{k}\{X\}'\mathbf{k}\{X\}' + I_M}{I_M}$$

  where the first map is the inclusion, the second map is the quotient map and the last quotient is a **k**-submodule of $\frac{\mathbf{k}\{X\}}{I_M}$.
- Call M **pre-bijective** if $\bar{\rho}_M$ is bijective for every set X.
- Let M be pre-bijective. Define the map

$$\bar{\mathfrak{m}} : \mathbf{k}\{X\}' \times \mathbf{k}\{X\}' \xrightarrow{conc} \mathbf{k}\{X\}'\mathbf{k}\{X\}' \xrightarrow{\pi_M} (\mathbf{k}\{X\}'\mathbf{k}\{X\}' + I_M)/I_M \xrightarrow{\bar{\rho}_M^{-1}} \mathbf{k}\{X\}'$$

  Here *conc* is the concatenation product in $\mathbf{k}\{X\}$.
- Call a pre-bijective M **associative** if $\bar{\mathfrak{m}}$ is associative in the sense that

$$\bar{\mathfrak{m}}(\bar{\mathfrak{m}}(x, y), z) = \bar{\mathfrak{m}}(x, \bar{\mathfrak{m}}(y, z))$$

  in $\mathbf{k}\{x, y, z\}'$.

Examples

- $M(x, y) = x[y] + [x]y + \lambda xy$ is both bijective and associative.
- $M(x, y) = x[[y]]$ is pre-bijective, but is neither bijective nor associative.
- $M(x, y) = x[y] + [x]y - [x][y]$ (Reynolds operator) is not pre-bijective.

Several equivalent properties

- Let $M \in \mathbf{k}\{X\}'$ be pre-bijective. For any set X, define

$$\diamond_M : \mathbf{k}\{X\}' \times \mathbf{k}\{X\}' \to \mathbf{k}\{X\}'$$

by
$$\mathbf{x} \diamond_M \mathbf{x}' = \begin{cases} \lfloor \bar{\mathfrak{m}}(\bar{\mathbf{x}}, \bar{\mathbf{x}}') \rfloor, & \mathbf{x} = \lfloor \bar{\mathbf{x}} \rfloor, \mathbf{x}' = \lfloor \bar{\mathbf{x}}' \rfloor, \\ \mathbf{x}\,\mathbf{x}', & \text{otherwise} \end{cases} \tag{4}$$

if $\mathbf{x}$ and $\mathbf{x}'$ are indecomposable and

$$\mathbf{x} \diamond_M \mathbf{x}' = \mathbf{x}_1 \cdots \mathbf{x}_{m-1} (\mathbf{x}_m \diamond_M \mathbf{x}'_1) \mathbf{x}'_2 \cdots \mathbf{x}'_n \tag{5}$$

in general. Here $\mathbf{x} = \mathbf{x}_1 \cdots \mathbf{x}_m$ and $\mathbf{x}' = \mathbf{x}'_1 \cdots \mathbf{x}'_n$ are the standard decompositions.
- **Theorem** Let $M \in \mathbf{k}\{x, y\}'$ be pre-bijective. The following statements are equivalent.
  1. M is bijective.
  2. M is associative.
  3. For any set X, the triple $(\mathbf{k}\{X\}', \diamond_M, \lfloor\ \rfloor)$ is a ϕ_M-algebra. Here $\lfloor\ \rfloor$ is the restriction of $\lfloor\ \rfloor : \mathbf{k}\{X\} \to \mathbf{k}\{X\}$ to $\mathbf{k}\{X\}'$
  4. The triple $(\mathbf{k}\{X\}', \diamond_M, \lfloor\ \rfloor)$ above, together with the embedding $i_X : X \to \mathbf{k}\{X\}'$, is a free ϕ_M-algebra on X.

- Let $M \in \mathbf{k}\{x, y\}'$ be pre-bijective. If either (and so all) of the equivalent conditions in Theorem is satisfied, then M is called a **shuffle type product** and the corresponding ϕ_M-algebra (resp. ϕ_M-operator) is called a **Rota-Baxter type algebra** (resp. **Rota-Baxter type operator**).
- **(Classification Problem of Rota-Baxter type operators)** Find all Rota-Baxter type operators. In other words, find all shuffle type $M(x, y) \in \mathbf{k}\{x, y\}'$.

Classification of Rota-Baxter type operators

- **Conjecture** The following is the list of all shuffle type products $M(x,y)$.

$dxy + exP(1)y$	multiplicative operator
$xP(y)$	average operator
$P(x)y$	inverse average operator
$xP(y) - xP(1)y$	*average TD operator
$P(x)y - xP(1)y$	*inverse average TD operator
$xP(y) + P(x)y + \lambda xy$	Rota-Baxter operator of weight λ
$xP(y) + P(x)y - P(xy)$	Nijenhuis operator
$xP(y) + P(x)y - xP(1)y + \lambda xy$	*TD operator of weight λ
$xP(y) + P(x)y - P(xy) - xP(1)y + \lambda xy$	*complete operator of weight λ

- In other words, all Rota-Baxter type operators are of the form

$$P(x)P(y) = P(M(x,y))$$

for some $M(x,y)$ from the above list.

- The operators marked by a $*$ in the table, except the Leroux operator when $\lambda = 0$, are not known previously.

What about differential type operators?

- The differential operator is defined by the identity

$$[xy] = [x]y + x[y]$$

So it is the form $[xy] = M(x, y)$ where

1. $M(x, y) \in \mathbf{k}\{x, y\}$ does not contain $[uv]$;
2. $M(x, y)$ is associative modulo I_M:

$$M(uv, w) = M(u, vw) \mod I_M.$$

3. The free object on X is defined by the noncommutative polynomial algebra $\mathbf{k}\langle X, [X], [[X]], [[[X]]], \cdots \rangle$ with a suitable derivation.
4. The restriction

$$\mathbf{k}\langle X, [X], [[X]], [[[X]]], \cdots \rangle \hookrightarrow \mathbf{k}\{X\} \to \mathbf{k}\{X\}/I_M$$

is bijective.

An operator is called a **differential type operator** if it satisfies these properties.

Associativity of differential operator

- Let $M(x,y) = [x]y + x[y]$. Then modulo I_M, we have

$$M(uv,w) = [uv]w + uv[w] = [u]vw + u[v]w + uv[w] = M(u,vw).$$