Rota-Baxter Operators on the Polynomial Algebra

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The general concept of a Rota-Baxter operator originated from G. Baxter’s probability study in 1960. It is a linear operator on an algebra $R$ satisfying

$$P(u)P(v) = P(P(u)v) + P(uP(v)) + \lambda P(uv), \quad \forall u, v \in R.$$ 

At that time, Baxter already observed that in the case of $\lambda = 0$, the Rota-Baxter equation is an (integrally) purified version of the integral by parts formula:

$$FG|_0^x = \int_0^x F'G + \int_0^x FG'.$$ 

There are other Rota-Baxter operators that can be defined analytically.

For example, the operators $J_a(f)(x) := \int_a^x f(t) \, dt$, $a \in \mathbb{R}$ and $P_r(f) := P(rf)$, $r \in R$ are also a Rota-Baxter operator of weight zero.
Main Problem

- **Natural question:** What kind of Rota-Baxter operators can be defined analytically?
- For the underlying algebra where the operators are defined, we take $\mathbb{R}[x]$ since its fundamental importance
  - in analysis, as approximations of analytic functions;
  - in algebra, as the free commutative algebra;
  - in Rota-Baxter algebra, as the initial object.
- So $\mathbb{R}[x]$ provides an ideal testing ground to compare the analysis and algebra.
- For $k = \mathbb{R}$, $J_a(x^n) := \int_a^x t^n \, dt = x^{n+1} - a^{n+1} \frac{n+1}{n+1}$. For general $k$ of characteristic zero, $J_a(x^n) := x^{n+1} - a^{n+1} \frac{n+1}{n+1}$.
- Its premultiplication $J_a r, r \in k[x]$, is still a Rota-Baxter operator.
- These operators are called **analytically modeled**.
- We will consider two classes of Rota-Baxter operators:
  - *monomial* Rota-Baxter operators, for a simple special case, and its independent interest;
  - *injective* Rota-Baxter operators, since we suspect that they give all analytically modeled Rota-Baxter operators.
Monomial Rota-Baxter operators

- A linear operator $P : k[x] \to k[x]$ is called **monomial** if $P(x^n) = \beta(n)x^{\theta(n)}, \forall n \geq 0$.

- So we have $\beta : \mathbb{N} \to k$ and $\theta : \mathbb{N} \to \mathbb{N}$.

- $P$ is called **degenerate** if $\beta(n) = 0$ for some $n$.

- Take $\theta(n) = 0$ if $\beta(n) = 0$.

- **Zero set** of $\phi : A \to B$: $\mathcal{Z}(\phi) := \{a \in A | \phi(a) = 0\}$.

- **Support** of $\phi$: $S(\phi) := A \setminus \mathcal{Z}$.

- **Theorem** A monomial operator $P(x^n) = \beta(n)x^{\theta(n)}$ is a Rota-Baxter operator if $\theta$ and $\beta$ satisfy
  1. $\mathcal{Z}_\beta + \theta(S_\beta) \subseteq \mathcal{Z}_\beta$;
  2. $\theta(m) + \theta(n) = \theta(m + \theta(n)) = \theta(\theta(m) + n)$,
     $\beta(m)\beta(n) = \beta(m + \theta(n))\beta(n) + \beta(n + \theta(m))\beta(m)$,

   for all $m, n \in S_\beta$.

Under the assumption that $S_\beta + \theta(S_\beta) \subseteq S_\beta$, if $P$ is a Rota-Baxter operator then the above conditions hold.
Nondegenerate Monomial Rota-Baxter Operators

- If $P$ is a nondegenerate monomial operator on $k[x]$, then $P$ is a Rota-Baxter operator if and only if
  \[
  \theta(m) + \theta(n) = \theta(m + \theta(n)) = \theta(\theta(m) + n),
  \]
  \[
  \beta(m)\beta(n) = \beta(m + \theta(n))\beta(n) + \beta(n + \theta(m))\beta(m),
  \]

- A map $\theta : S \to S$ on a semigroup $S$ is called an averaging operator if
  \[
  \theta(m\theta(n)) = \theta(m)\theta(n) \quad \text{for all } m, n \in S.
  \]

- A linear map $\Theta : R \to R$ on a $k$-algebra $R$ is called an averaging operator if $\Theta$ is an averaging operator on the multiplicative semigroup of $R$.

- Its study can be tracked back to Reynolds (1895, turbulence theory) and Birkhoff;

- Thus determining $\theta$ for all nondegenerate monomial Rota-Baxter operator on $k[x]$ is equivalent to determine all averaging operators on the semigroup $\mathbb{N}$ with image in $\mathbb{N}^\times := \mathbb{N}\setminus\{0\}$. 
Averaging operators

Let $\mathcal{A}$ denote the set of averaging operators $\theta : \mathbb{N} \rightarrow \mathbb{N}\setminus\{0\}$.

Let $S(\mathbb{N}^\times)$ denote the free semigroup on the set $\mathbb{N}^\times$. So $S(\mathbb{N}^\times) = \coprod_{n \geq 1} (\mathbb{N}^\times)^n$.

**Theorem.** There is a bijection

$$\Phi : \mathcal{A} \rightarrow S(\mathbb{N}^\times)$$

given by

$$\Phi(\theta) = \left( \theta(0), \ldots , \theta(d - 1) \right) / d$$

with

$$d := \min \{ j \in \mathbb{N}^\times \mid \theta(r + j) = \theta(r) + j \text{ for all } r \in \mathbb{N} \}.$$

The inverse map sends $\sigma := (\sigma_0, \cdots , \sigma_{d - 1}) \in S(\mathbb{N}^\times)$ to the map $\theta : \mathbb{N} \rightarrow \mathbb{N}^\times$ defined by $\theta(n) = (\ell + \sigma_j) d$ for $n = \ell d + j$ with $\ell \in \mathbb{N}$ and $0 \leq j < d$. Moreover, we have $\text{im}(\theta) = d\mathbb{N}_{\geq s}$ for $s := \min\{\sigma_0, \cdots , \sigma_{d - 1}\}$.
Algorithm to Determine $\theta$

- Every averaging operator $\theta : \mathbb{N} \to \mathbb{N}^\times$ can be defined as follows.

- Fix $d \geq 1$;

- For each $1 \leq j \leq d$, fix $\sigma_j \in \mathbb{N}^\times$;

- For $n \in \mathbb{N}$ with $n = \ell d + \bar{n}$ where $\bar{n} \in \{0 \ldots d - 1\}$ is the remainder of $n$ modulo $d$, define

$$\theta(n) := n + \sigma_{\bar{n}} d - \bar{n} = \ell d + \sigma_{\bar{n}} d.$$ 

- Theorem  Let $P$ be a monomial Rota-Baxter operator on $k[x]$. The following statements are equivalent.

  1. The operator $P$ is injective.
  2. The map $\theta$ from $P$ satisfies $\theta(n) = n + k$ for some $k \in \mathbb{N}^\times$.
  3. There are $k \in \mathbb{N}^\times$ and $c \in k^\times$ such that $P(x^n) = c \frac{x^{n+k}}{n+k}$ and hence $P = cJ_0 x^{k-1}$. 

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General Monomial Rota-Baxter Operators

We also got partial results in the degenerated case. Improving on these, Houyi Yu obtained the following classification theorem.

**Theorem** Let $P$ be a monomial linear operator on $\mathbf{k}[x]$ defined by $P(x^n) = \beta(n)x^{\theta(n)}$, $n \geq 0$. Then $P$ is a Rota-Baxter operator if and only if there exist a positive integer $d$; nonnegative integers $c_0, c_1, \cdots, c_{d-1}$, and $b_0, b_1, \cdots, b_{d-1} \in \mathbf{k}$ such that

1. $b_i = 0$ if and only if $c_i = 0$, $1 \leq i \leq d - 1$;
2. for all $n \in \mathbb{N}$, we have
   
   \[
   \theta(n) = \begin{cases}
   0, & b_n = 0, \\
   c_n d + n - \bar{n}, & b_n \neq 0;
   \end{cases}
   \]
3. for all $n \in \mathbb{N}$, we have
   
   \[
   \beta(n) = \begin{cases}
   0, & b_n = 0, \\
   \frac{b_n c_n d}{c_n d + n - \bar{n}}, & b_n \neq 0,
   \end{cases}
   \]

where $n \in \{0, 1, \cdots, d - 1\}$ is the remainder of $n$ modulo $d$. 


Injective Rota-Baxter Operators

- An important subclass of Rota-Baxter operators $P$ on $k[x]$ are those associated with the standard derivation $\partial$ in the sense that $\partial \circ P = 1_{k[x]}$.
- More generally, for arbitrary $r \in k[x]^\times$ to the differential law $\partial \circ P = r$, where $r$ denotes the corresponding multiplication operator.
- Define
  $$\text{RBO}_r(k[x]) := \{ P \in \text{RBO}(k[x]) | \partial \circ P = r \}.$$  
  These operators are all injective.
- Let $\text{RBO}^*_r(k[x])$ denote the set of injective Rota-Baxter operators on $k[x]$.
- **Theorem** We have $\text{RBO}^*_r(k[x]) = \bigcup_{r \in k[x]^\times} \text{RBO}_r(k[x])$.
- **Conjecture** For any $r \in k[x]^\times$, we have $\text{RBO}_r(k[x]) = \{ J_a r | a \in k \}$. (At least for $k = \mathbb{R}$.)
Some notations

- for a Rota-Baxter algebra \((R, P)\), the multiplication

\[ \star_P : R \otimes R \to R, \quad u \star_P v := P(u)v + uP(v) \] for all \(u, v \in R\),

is an associative product on \(R\), called the **double multiplication**. Moreover, \((R, \star_P, P)\) is a Rota-Baxter algebra and \(P : (R, \star_P, P) \to (R, P)\) is a homomorphism of nonunitary Rota-Baxter algebras.

- If \(A\) is a \(k\)-module, its (linear) **dual** is denoted by \(A^*\). If \(A\) is moreover a \(k\)-algebra, we use the notation

\[ A^\bullet := \{ \phi \in A^* \mid \phi(uv) = \phi(u)\phi(v) \} \]

for the set of **multiplicative functionals**.

- Through the structure map \(k \to A\) we may also view the elements of \(A^*\) as \(k\)-linear operators on \(A\), and those of \(A^\bullet\) as \(k\)-algebra endomorphisms on \(A\).
We use the abbreviation \( \star r, a \) for \( \star J_a r \), and \( \star r \) for \( \star r, 0 \).

**Theorem**  Let \( r \in k[x]^\times \) and \( a \in k \) be arbitrary. Then the map defined by \( P \mapsto J_a r - P \) is a bijection between \( RBO_r(k[x]) \) and \( (k[x], \star r, a)^. \)

For any \( k \in \mathbb{N} \), we have the isomorphism \( (k[x], \star_k) \cong x^{k+1}k[x] \) of nonunitary algebras.

We have \( RBO_{x^k} (\mathbb{R}[x]) = \{ J_a x^k \mid a \in \mathbb{R} \} \) for any \( k \in \mathbb{N} \).