Free Commutative Integro-differential Algebras

Li GUO

Rutgers University

(Joint work with G. Regensburger and M. Rosenkranz, and X. Gao and S. Zheng)

Free differential algebras

- Differential algebra

$$d(xy) = d(x)y + xd(y) + \lambda d(x)d(y).$$

$$d(uv) \mapsto_\phi d(u)v + ud(v) + \lambda d(u)d(v), \forall u, v \in R.$$

This leads to normal forms w with no products in d.

Free differential algebras

- Differential algebra

$$d(xy) = d(x)y + xd(y) + \lambda d(x)d(y).$$

$$d(uv) \mapsto_\phi d(u)v + ud(v) + \lambda d(u)d(v), \forall u, v \in R.$$

This leads to normal forms w with no products in d.

- Thus free commutative differential algebra (of weight λ) on a set X is of the form

$$\mathbf{k}\{X\} := \mathbf{k}[\Delta X], \quad \Delta X := \{x^{(n)} \mid x \in X, n \geq 0\}$$

with concatenation product. Define $d_X \colon \mathbf{k}\{X\} \to \mathbf{k}\{X\}$ as follows. Let $w = u_1 \cdots u_k, u_i \in \Delta(X), 1 \leq i \leq k$, be a commutative word from the alphabet set $\Delta(X)$. If $k = 1$, so that $w = x^{(n)} \in \Delta(X)$, define $d_X(w) = x^{(n+1)}$. If $k > 1$, recursively define

$$d_X(w) = d_X(u_1)u_2 \cdots u_k + u_1 d_X(u_2 \cdots u_k) + \lambda d_X(u_1) d_X(u_2 \cdots u_k).$$

Further define $d_X(1) = 0$. Then $(\mathbf{k}\{X\}, d_X)$ is the free commutative differential algebra of weight λ on the set X.

Integral algebra

- What about integral algebra?

Integral algebra

- What about integral algebra?
- What is an integral algebra? It is a special case of Rota-Baxter algebra.

Integral algebra

- What about integral algebra?
- What is an integral algebra? It is a special case of Rota-Baxter algebra.
- Let $\mathbf{k}$ be a commutative ring. Let $\lambda \in \mathbf{k}$ be fixed. A Rota-Baxter operator of weight λ on a $\mathbf{k}$-algebra R is a linear map $P : R \to R$ such that

$$P(x)P(y) = P(xP(y)) + P(P(x)y) + \lambda P(xy), \ \forall x, y \ \in R.$$

Integral algebra

- What about integral algebra?
- What is an integral algebra? It is a special case of Rota-Baxter algebra.
- Let **k** be a commutative ring. Let $\lambda \in \mathbf{k}$ be fixed. A Rota-Baxter operator of weight λ on a **k**-algebra R is a linear map $P : R \to R$ such that

$$P(x)P(y) = P(xP(y)) + P(P(x)y) + \lambda P(xy), \ \forall\, x, y \in R.$$

- References:

1. L. Guo, WHAT IS a Rota-Baxter Algebra, *Notice of Amer. Math. Soc.* **56** (2009), 1436-1437.
2. L. Guo, An Introduction to Rota-Baxter Algebra, International Press, 2012.

The integration operator I

- For continuous functions $f(x)$ and $g(x)$, define

$$F(x) := I[f](x) := \int_0^x f(s)ds, G(x) := I[g](x) := \int_0^x g(s)ds. \quad (1)$$

Then $F'(x) = f(x), G'(x) = g(x)$.

The integration operator I

- For continuous functions $f(x)$ and $g(x)$, define

$$F(x) := I[f](x) := \int_0^x f(s)ds, G(x) := I[g](x) := \int_0^x g(s)ds. \tag{1}$$

  Then $F'(x) = f(x), G'(x) = g(x)$.
- The integration by parts formula

$$\int_0^x F'(t)G(t)dt = F(t)G(t)\big|_0^x - \int_0^x F(t)G'(t)\,dt$$

  can be "rewritten" as

$$\int_0^x f(t)G(t)\,dt = F(x)G(x) - \int_0^x F(t)g(t)\,dt.$$

The integration operator I

- For continuous functions $f(x)$ and $g(x)$, define

$$F(x) := I[f](x) := \int_0^x f(s)ds, G(x) := I[g](x) := \int_0^x g(s)ds. \qquad (1)$$

Then $F'(x) = f(x), G'(x) = g(x)$.

- The integration by parts formula

$$\int_0^x F'(t)G(t)dt = F(t)G(t)\big|_0^x - \int_0^x F(t)G'(t)\,dt$$

can be "rewritten" as

$$\int_0^x f(t)G(t)\,dt = F(x)G(x) - \int_0^x F(t)g(t)\,dt.$$

- Using Eq. (1), get $I[f\,I[g]](x) = I[f]I[g](x) - I[I[f]\,g](x)$.

$$I[f\,I[g]] = I[f]I[g] - I[I[f]\,g], \quad I[f]I[g] = I[f\,I[g]] + I[I[f]\,g].$$

The integration operator I

- For continuous functions $f(x)$ and $g(x)$, define

$$F(x) := I[f](x) := \int_0^x f(s)ds, G(x) := I[g](x) := \int_0^x g(s)ds. \qquad (1)$$

Then $F'(x) = f(x), G'(x) = g(x)$.

- The integration by parts formula

$$\int_0^x F'(t)G(t)dt = F(t)G(t)\Big|_0^x - \int_0^x F(t)G'(t)\,dt$$

can be "rewritten" as

$$\int_0^x f(t)G(t)\,dt = F(x)G(x) - \int_0^x F(t)g(t)\,dt.$$

- Using Eq. (1), get $I[f\,I[g]](x) = I[f]I[g](x) - I[I[f]\,g](x)$.

$$I[f\,I[g]] = I[f]I[g] - I[I[f]\,g], \quad I[f]I[g] = I[f\,I[g]] + I[I[f]\,g].$$

- An integral algebra is an algebra R together with a linear operator $I : R \to R$ that satisfies

$$I[f]I[g] = I[f\,I[g]] + I[g\,I[f]], \quad \forall f, g \in R.$$

Free commutative Rota-Baxter algebras

- Rota-Baxter algebra

$$P(x)P(y) = P(P(x)y) + P(xP(y)) + \lambda P(xy).$$

$$P(x)P(y) \mapsto_P P(P(x)y) + P(xP(y)) + \lambda P(xy).$$

This leads to normal forms w with no subwords of the form $P(x)P(y)$.

Free commutative Rota-Baxter algebras

- Rota-Baxter algebra

$$P(x)P(y) = P(P(x)y) + P(xP(y)) + \lambda P(xy).$$

$$P(x)P(y) \mapsto_P P(P(x)y) + P(xP(y)) + \lambda P(xy).$$

This leads to normal forms w with no subwords of the form $P(x)P(y)$.

- In a commutative Rota-Baxter algebra, this means

$$\mathfrak{a} = a_0 P(a_1 P(a_2 P(\cdots P(a_n)\cdots))), a_i \in A.$$

$$\mathfrak{a} = a_0 \otimes a_1 \otimes \cdots \otimes a_n \in A^{\otimes(n+1)}.$$

The product is given by

$$\mathfrak{a}\mathfrak{b} = (a_0 b_0) \otimes \big((a_1 \otimes \cdots a_n) ш_\lambda (b_1 \otimes \cdots \otimes b_m)\big).$$

$ш_\lambda$ is a shuffle like product, called mixable shuffle product.

Mixable shuffle product

- Let A be a commutative **k**-algebra. Let $Ш^+(A) = \bigoplus_{n\geq 0} A^{\otimes n} (= T(A))$. Consider the following products on $Ш^+(A)$.

Mixable shuffle product

- Let A be a commutative $\mathbf{k}$-algebra. Let $Ш^+(A) = \bigoplus_{n\geq 0} A^{\otimes n} (= T(A))$. Consider the following products on $Ш^+(A)$.
- A shuffle of $\mathfrak{a} = a_1 \otimes \ldots \otimes a_m$ and $\mathfrak{b} = b_1 \otimes \ldots \otimes b_n$ is a tensor list of a_i and b_j without change the order of the a_is and b_js.

Mixable shuffle product

- Let A be a commutative **k**-algebra. Let $Ш^+(A) = \bigoplus_{n \geq 0} A^{\otimes n} (= T(A))$. Consider the following products on $Ш^+(A)$.
- A shuffle of $\mathfrak{a} = a_1 \otimes \ldots \otimes a_m$ and $\mathfrak{b} = b_1 \otimes \ldots \otimes b_n$ is a tensor list of a_i and b_j without change the order of the a_is and b_js.
- A mixable shuffle is a shuffle in which some pairs $a_i \otimes b_j$ are merged into $a_i b_j$.

  Define $(a_1 \otimes \ldots \otimes a_m) Ш_\lambda (b_1 \otimes \ldots \otimes b_n)$ to be the sum of mixable shuffles of $a_1 \otimes \ldots \otimes a_m$ and $b_1 \otimes \ldots \otimes b_n$.

Mixable shuffle product

- Let A be a commutative **k**-algebra. Let $Ш^+(A) = \bigoplus_{n\geq 0} A^{\otimes n} (= T(A))$. Consider the following products on $Ш^+(A)$.
- A shuffle of $\mathfrak{a} = a_1 \otimes \ldots \otimes a_m$ and $\mathfrak{b} = b_1 \otimes \ldots \otimes b_n$ is a tensor list of a_i and b_j without change the order of the a_is and b_js.
- A mixable shuffle is a shuffle in which some pairs $a_i \otimes b_j$ are merged into $a_i b_j$.
  Define $(a_1 \otimes \ldots \otimes a_m) Ш_\lambda (b_1 \otimes \ldots \otimes b_n)$ to be the sum of mixable shuffles of $a_1 \otimes \ldots \otimes a_m$ and $b_1 \otimes \ldots \otimes b_n$.
- **Example:**

$$
\begin{aligned}
& a_1 Ш_\lambda (b_1 \otimes b_2) \\
& = a_1 \otimes b_1 \otimes b_2 + b_1 \otimes a_1 \otimes b_2 + b_1 \otimes b_2 \otimes a_1 \quad \text{(shuffles)} \\
& + a_1 b_1 \otimes b_2 + b_1 \otimes a_1 b_2 \quad \text{(merged shuffles)}.
\end{aligned}
$$

Quasi-shuffle product

- The mixable shuffle product can also be defined recursively by the **quasi-shuffle product** (Hoffman, 2000).

Quasi-shuffle product

- The mixable shuffle product can also be defined recursively by the **quasi-shuffle product** (Hoffman, 2000).
- Let $Ш^+(A) = \bigoplus_{n\geq 0} A^{\otimes n} (= T(A))$.

Quasi-shuffle product

- The mixable shuffle product can also be defined recursively by the **quasi-shuffle product** (Hoffman, 2000).
- Let $\text{Ш}^+(A) = \bigoplus_{n\geq 0} A^{\otimes n} (= T(A))$.
- Let $\mathbf{1_k} \in \mathbf{k}$ denote the unit. Let $\mathfrak{a} = a_1 \otimes \cdots \otimes a_m \in A^{\otimes m}$ and $\mathfrak{b} = b_1 \otimes \cdots \otimes b_n \in A^{\otimes n}$. Write $\mathfrak{a} = a_1 \otimes \mathfrak{a}'$, $\mathfrak{b} = b_1 \otimes \mathfrak{b}'$. We have

$$(a_1 \otimes \mathfrak{a}') \text{Ш} (b_1 \otimes \mathfrak{b}') = a_1 \otimes (\mathfrak{a}' \text{Ш} (b_1 \otimes \mathfrak{b}')) + b_1 \otimes ((a_1 \otimes \mathfrak{a}') \text{Ш} \mathfrak{b}') + a_1 b_1 \otimes (\mathfrak{a}' \text{Ш} \mathfrak{b}'),$$

with the convention that if $\mathfrak{a} = a_1$, then $\mathfrak{a}'$ multiples as the identity.

Quasi-shuffle product

- The mixable shuffle product can also be defined recursively by the **quasi-shuffle product** (Hoffman, 2000).
- Let $Ш^+(A) = \bigoplus_{n\geq 0} A^{\otimes n} (= T(A))$.
- Let $\mathbf{1_k} \in \mathbf{k}$ denote the unit. Let $\mathfrak{a} = a_1 \otimes \cdots \otimes a_m \in A^{\otimes m}$ and $\mathfrak{b} = b_1 \otimes \cdots \otimes b_n \in A^{\otimes n}$. Write $\mathfrak{a} = a_1 \otimes \mathfrak{a}'$, $\mathfrak{b} = b_1 \otimes \mathfrak{b}'$. We have

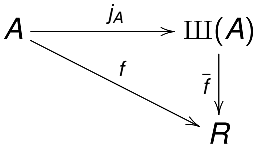
$$(a_1 \otimes \mathfrak{a}') Ш (b_1 \otimes \mathfrak{b}') = a_1 \otimes (\mathfrak{a}' Ш (b_1 \otimes \mathfrak{b}')) + b_1 \otimes ((a_1 \otimes \mathfrak{a}') Ш \mathfrak{b}') + a_1 b_1 \otimes (\mathfrak{a}' Ш \mathfrak{b}'),$$

  with the convention that if $\mathfrak{a} = a_1$, then $\mathfrak{a}'$ multiples as the identity.
- **Example.**

$$a_1 Ш (b_1 \otimes b_2) = a_1 \otimes (\mathfrak{a}' Ш (b_1 \otimes b_2)) + b_1 \otimes (a_1 Ш b_2) + (a_1 b_1) \otimes (\mathfrak{a}' Ш b_2)$$
$$= a_1 \otimes (b_1 \otimes b_2) + b_1 \otimes (a_1 Ш b_2) + (a_1 b_1) \otimes b_2.$$
$$= a_1 \otimes b_1 \otimes b_2 + b_1 \otimes a_1 \otimes b_2 + b_1 \otimes b_2 \otimes a_1 + b_1 \otimes a_1 b_2 + a_1 b_1 \otimes b_2.$$

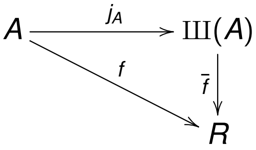
Free commutative Rota-Baxter algebras

- A **free Rota-Baxter algebra over another algebra A** is a Rota-Baxter algebra $Ш(A)$ with an algebra homomorphism $j_A : A \to Ш(A)$ such that for any Rota-Baxter algebra R and algebra homomorphism $f : A \to R$, there is a unique Rota-Baxter algebra homomorphism making the diagram commute



Free commutative Rota-Baxter algebras

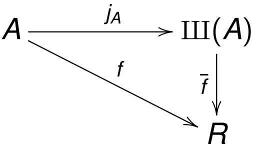
- A **free Rota-Baxter algebra over another algebra A** is a Rota-Baxter algebra $Ш(A)$ with an algebra homomorphism $j_A : A \to Ш(A)$ such that for any Rota-Baxter algebra R and algebra homomorphism $f : A \to R$, there is a unique Rota-Baxter algebra homomorphism making the diagram commute



- When $A = \mathbf{k}[X]$, we have the free Rota-Baxter algebra over X.

Free commutative Rota-Baxter algebras

- A **free Rota-Baxter algebra over another algebra A** is a Rota-Baxter algebra $\text{Ш}(A)$ with an algebra homomorphism $j_A : A \to \text{Ш}(A)$ such that for any Rota-Baxter algebra R and algebra homomorphism $f : A \to R$, there is a unique Rota-Baxter algebra homomorphism making the diagram commute



- When $A = \mathbf{k}[X]$, we have the free Rota-Baxter algebra over X.
- Recall $(\text{Ш}^+(A), ш_\lambda)$ is an associative algebra. Then the tensor product algebra (scalar extension) $\text{Ш}(A) := A \otimes \text{Ш}^+(A)$ is an A-algebra.
  Theorem (Guo-Keigher) $\text{Ш}(A)$ with the shift operator $P(\mathfrak{a}) := 1 \otimes \mathfrak{a}$ is the free commutative RBA over A.

Examples

- The free commutative Rota-Baxter algebra on **k** (i.e., on the empty set) is

$$Ш(\emptyset) = \oplus_{k\geq 1}\mathbf{k}a_k,$$

$$a_m a_n = \sum_{r=0}^{\min(m,n)} \binom{m+n-r}{m}\binom{m}{r}\lambda^r \mathbf{1}^{\otimes(m+n-r)}.$$

When $\lambda = 0$, we obtain the divided powers.

Examples

- The free commutative Rota-Baxter algebra on $\mathbf{k}$ (i.e., on the empty set) is
$$Ш(\emptyset) = \oplus_{k\geq 1}\mathbf{k}a_k,$$
$$a_m a_n = \sum_{r=0}^{\min(m,n)} \binom{m+n-r}{m}\binom{m}{r}\lambda^r \mathbf{1}^{\otimes(m+n-r)}.$$
When $\lambda = 0$, we obtain the divided powers.
- The free commutative integral algebra (Rota-Baxter algebra of weight 0) on $\mathbf{k}[x]$ (i.e., on one generator x):
Let $\mathfrak{I} := \coprod_{k\geq 1}\mathbb{N}_{\geq 0}^k$. For $I = (i_0, \cdots, i_k) \in \mathfrak{I}$, denote
$$x^{\otimes I} := x^{i_0} \otimes \cdots \otimes x^{i_k}.$$
Then $\quad Ш(\mathbf{k}[x]) = \oplus_{I\in\mathfrak{I}}\mathbf{k}x^{\otimes I}.$
For $x^{\otimes I} = x^{i_0} \otimes x^{\bar{I}}$ and $x^{\otimes J} = x^{j_0} \otimes x^{\bar{J}}$, we have
$$x^{\otimes I}x^{\otimes J} = x^{i_0+j_0} \otimes \left(x^{\bar{I}} Ш x^{\bar{J}}\right),$$
where Ш is the shuffle product (partitions and multiple zeta values).

Differential Rota-Baxter algebra

- A differential Rota-Baxter algebra (DRB) is a triple (R, d, P) where (R, d) is a differential algebra (of weight λ), (R, P) is a Rota-Baxter algebra (of weight λ) such that $d \circ P = \mathrm{id}_R$.

Differential Rota-Baxter algebra

- A differential Rota-Baxter algebra (DRB) is a triple (R, d, P) where (R, d) is a differential algebra (of weight λ), (R, P) is a Rota-Baxter algebra (of weight λ) such that $d \circ P = \mathrm{id}_R$.
- These give three rewriting rules that imply that a normal form for the DRB algebra is of the form $\mathfrak{x} := x_0 \otimes x_1 \otimes \cdots \otimes x_n, x_i \in \Delta X$.

Differential Rota-Baxter algebra

- A differential Rota-Baxter algebra (DRB) is a triple (R, d, P) where (R, d) is a differential algebra (of weight λ), (R, P) is a Rota-Baxter algebra (of weight λ) such that $d \circ P = \mathrm{id}_R$.
- These give three rewriting rules that imply that a normal form for the DRB algebra is of the form $\mathfrak{x} := x_0 \otimes x_1 \otimes \cdots \otimes x_n, x_i \in \Delta X$.
- More generally, let (A, d) be a differential algebra of weight λ. On the free commutative Rota-Baxter algebra $(Ш(A), P_A)$, define

$$d_A : Ш(A) \to Ш(A),$$

$$\begin{aligned} d_A(x_0 \otimes x_1 \otimes \cdots \otimes x_n) &= d_0(x_0) \otimes x_1 \otimes \ldots \otimes x_n \\ &+ x_0 x_1 \otimes x_2 \otimes \ldots \otimes x_n + \lambda d_0(x_0) x_1 \otimes x_2 \otimes \ldots \otimes x_n. \end{aligned}$$

Then $(Ш(A), d_A, P_A)$ is the free commutative differential Rota-Baxter algebra on A.

Differential Rota-Baxter algebra

- A differential Rota-Baxter algebra (DRB) is a triple (R, d, P) where (R, d) is a differential algebra (of weight λ), (R, P) is a Rota-Baxter algebra (of weight λ) such that $d \circ P = \mathrm{id}_R$.
- These give three rewriting rules that imply that a normal form for the DRB algebra is of the form $\mathfrak{x} := x_0 \otimes x_1 \otimes \cdots \otimes x_n, x_i \in \Delta X$.
- More generally, let (A, d) be a differential algebra of weight λ. On the free commutative Rota-Baxter algebra $(Ш(A), P_A)$, define

$$d_A : Ш(A) \to Ш(A),$$

$$\begin{aligned} d_A(x_0 \otimes x_1 \otimes \cdots \otimes x_n) &= d_0(x_0) \otimes x_1 \otimes \ldots \otimes x_n \\ &+ x_0 x_1 \otimes x_2 \otimes \ldots \otimes x_n + \lambda d_0(x_0) x_1 \otimes x_2 \otimes \ldots \otimes x_n. \end{aligned}$$

  Then $(Ш(A), d_A, P_A)$ is the free commutative differential Rota-Baxter algebra on A.
- Theorem (Guo-Keigher) Let X be a set. The differential Rota-Baxter algebra $(Ш(\mathbf{k}\{X\}), d_{\mathbf{k}\{X\}}, P_{\mathbf{k}\{X\}})$ is the free differential Rota-Baxter algebra on X.

Integro-differential algebras

- Note that the "integral by parts" formula in Rota-Baxter algebra

$$I(f)I(g) = I(fI(g)) + I(I(f)g)$$

is a "purified" version of the original formula

$$FG = I(F'G) + I(FG')$$

by taking the differentiation out of the picture. This needs to be put back in order to understand fully the algebraic structure in differential equations.

Definition of Integro-differential Algebras

- An integro-differential **k**-algebra of weight λ (also called a λ-integro-differential **k**-algebra) is a differential **k**-algebra (R, D) of weight λ with a linear operator $\Pi\colon R \to R$ such that

$$D \circ \Pi = \mathrm{id}_R$$

and the initialization

$$J\colon = \Pi \circ D$$

satisfies

$$J(x)J(y) = J(x)y + xJ(y) - J(xy) \quad \text{for all } x, y \in R.$$

Equivalent conditions

- Let (R, D) be a differential algebra of weight λ with a linear operator Π on R such that $D \circ \Pi = \mathrm{id}_R$. Denote $J = \Pi \circ D$, called the initialization, and $E = \mathrm{id}_R - J$, called the evaluation. Then the following statements are equivalent:
  1. (R, D, Π) is an integro-differential algebra;
  2. $E(xy) = E(x)E(y)$ for all $x, y \in R$;
  3. $\ker E = \mathrm{im} J$ is an ideal;
  4. $J(xJ(y)) = xJ(y)$ and $J(J(x)y) = J(x)y$ for all $x, y \in R$;
  5. $J(x\Pi(y)) = x\Pi(y)$ and $J(\Pi(x)y) = \Pi(x)y$ for all $x, y \in R$;
  6. $x\Pi(y) = \Pi(D(x)\Pi(y)) + \Pi(xy) + \lambda\Pi(D(x)y)$ and $\Pi(x)y = \Pi(\Pi(x)D(y)) + \Pi(xy) + \lambda\Pi(xD(y))$ for all $x, y \in R$;
  7. (R, D, Π) is a differential Rota-Baxter algebra and $\Pi(E(x)y) = E(x)\Pi(y)$ and $\Pi(xE(y)) = \Pi(x)E(y)$ for all $x, y \in R$;
  8. (R, D, Π) is a differential Rota-Baxter algebra and $J(E(x)J(y)) = E(x)J(y)$ and $J(J(x)E(y)) = J(x)E(y)$ for all $x, y \in R$.

Equivalent conditions

- Let (R, D) be a differential algebra of weight λ with a linear operator Π on R such that $D \circ \Pi = \mathrm{id}_R$. Denote $J = \Pi \circ D$, called the initialization, and $E = \mathrm{id}_R - J$, called the evaluation. Then the following statements are equivalent:
  1. (R, D, Π) is an integro-differential algebra;
  2. $E(xy) = E(x)E(y) \quad$ for all $x, y \in R$;
  3. $\ker E = \mathrm{im} J$ is an ideal;
  4. $J(xJ(y)) = xJ(y)$ and $J(J(x)y) = J(x)y \quad$ for all $x, y \in R$;
  5. $J(x\Pi(y)) = x\Pi(y)$ and $J(\Pi(x)y) = \Pi(x)y \quad$ for all $x, y \in R$;
  6. $x\Pi(y) = \Pi(D(x)\Pi(y)) + \Pi(xy) + \lambda\Pi(D(x)y)$ and $\Pi(x)y = \Pi(\Pi(x)D(y)) + \Pi(xy) + \lambda\Pi(xD(y)) \quad$ for all $x, y \in R$;
  7. (R, D, Π) is a differential Rota-Baxter algebra and $\Pi(E(x)y) = E(x)\Pi(y)$ and $\Pi(xE(y)) = \Pi(x)E(y) \quad$ for all $x, y \in R$;
  8. (R, D, Π) is a differential Rota-Baxter algebra and $J(E(x)J(y)) = E(x)J(y)$ and $J(J(x)E(y)) = J(x)E(y) \quad$ for all $x, y \in R$.
- We will focus on 6: $\Pi(D(x)\Pi(y)) = x\Pi(y) - \Pi(xy) - \lambda\Pi(D(x)y)$.

Integral by parts revisited

- (R, D, Π) is an integro-differential algebra if and only if (R, D) is a differential algebra, $D \circ \Pi = \mathrm{id}_R$ and

$$\Pi(D(x)\Pi(y)) - x\Pi(y) + \Pi(xy) + \lambda\Pi(D(x)y) = 0, \quad \forall x, y \in R.$$

Integral by parts revisited

- (R, D, Π) is an integro-differential algebra if and only if (R, D) is a differential algebra, $D \circ \Pi = \mathrm{id}_R$ and
$$\Pi(D(x)\Pi(y)) - x\Pi(y) + \Pi(xy) + \lambda\Pi(D(x)y) = 0, \quad \forall x, y \in R.$$

- Theorem Let (A, D) be a differential algebra. Let I_{ID} be the differential Rota-Baxter ideal of $\text{Ш}(A)$ generated by elements in the above equations. Then the quotient differential Rota-Baxter algebra $\text{Ш}(A)/I_{ID}$ is the free integro-differential algebra on (A, D).

Integral by parts revisited

- (R, D, Π) is an integro-differential algebra if and only if (R, D) is a differential algebra, $D \circ \Pi = \mathrm{id}_R$ and

$$\Pi(D(x)\Pi(y)) - x\Pi(y) + \Pi(xy) + \lambda\Pi(D(x)y) = 0, \quad \forall x, y \in R.$$

- Theorem Let (A, D) be a differential algebra. Let I_{ID} be the differential Rota-Baxter ideal of $Ш(A)$ generated by elements in the above equations. Then the quotient differential Rota-Baxter algebra $Ш(A)/I_{ID}$ is the free integro-differential algebra on (A, D).
- The last equation suggests the rewriting rule

$$\Pi(D(x)\Pi(y)) \mapsto_{ID} x\Pi(y) - \Pi(xy) - \lambda\Pi(D(x)y).$$

Working in the free differential Rota-Baxter algebra $Ш(A)$ where (A, d) is a differential algebra, this means that $d(x)$ should not appear in Π. More precisely, in $\mathfrak{a} = a_0 \otimes a_1 \otimes \cdots \otimes a_n$, $a_1, \cdots, a_{n-1} \in A$ should be "in complement of" $d(A)$, i.e., in A_T such that $A = \mathrm{im} d \oplus A_T$. Such an A is called regular.

Regular differential algebras

- Let (A, d) be a differential algebra. A linear map $Q : A \to A$ is called a quasi-antiderivative if $d \circ Q \circ d = d$ and $Q \circ d \circ Q = Q$ (and $\ker Q \leq A$ if $\lambda \neq 0$). Then (A, d) is called regular. Take $T := \mathrm{Id} - d \circ Q$.

Regular differential algebras

- Let (A, d) be a differential algebra. A linear map $Q : A \to A$ is called a quasi-antiderivative if $d \circ Q \circ d = d$ and $Q \circ d \circ Q = Q$ (and $\ker Q \leq A$ if $\lambda \neq 0$). Then (A, d) is called regular. Take $T := \mathrm{Id} - d \circ Q$.
- (A, d) is regular if and only if $A = A_T \oplus \mathrm{im} d = A_J \oplus \ker d$.

Regular differential algebras

- Let (A, d) be a differential algebra. A linear map $Q : A \to A$ is called a quasi-antiderivative if $d \circ Q \circ d = d$ and $Q \circ d \circ Q = Q$ (and $\ker Q \leq A$ if $\lambda \neq 0$). Then (A, d) is called regular. Take $T := \mathrm{Id} - d \circ Q$.
- (A, d) is regular if and only if $A = A_T \oplus \mathrm{im} d = A_J \oplus \ker d$.
- Let X be a well ordered set. For $x_1^{(i_1)}, x_2^{(i_2)} \in \Delta X$ with $x_1, x_2 \in X$ and $i_1, i_2 \geq 0$, define

$$x_1^{(i_1)} \leq x_2^{(i_2)} \Leftrightarrow (x_1, -i_1) \leq (x_2, -i_2) \text{ lexicographically.}$$

For example $x^{(2)} < x^{(1)} < x$. Also, $x_1 < x_2$ implies $x_1^{(i_1)} < x_2^{(i_2)}$ for all $i_1, i_2 \geq 0$.

Regular differential algebras

- Let (A, d) be a differential algebra. A linear map $Q : A \to A$ is called a quasi-antiderivative if $d \circ Q \circ d = d$ and $Q \circ d \circ Q = Q$ (and $\ker Q \leq A$ if $\lambda \neq 0$). Then (A, d) is called regular. Take $T := \mathrm{Id} - d \circ Q$.
- (A, d) is regular if and only if $A = A_T \oplus \mathrm{im} d = A_J \oplus \ker d$.
- Let X be a well ordered set. For $x_1^{(i_1)}, x_2^{(i_2)} \in \Delta X$ with $x_1, x_2 \in X$ and $i_1, i_2 \geq 0$, define

$$x_1^{(i_1)} \leq x_2^{(i_2)} \Leftrightarrow (x_1, -i_1) \leq (x_2, -i_2) \text{ lexicographically.}$$

  For example $x^{(2)} < x^{(1)} < x$. Also, $x_1 < x_2$ implies $x_1^{(i_1)} < x_2^{(i_2)}$ for all $i_1, i_2 \geq 0$.
- Let $u \in C(\Delta X)$ (free commutative monoid) be in the form

$$u = u_0^{j_0} \cdots u_k^{j_k}, \text{ where } u_0, \cdots, u_k \in \Delta X, u_0 > \cdots > u_k \text{ and } j_0, \cdots, j_k \geq 1.$$

  Call u functional if either $u \in C(X)$ or $j_k > 1$. Let $A = \mathbf{k}[\Delta X]$ and A_T be the linear span of the functional monomials. Then $\mathbf{k}[\Delta X] = A_T \oplus \mathrm{im} d$ and $\mathbf{k}\{X\}$ is a regular differential algebra.

Construction of free integro-differential algebras

- Let (A, d) be a regular differential algebra. Thus $A = \mathrm{im} d \oplus A_T$ and A_T is a nonunitary subalgebra of A when $\lambda \neq 0$.

Construction of free integro-differential algebras

- Let (A, d) be a regular differential algebra. Thus $A = \mathrm{im} d \oplus A_T$ and A_T is a nonunitary subalgebra of A when $\lambda \neq 0$.
- Let
$$Ш_T(A) = \oplus_{n \geq 0} A \otimes A_T^{\otimes n} = A \oplus (A \otimes A_T) \oplus \cdots$$
be the **k**-submodule of $Ш(A)$. Since A_T is assumed to be a nonunitary subalgebra if $\lambda \neq 0$, $Ш_T(A)$ is a subalgebra of $Ш(A)$.

Construction of free integro-differential algebras

- Let (A, d) be a regular differential algebra. Thus $A = \mathrm{im} d \oplus A_T$ and A_T is a nonunitary subalgebra of A when $\lambda \neq 0$.
- Let
$$Ш_T(A) = \oplus_{n\geq 0} A \otimes A_T^{\otimes n} = A \oplus (A \otimes A_T) \oplus \cdots$$
be the **k**-submodule of $Ш(A)$. Since A_T is assumed to be a nonunitary subalgebra if $\lambda \neq 0$, $Ш_T(A)$ is a subalgebra of $Ш(A)$.
- Let $A_\varepsilon := \{\varepsilon(a) \,|\, a \in A\}$ be another copy of the algebra A, but with the zero derivation. Then both A and A_ε are K-algebras for $K := \ker d$.

Construction of free integro-differential algebras

- Let (A, d) be a regular differential algebra. Thus $A = \mathrm{im} d \oplus A_T$ and A_T is a nonunitary subalgebra of A when $\lambda \neq 0$.
- Let
$$Ш_T(A) = \oplus_{n\geq 0} A \otimes A_T^{\otimes n} = A \oplus (A \otimes A_T) \oplus \cdots$$
be the **k**-submodule of $Ш(A)$. Since A_T is assumed to be a nonunitary subalgebra if $\lambda \neq 0$, $Ш_T(A)$ is a subalgebra of $Ш(A)$.
- Let $A_\varepsilon := \{\varepsilon(a) \mid a \in A\}$ be another copy of the algebra A, but with the zero derivation. Then both A and A_ε are K-algebras for $K := \ker d$.
- Define $ID(A) := A_\varepsilon \otimes_K Ш_T(A) = A_\varepsilon \otimes_K (A \otimes Ш^+(A_T))$ to be the tensor product algebra.

Construction of free integro-differential algebras

- Let (A, d) be a regular differential algebra. Thus $A = \operatorname{im} d \oplus A_T$ and A_T is a nonunitary subalgebra of A when $\lambda \neq 0$.
- Let
$$Ш_T(A) = \oplus_{n \geq 0} A \otimes A_T^{\otimes n} = A \oplus (A \otimes A_T) \oplus \cdots$$
be the **k**-submodule of $Ш(A)$. Since A_T is assumed to be a nonunitary subalgebra if $\lambda \neq 0$, $Ш_T(A)$ is a subalgebra of $Ш(A)$.
- Let $A_\varepsilon := \{\varepsilon(a) \mid a \in A\}$ be another copy of the algebra A, but with the zero derivation. Then both A and A_ε are K-algebras for $K := \ker d$.
- Define $ID(A) := A_\varepsilon \otimes_K Ш_T(A) = A_\varepsilon \otimes_K (A \otimes Ш^+(A_T))$ to be the tensor product algebra.
- Define d_A on $ID(A)$ by the product rule on the tensor product.

Construction of free integro-differential algebras

- Let (A, d) be a regular differential algebra. Thus $A = \mathrm{im} d \oplus A_T$ and A_T is a nonunitary subalgebra of A when $\lambda \neq 0$.
- Let
$$Ш_T(A) = \oplus_{n\geq 0} A \otimes A_T^{\otimes n} = A \oplus (A \otimes A_T) \oplus \cdots$$
be the **k**-submodule of $Ш(A)$. Since A_T is assumed to be a nonunitary subalgebra if $\lambda \neq 0$, $Ш_T(A)$ is a subalgebra of $Ш(A)$.
- Let $A_\varepsilon := \{\varepsilon(a) \mid a \in A\}$ be another copy of the algebra A, but with the zero derivation. Then both A and A_ε are K-algebras for $K := \ker d$.
- Define $ID(A) := A_\varepsilon \otimes_K Ш_T(A) = A_\varepsilon \otimes_K (A \otimes Ш^+(A_T))$ to be the tensor product algebra.
- Define d_A on $ID(A)$ by the product rule on the tensor product.
- For $a = d(Q(a)) + T(a) \in A$, define
$P_A(a) = Q(a) - \varepsilon(Q(a)) + 1 \otimes T(a)$.
For $\mathfrak{a} := a_0 \otimes \cdots \otimes a_n \in A \otimes (A_\psi)^{\otimes n}$, write $\mathfrak{a} = a_0 \otimes \overline{\mathfrak{a}}, \overline{\mathfrak{a}} \in A_\psi^{\otimes n}$. Define
$$P_A(\mathfrak{a}) = Q(a_0) \otimes \overline{\mathfrak{a}} - P_A(Q(a_0)\overline{\mathfrak{a}}) + 1 \otimes T(a_0) \otimes \overline{\mathfrak{a}}.$$

Construction of free integro-differential algebras

- Let (A, d) be a regular differential algebra. Thus $A = \mathrm{im} d \oplus A_T$ and A_T is a nonunitary subalgebra of A when $\lambda \neq 0$.
- Let
$$Ш_T(A) = \oplus_{n \geq 0} A \otimes A_T^{\otimes n} = A \oplus (A \otimes A_T) \oplus \cdots$$
be the **k**-submodule of $Ш(A)$. Since A_T is assumed to be a nonunitary subalgebra if $\lambda \neq 0$, $Ш_T(A)$ is a subalgebra of $Ш(A)$.
- Let $A_\varepsilon := \{\varepsilon(a) \,|\, a \in A\}$ be another copy of the algebra A, but with the zero derivation. Then both A and A_ε are K-algebras for $K := \ker d$.
- Define $ID(A) := A_\varepsilon \otimes_K Ш_T(A) = A_\varepsilon \otimes_K (A \otimes Ш^+(A_T))$ to be the tensor product algebra.
- Define d_A on $ID(A)$ by the product rule on the tensor product.
- For $a = d(Q(a)) + T(a) \in A$, define
$P_A(a) = Q(a) - \varepsilon(Q(a)) + 1 \otimes T(a)$.
For $\mathfrak{a} := a_0 \otimes \cdots \otimes a_n \in A \otimes (A_\psi)^{\otimes n}$, write $\mathfrak{a} = a_0 \otimes \overline{\mathfrak{a}}, \overline{\mathfrak{a}} \in A_\psi^{\otimes n}$. Define
$$P_A(\mathfrak{a}) = Q(a_0) \otimes \overline{\mathfrak{a}} - P_A(Q(a_0)\overline{\mathfrak{a}}) + 1 \otimes T(a_0) \otimes \overline{\mathfrak{a}}.$$

- Theorem (Guo-Regensburger-Rosenkranz) The triple $(ID(A), d_u, P_u)$ is the free commutative integro-differential algebra on (A, d).

Normal forms of integro-differential algebras

- Back to the rewriting rule

$$\Pi(D(x)\Pi(y)) \mapsto_{ID} x\Pi(y) - \Pi(xy) - \lambda\Pi(D(x)y).$$

Normal forms of integro-differential algebras

- Back to the rewriting rule

$$\Pi(D(x)\Pi(y)) \mapsto_{ID} x\Pi(y) - \Pi(xy) - \lambda\Pi(D(x)y).$$

- Then in the free differential Rota-Baxter algebra $Ш(\mathbf{k}\{X\})$. Elements of the form $d(x)$ should not appear in $\Pi(-\Pi(v))$. So for $\mathfrak{a} = a_0 \otimes a_1 \otimes \cdots \otimes a_n$ to be normal, we should have $a_i \in A_T, 1 \leq i \leq n-1$. This is quite hard to verify directly.

Free integro-differential algebras by normal forms

- By the method of Gröbner-Shirshov basis, we obtain.
  Theorem(Gao-Guo-Zheng) Let X be a nonempty well-ordered set and $A := \mathbf{k}\{X\}$. Let $\text{Ш}(\mathbf{k}\{X\}) = \text{Ш}(\mathbf{k}[\Delta X])$, with the derivation d and Rota-Baxter operator P, be the free commutative differential Rota-Baxter algebra of weight λ on X. Let I_{ID} be the differential Rota-Baxter ideal of $\text{Ш}(\mathbf{k}\{X\})$ generated by

  $$S := \{P(d(u)P(v)) - uP(v) + P(uv) + \lambda P(d(u)v) \mid u, v \in \text{Ш}(\mathbf{k}\{X\})\}.$$

  Let A_T be the submodule of $A = \mathbf{k}\{X\}$ spanned by functional monomials. Then the composition

  $$\text{Ш}(A)_T := A \oplus \left(\bigoplus_{k \geq 0} A \otimes A_T^{\otimes k} \otimes A \right) \hookrightarrow \text{Ш}(A) \to \text{Ш}(A)/I_{ID}$$

  of the inclusion and the quotient map is a linear bijection. Thus $\text{Ш}(A)_T$ gives an explicit construction of the free integro-differential algebra $\text{Ш}(A)/I_{ID}$.

- References:

* L. Guo and W. Keigher, Baxter algebras and shuffle products, *Adv. Math* **150** (2000), 117-149.
* L. Guo and W. Keigher, On differential Rota-Baxter algebras, *J. Pure Appl. Algebra* **212** (2008), 522-540.
* M. Rosenkranz and G. Regensburger, Solving and factoring boundary problems for linear ordinary differential equations in differential algebra, *J. Symbolic Comput.* **43** (2008), 515-544.
* L. Guo, G. Regensburger and M. Rosenkranz, On integro-differential algebras, *J. Pure Appl. Algebra* **218** (2014), 456-471.
* X. Gao, L. Guo and S. Zheng, Construction of free commutative integro-differential algebras by the method of Gröbner-Shirshov bases, *Jour Algebra and Its Applications* **13** (2014), 1350160.
* X. Gao and L. Guo, Constructions of free commutative integro-Differential algebras, *Lecture Notes in Computer Science* **8372** (2014), 1-22.

- References:

* L. Guo and W. Keigher, Baxter algebras and shuffle products, *Adv. Math* **150** (2000), 117-149.
* L. Guo and W. Keigher, On differential Rota-Baxter algebras, *J. Pure Appl. Algebra* **212** (2008), 522-540.
* M. Rosenkranz and G. Regensburger, Solving and factoring boundary problems for linear ordinary differential equations in differential algebra, *J. Symbolic Comput.* **43** (2008), 515-544.
* L. Guo, G. Regensburger and M. Rosenkranz, On integro-differential algebras, *J. Pure Appl. Algebra* **218** (2014), 456-471.
* X. Gao, L. Guo and S. Zheng, Construction of free commutative integro-differential algebras by the method of Gröbner-Shirshov bases, *Jour Algebra and Its Applications* **13** (2014), 1350160.
* X. Gao and L. Guo, Constructions of free commutative integro-Differential algebras, *Lecture Notes in Computer Science* **8372** (2014), 1-22.

- **Thank You!**