Rota-Baxter Operators on the Polynomial Algebra

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The analytic origin of Rota-Baxter operators

- The general concept of a Rota-Baxter operator originated from G. Baxter's probability study in 1960. It is a linear operator on an algebra R satisfying

$$P(u)P(v) = P(P(u)v) + P(uP(v)) + \lambda P(uv), \quad \forall u, v \in R.$$

- At that time, Baxter already observed that in the case of $\lambda = 0$, the Rota-Baxter equation is an (integrally) purified version of the integral by parts formula:

$$FG|_0^x = \int_0^x F'G + \int_0^x FG'.$$

- There are other Rota-Baxter operators that can be defined *analytically*.
- For example, the operators $J_a(f)(x) := \int_a^x f(t), dt, a \in \mathbb{R}$ and $P_r(f) := P(rf), r \in R$ are also a Rota-Baxter operator of weight zero.

Main Problem

- Natural question: What kind of Rota-Baxter operators can be defined analytically?
- For the underlying algebra where the operators are defined, we take $\mathbb{R}[x]$ since its fundamental importance
  - in analysis, as approximations of analytic functions;
  - in algebra, as the free commutative algebra;
  - in Rota-Baxter algebra, as the initial object.
- So $\mathbb{R}[x]$ provides an ideal testing ground to compare the analysis and algebra.
- For $\mathbf{k} = \mathbb{R}$, $J_a(x^n) := \int_a^x t^n\,dt = \frac{x^{n+1}-a^{n+1}}{n+1}$. For general $\mathbf{k}$ of characteristic zero, $J_a(x^n) := \frac{x^{n+1}-a^{n+1}}{n+1}$.
- Its premultiplication $J_a r, r \in \mathbf{k}[x]$, is still a Rota-Baxter operator.
- These operators are called analytically modeled.
- We will consider two classes of Rota-Baxter operators:
  - monomial Rota-Baxter operators, for a simple special case, and its independent interest;
  - injective Rota-Baxter operators, since we suspect that they give all analytically modeled Rota-Baxter operators.

Monomial Rota-Baxter operators

- A linear operator $P : \mathbf{k}[x] \to \mathbf{k}[x]$ is called monomial if $P(x^n) = \beta(n)x^{\theta(n)}, \forall n \geq 0$.
- So we have $\beta : \mathbb{N} \to \mathbf{k}$ and $\theta : \mathbb{N} \to \mathbb{N}$.
- P is called degenerate if $\beta(n) = 0$ for some n.
- Take $\theta(n) = 0$ if $\beta(n) = 0$.
- Zero set of $\phi : A \to B$: $\mathcal{Z}(\phi) := \{a \in A \,|\, \phi(a) = 0\}$.
- Support of ϕ: $\mathcal{S}(\phi) := A \backslash \mathcal{Z}$
- **Theorem** A monomial operator $P(x^n) = \beta(n)x^{\theta(n)}$ is a Rota-Baxter operator if θ and β satisfy
  1. $\mathcal{Z}_\beta + \theta(\mathcal{S}_\beta) \subseteq \mathcal{Z}_\beta$;
  2. $$\begin{aligned} \theta(m) + \theta(n) &= \theta(m + \theta(n)) = \theta(\theta(m) + n), \\ \beta(m)\beta(n) &= \beta(m + \theta(n))\beta(n) + \beta(n + \theta(m))\beta(m), \end{aligned}$$

     for all $m, n \in \mathcal{S}_\beta$.

  Under the assumption that $\mathcal{S}_\beta + \theta(\mathcal{S}_\beta) \subseteq \mathcal{S}_\beta$, if P is a Rota-Baxter operator then the above conditions hold.

Nondegenerate Monomial Rota-Baxter Operators

- If P is a nondegenerate monomial operator on $\mathbf{k}[x]$, then P is a Rota-Baxter operator if and only if

$$\theta(m)+\theta(n)=\theta(m+\theta(n))=\theta(\theta(m)+n),$$
$$\beta(m)\beta(n)=\beta(m+\theta(n))\beta(n)+\beta(n+\theta(m))\beta(m),$$

- A map $\theta : S \to S$ on a semigroup S is called an averaging operator if

$$\theta(m\theta(n))=\theta(m)\theta(n) \quad \text{for all } m,n\in S.$$

- A linear map $\Theta : R \to R$ on a $\mathbf{k}$-algebra R is called an averaging operator if Θ is an averaging operator on the multiplicative semigroup of R.
- Its study can be tracked back to Reynolds (1895, turbulence theory) and Birkhoff;
- Thus determining θ for all nondegenerate monomial Rota-Baxter operator on $\mathbf{k}[x]$ is equivalent to determine all averaging operators on the semigroup $\mathbb{N}$ with image in $\mathbb{N}^{\times} := \mathbb{N}\backslash\{0\}$.

Averaging operators

- Let $\mathcal{A}$ denote the set of averaging operators $\theta : \mathbb{N} \to \mathbb{N}\backslash\{0\}$.
- Let $S(\mathbb{N}^\times)$ denote the free semigroup on the set $\mathbb{N}^\times$. So $S(\mathbb{N}^\times) = \coprod_{n\geq 1}(\mathbb{N}^\times)^n$.
- **Theorem.** There is a bijection

$$\Phi\colon \mathcal{A} \to S(\mathbb{N}^\times)$$

given by

$$\Phi(\theta) = \Big(\theta(0), \cdots, \theta(d-1)\Big) \Big/ d$$

with

$$d := \min\{j \in \mathbb{N}^\times \mid \theta(r+j) = \theta(r) + j \text{ for all } r \in \mathbb{N}\}.$$

The inverse map sends $\sigma := (\sigma_0, \cdots, \sigma_{d-1}) \in S(\mathbb{N}^\times)$ to the map $\theta\colon \mathbb{N} \to \mathbb{N}^\times$ defined by $\theta(n) = (\ell + \sigma_j)\,d$ for $n = \ell d + j$ with $\ell \in \mathbb{N}$ and $0 \leq j < d$. Moreover, we have $\mathrm{im}(\theta) = d\mathbb{N}_{\geq s}$ for $s := \min\{\sigma_0, \cdots, \sigma_{d-1}\}$.

Algorithm to Determine θ

- Every averaging operator $\theta : \mathbb{N} \to \mathbb{N}^{\times}$ can be defined as follows.
- Fix $d \geq 1$;
- For each $1 \leq j \leq d$, fix $\sigma_j \in \mathbb{N}^{\times}$;
- For $n \in \mathbb{N}$ with $n = \ell d + \bar{n}$ where $\bar{n} \in \{0 \ldots d-1\}$ is the remainder of n modulo d, define

$$\theta(n) := n + \sigma_{\bar{n}} d - \bar{n} = \ell d + \sigma_{\bar{n}} d.$$

- Theorem Let P be a monomial Rota-Baxter operator on $\mathbf{k}[x]$. The following statements are equivalent.
  1. The operator P is injective.
  2. The map θ from P satisfies $\theta(n) = n + k$ for some $k \in \mathbb{N}^{\times}$.
  3. There are $k \in \mathbb{N}^{\times}$ and $c \in \mathbf{k}^{\times}$ such that $P(x^n) = c\frac{x^{n+k}}{n+k}$ and hence $P = cJ_0 x^{k-1}$.

General Monomial Rota-Baxter Operators

- We also got partial results in the degenerated case. Improving on these on these, Houyi Yu obtained the following classification theorem.
- Theorem Let P be a monomial linear operator on $\mathbf{k}[x]$ defined by $P(x^n) = \beta(n)x^{\theta(n)}, n \geq 0$. Then P is a Rota-Baxter operator if and only if there exist a positive integer d; nonnegative integers $c_0, c_1, \cdots, c_{d-1}$, and $b_0, b_1, \cdots, b_{d-1} \in \mathbf{k}$ such that
  1. $b_i = 0$ if and only if $c_i = 0, 1 \leq i \leq d-1$;
  2. for all $n \in \mathbb{N}$, we have
  $$\theta(n) = \begin{cases} 0, & b_n = 0, \\ c_n d + n - \bar{n}, & b_n \neq 0; \end{cases}$$
  3. for all $n \in \mathbb{N}$, we have
  $$\beta(n) = \begin{cases} 0, & b_n = 0, \\ \frac{b_n c_n d}{c_n d + n - \bar{n}}, & b_n \neq 0, \end{cases}$$

  where $n \in \{0, 1, \cdots, d-1\}$ is the remainder of n modulo d.

Injective Rota-Baxter Operators

- An important subclass of Rota-Baxter operators P on $\mathbf{k}[x]$ are those associated with the standard derivation ∂ in the sense that $\partial \circ P = 1_{\mathbf{k}[x]}$.
- More generally, for arbitrary $r \in \mathbf{k}[x]^\times$ to the *differential law* $\partial \circ P = r$, where r denotes the corresponding multiplication operator.
- Define
$$\mathrm{RBO}_r(\mathbf{k}[x]) := \{P \in \mathrm{RBO}(\mathbf{k}[x]) \mid \partial \circ P = r\}.$$
These operators are all injective.
- Let $\mathrm{RBO}_*(\mathbf{k}[x])$ denote the set of injective Rota-Baxter operators on $\mathbf{k}[x]$.
- Theorem We have $\mathrm{RBO}_*(\mathbf{k}[x]) = \bigcup_{r \in \mathbf{k}[x]^\times} \mathrm{RBO}_r(\mathbf{k}[x])$.
- Conjecture For any $r \in \mathbf{k}[x]^\times$, we have $\mathrm{RBO}_r(\mathbf{k}[x]) = \{J_a r \mid a \in \mathbf{k}\}$. (At least for $\mathbf{k} = \mathbb{R}$).

Some notations

- for a Rota-Baxter algebra (R, P), the multiplication

$$\star_P : R \otimes R \to R, \quad u \star_P v := P(u)v + uP(v) \text{ for all } u, v \in R,$$

  is an associative product on R, called the **double multiplication.** Moreover, $(R, \star_P, P)$ is a Rota-Baxter algebra and $P\colon (R, \star_P, P) \to (R, P)$ is a homomorphism of nonunitary Rota-Baxter algebras.
- If A is a **k**-module, its (linear) *dual* is denoted by A^*. If A is moreover a **k**-algebra, we use the notation

$$A^\bullet := \{\phi \in A^* \mid \phi(uv) = \phi(u)\phi(v)\}$$

  for the set of *multiplicative functionals*.
- Through the structure map $\mathbf{k} \to A$ we may also view the elements of A^* as **k**-linear operators on A, and those of $A^\bullet$ as **k**-algebra endomorphisms on A.

Relation with derived product

- We use the abbreviation $\star_{r,a}$ for $\star_{J_a r}$, and $\star_r$ for $\star_{r,0}$.
- Theorem Let $r \in \mathbf{k}[x]^\times$ and $a \in \mathbf{k}$ be arbitrary. Then the map defined by $P \mapsto J_a r - P$ is a bijection between $\mathrm{RBO}_r(\mathbf{k}[x])$ and $(\mathbf{k}[x], \star_{r,a})^\bullet$.
- For any $k \in \mathbb{N}$, we have the isomorphism $(\mathbf{k}[x], \star_k) \cong x^{k+1}\mathbf{k}[x]$ of nonunitary algebras.
- We have $\mathrm{RBO}_{x^k}(\mathbb{R}[x]) = \left\{ J_a x^k \mid a \in \mathbb{R} \right\}$ for any $k \in \mathbb{N}$.