Differential Type Operators, Rewriting Systems and Gröbner-Shirshov Bases

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Motivation: Classification of Linear Operators

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- Well-known examples include Galois theory where a field is studied by its automorphisms (the Galois group),
- and analysis and geometry where functions and manifolds are studied through their derivations, integrals and related vector fields.

Rota's Question

- By the 1970s, several other operators had been discovered from studies in analysis, probability and combinatorics.

  Average operator $P(x)P(y) = P(xP(y))$,

  Inverse average operator $P(x)P(y) = P(P(x)y)$,

  (Rota-)Baxter operator $P(x)P(y) = P(xP(y) + P(x)y + \lambda xy)$,

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  Reynolds operator $P(x)P(y) = P(xP(y) + P(x)y - P(x)P(y))$.

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- Rota posed the question of finding all the identities that could be satisfied by a linear operator defined on *associative algebras*. He also suggested that there should not be many such operators other than these previously known ones.

Quotation from Rota and Known Operators

- ”In a series of papers, I have tried to show that other linear operators satisfying algebraic identities may be of equal importance in studying certain algebraic phenomena, and I have posed the problem of finding all possible algebraic identities that can be satisfied by a linear operator on an algebra. Simple computations show that the possibility are very few, and the problem of classifying all such identities is very probably completely solvable. A partial (but fairly complete) list of such identities is the following. Besides endomorphisms and derivations, one has averaging operators, Reynolds operators and Baxter operators.”

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- Little progress was made on finding all such operators while new operators have merged from physics and combinatorial studies, such as

$$\text{Nijenhuis operator} \quad P(x)P(y) = P(xP(y) + P(x)y - P(xy)),$$
$$\text{Leroux's TD operator} \quad P(x)P(y) = P(xP(y) + P(x)y - xP(1)y).$$

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- Vast theories were established for differential algebra and difference algebra, with wide applications, including Wen-Tsun Wu's mechanical proof of geometric theorems and mathematics mechanization (based on work of Ritt).
- Rota-Baxter algebra has found applications in classical Yang-Baxter equations, operads, combinatorics, and most prominently, the renormalization of quantum field theory through the Hopf algebra framework of Connes and Kreimer.

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- A **k**-algebra R is called a PI algebra (Procesi, Rowen, ...) if there is a fixed element $f(x_1, \cdots, x_n)$ in the noncommutative polynomial algebra (that is, the free algebra) $\mathbf{k}\langle x_1, \cdots, x_n\rangle$ such that

$$f(a_1, \cdots, a_n) = 0, \quad \forall a_1, \cdots, a_n \in R.$$

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  Thus an algebraic identity satisfied by an algebra is an element in the free algebra.
- Then an algebraic identity satisfied by a linear operator should be an element in a free algebra with an operator, a so called free operated algebra.

Operated algebras

Operated algebras

- An operated $\mathbf{k}$-algebra is a $\mathbf{k}$-algebra R with a linear operator P on R. Examples are given by differential algebras and Rota-Baxter algebras. We can also consider algebras with multiple operators, such as differential-difference algebras, differential Rota-Baxter algebras and integro-differential algebras.

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- The adjoint functor of the forgetful functor from the category of operated algebras to the category of sets gives the free operated **k**-algebras.
- More precisely, a free operated **k**-algebra on a set X is an operated **k**-algebra $(\mathbf{k}\|X\|, \alpha_X)$ together with a map $j_X : X \to \mathbf{k}\|X\|$ with the property that, for any operated algebra (R, β) together with a map $f : X \to R$, there is a unique morphism $\bar{f} : (\mathbf{k}\|X\|, \alpha_X) \to (R, \beta)$ of operated algebras such that $f = \bar{f} \circ j_X$.

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  - counting from the left, the number of $\lfloor$ is always greater or equal to the number of $\rfloor$.
- $\mathfrak{M}(X)$ can also be constructed by rooted trees and Motzkin paths.

- Theorem. 1. The set $\mathfrak{M}(X)$, equipped with the concatenation product, the operator $w \mapsto \lfloor w \rfloor, w \in \mathfrak{M}(X)$ and the natural embedding $j_X : X \to \mathfrak{M}(X)$, is the free operated monoid on X.
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$$\Delta(Z) := Z \cup \lfloor Z \rfloor \cup \lfloor \lfloor Z \rfloor \rfloor \cup \cdots .$$

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- Note that $\mathfrak{D}(Z)$ is closed under multiplication by definition, but not under the operator $\lfloor\ \rfloor$.

Operated Polynomial Identities

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- Examples

  1. When $\phi = [xy] - x[y] - [x]y$, a ϕ-operator (resp. algebra) is a differential operator (resp. algebra).
  2. When $\phi = [x][y] - [x[y]] - [[x]y] - \lambda[xy]$, a ϕ-operator (resp. ϕ-algebra) is a Rota-Baxter operator (resp. algebra) of weight λ.
  3. When $\phi = [x] - x$, then a ϕ-algebra is just an associative algebra. Together with a second identity from the noncommutative polynomial algebra $\mathbf{k}\langle X\rangle$, we get a PI-algebra.

Free ϕ-algebras

- **Proposition** Let $\phi = \phi(x_1, \cdots, x_k) \in \mathbf{k}\llbracket X \rrbracket$ be given. For any set Z, the free ϕ-algebra on Z is given by the quotient operated algebra $\mathbf{k}\llbracket Z \rrbracket / I_{\phi,Z}$ where $I_{\phi,Z}$ is the operated ideal of $\mathbf{k}\llbracket Z \rrbracket$ generated by the set
$$\{\phi(u_1, \cdots, u_k) \mid u_1, \cdots, u_k \in \mathbf{k}\llbracket Z \rrbracket\}.$$

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- Examples
  - When $\phi = [x] - x$, then the quotient $\mathbf{k}\|Z\|/I_{\phi,Z}$ gives the free algebra $\mathbf{k}\langle Z\rangle$ on Z.
  - When $\phi = [xy] - x[y] - [x]y$, then the quotient gives the free noncommutative polynomial differential algebra $\mathbf{k}\langle \mathfrak{D}(Z)\rangle$ on Z.

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- What does this mean?

Examples of compatibility with associativity

- Example 1: For $\phi(x,y) = [xy] - [x]y - x[y]$, we have

$$[xy] \mapsto [x]y + x[y].$$

Thus

$$[(xy)z] \mapsto [xy]z + (xy)[z] \mapsto [x]yz + x[y]z + xy[z].$$

$$[x(yz)] \mapsto [x](yz) + x[yz] \mapsto [x]yz + x[y]z + xy[z].$$

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- Example 3: Suppose $\phi(x,y) = [xy] - [y]x$. Then $[xy] \mapsto [y]x$. So

$$[w]uv \leftarrow\!\shortmid [(uv)w] = [u(vw)] \mapsto [vw]u \mapsto [w]vu.$$

Thus a ϕ-algebra (R,δ) satisfies the weak commutativity:

$$\delta(w)(uv - vu) = 0, \forall u,v,w \in Z.$$

Differential type operators

- differential operator $[xy] = [x]y + x[y]$,
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- They are of the form $[xy] = N(x,y)$ where
  1. $N(x,y) \in \mathbf{k}[\![x,y]\!]$ is in DRF, namely, it does not contain $[uv], u, v \neq 1$, that is, $N(x,y)$ is in $\mathbf{k}\mathfrak{D}(x,y)$;
  2. $N(uv,w) = N(u,vw)$ is reduced to zero under the reduction $[xy] \mapsto N(x,y)$.

An operator identity $\phi(x,y) = 0$ is said of differential type if $\phi(x,y) = [xy] - N(x,y)$ where $N(x,y)$ satisfies these properties. We call $N(x,y)$ and an operator satisfying $\phi(x,y) = 0$ of differential type.

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- The above examples also satisfy
  1. The free ϕ-algebra on Z can be defined by the noncommutative polynomial algebra $\mathbf{k}\langle\Delta(Z)\rangle$ with a suitable operator. So $\mathfrak{D}(Z)$ is a canonical basis of the free object.
  2. The restriction $\mathbf{k}\langle\Delta(Z)\rangle \hookrightarrow \mathbf{k}\|Z\| \to \mathbf{k}\|Z\|/I_\phi(Z)$ is bijective.

Classification of differential type operators

- **(Rota's Problem: the Differential Case)** Find all operated polynomial identities of differential type by finding all expressions $N(x, y) \in \mathbf{k}\lfloor x, y \rfloor$ of differential type.

Classification of differential type operators

- **(Rota's Problem: the Differential Case)** Find all operated polynomial identities of differential type by finding all expressions $N(x,y) \in \mathbf{k}\|x,y\|$ of differential type.
- **Conjecture (OPIs of Differential Type)** Let $\mathbf{k}$ be a field of characteristic zero. Every expression $N(x,y) \in \mathbf{k}\|x,y\|$ of differential type takes one of the forms below for some $a, b, c, e \in \mathbf{k}$:
  1. $b(x\lfloor y\rfloor + \lfloor x\rfloor y) + c\lfloor x\rfloor\lfloor y\rfloor + exy$ where $b^2 = b + ce$,
  2. $ce^2yx + exy + c\lfloor y\rfloor\lfloor x\rfloor - ce(y\lfloor x\rfloor + \lfloor y\rfloor x)$,
  3. $axy\lfloor 1\rfloor + b\lfloor 1\rfloor xy + cxy$,
  4. $x\lfloor y\rfloor + \lfloor x\rfloor y + ax\lfloor 1\rfloor y + bxy$,
  5. $\lfloor x\rfloor y + a(x\lfloor 1\rfloor y - xy\lfloor 1\rfloor)$,
  6. $x\lfloor y\rfloor + a(x\lfloor 1\rfloor y - \lfloor 1\rfloor xy)$.

Rewriting systems

- $\phi(x,y) := \lfloor xy \rfloor - N(x,y) \in \mathbf{k}\llbracket x,y \rrbracket$ defines a rewriting system:

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- Theorem $\phi = [xy] - N(x,y)$ defines a differential type operator if and only if the rewriting system Σ_ϕ is convergent.

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- The Gröbner-Shirshov condition can be weakened to requiring for only intersection and including compositions.

Differential well ordering

- Let $>$ be a well order on a set Z. We extend $>$ to a well order on $\mathfrak{M}(Z) = \varinjlim \mathfrak{M}_n(Z)$ by inductively defining a well ordering $>$ on $\mathfrak{M}_n := \mathfrak{M}_n(Z)$, $n \geq 0$.

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- Suppose a well order $>$ has been defined on $\mathfrak{M}_n$ for $n \geq 0$. Then for $u, v \in Z \cup \lfloor \mathfrak{M}_n \rfloor$, define

$$u > v \Leftrightarrow \begin{cases} u, v \in X, \text{ and } u > v, \text{ or} \\ u \in \lfloor \mathfrak{M}_n \rfloor, v \in x, \text{ or} \\ u = \lfloor u' \rfloor, v = \lfloor v' \rfloor \in \lfloor \mathfrak{M}_n \rfloor \text{ and } u' > v'. \end{cases}$$

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- We obtain a well order, still denoted by $>$, on the direct limit $\mathfrak{M}(Z) = \varinjlim \mathfrak{M}_n$.

Differential well ordering (cont'd)

- Let $\deg_z(u)$ denote the number of $z \in Z$ in u. Denote the **weight** of u by
$$wt(u) = (\deg_z(u), u).$$
Define
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- This order is a monomial well ordering on $\mathfrak{M}(Z)$.
- Under this order, $\lfloor xy \rfloor$ is greater than elements in $\Delta(x, y)$. Thus $\lfloor xy \rfloor$ is the leading term for $\phi(x, y) = \lfloor xy \rfloor - N(x, y)$ when $N(x, y)$ is in DRF.

Differential type, rewriting systems and Gröbner-Shirshov bases

- Theorem. For $\phi(x,y) := \delta(xy) - N(x,y) \in \mathbf{k}\llbracket x,y \rrbracket$, the following statements are equivalent.

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- $\phi(x,y)$ is of differential type;
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- Let Z be a set with a well ordering. With the differential order $>$, the set

$$S := S_\phi := \{\phi(u,v) = \delta(uv) - N(u,v) |\, u, v \in \mathfrak{M}(Z)\backslash\{1\}\}$$

  is a Gröbner-Shirshov basis in $\mathbf{k}\llbracket Z \rrbracket$.
- The free ϕ-algebra on a set Z is the noncommutative polynomial $\mathbf{k}$-algebra $\mathbf{k}\langle\Delta(Z)\rangle$, together with the operator $d := d_Z$ on $\mathbf{k}\langle\Delta(Z)\rangle$ defined by the following recursion:
  Let $u = u_1u_2\cdots u_k \in M(\Delta(Z))$, where $u_i \in \Delta(Z), 1 \leq i \leq k$.
  1. If $k = 1$, i.e., $u = \delta^i(x)$ for some $i \geq 0, x \in Z$, then define $d(u) = \delta^{(i+1)}(x)$.
  2. If $k \geq 1$, then define $d(u) = N(u_1, u_2\cdots u_k)$.

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- Rota-Baxter type operators can be similarly characterized in terms of convergent rewriting systems and Gröbner-Shirshov bases.

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- Also, $M(u,v)$ is formally associative:
$$M(M(u,v),w) = M(u, M(v,w))$$
modulo the relation $\phi_M := [u][v] - [M(u,v)]$.
- Further, free algebras in the corresponding categories (of Rota-Baxter algebras, of average algebras, ...) have a special basis. More precisely, The map
$$\mathbf{k}\{Z\}' := \mathbf{k}\mathfrak{M}'(Z) \to \mathbf{k}\lfloor Z \rfloor \to \mathbf{k}\lfloor Z \rfloor / I_{\phi,Z}$$
is bijective. Thus a suitable multiplication on $\mathbf{k}\{Z\}'$ makes it the free ϕ_M-algebra on Z.
- As we will see, these properties are related.

Classification of Rota-Baxter type operators

- **Conjecture.** Any Rota-Baxter type operator is necessarily of the form
$$P(x)P(y) = P(M(x,y)),$$
for an $M(x,y)$ from the following list (new types in red).
  1. $xP(y)$ (average operator)
  2. $P(x)y$ (inverse average operator)
  3. $xP(y) + yP(x)$
  4. $P(x)y + P(y)x$
  5. $xP(y) + P(x)y - P(xy)$ (Nijenhuis operator)
  6. $xP(y) + P(x)y + e_1xy$ (RBA with weight e_1)
  7. $xP(y) - xP(1)y + e_1xy$
  8. $P(x)y - xP(1)y + e_1xy$
  9. $xP(y) + P(x)y - xP(1)y + e_1xy$ (TD operator with weight e_1)
  10. $xP(y) + P(x)y - xyP(1) - xP(1)y + e_1xy$
  11. $xP(y) + P(x)y - P(xy) - xP(1)y + e_1xy$
  12. $xP(y) + P(x)y - xP(1)y - P(1)xy + e_1xy$
  13. $d_0xP(1)y + e_1xy$ (generalized endomorphisms)
  14. $d_2yP(1)x + e_0yx$

Summary and outlook

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- In the framework of bracketed polynomials, operators of differential type are defined by the convergence of special cases of the rewriting system from the operator identity. The fact that these special cases are enough for the general convergence is proved by Gröbner-Shirshov bases.
- For operators of Rota-Baxter type (including Rota-Baxter, average, Nijenhuis, Leroux's TD), a similar conjecture and equivalence can be established.
- In general, the linear operators that interested Rota and maybe other mathematicians (good operators) should be the ones whose defining identities define convergent rewriting systems (good systems), or give Gröbner-Shirshov bases (good bases).

Thank You!