A GROTHENDIECK APPROACH TO DIFFERENTIAL AZUMAYA ALGEBRAS

RAYMOND HOOBLER

1. How Grothendieck thought about derivations

Grothendieck's approach to derivations was through deformation theory. (See [Gro64, §20] for the algebra and [Gro67, §16] for the geometrical picture.) His geometric picture looks like this. If $i : X \hookrightarrow Y$ is a closed embedding defined by a sheaf of ideals $\mathcal{I}$, let $X \hookrightarrow X^{(n)} \hookrightarrow Y$ denote the factorization of i through the n^{th} infinitesmal neighborhood, i.e. $X^{(n)} = V\left(\mathcal{I}^{n+1}\right)$. As a topological space, $X^{(n)}$ is the same as X. Only the sheaf of functions has changed essentially by taking 'normal derivatives to i up to order n'. Let $f : X \to S$, $\Delta_f = \Delta_{X/S} : X \to X \times_S X$, and $\Delta_2 : X \to X \times_S X \times_S X$ be the diagonal embeddings. Then we have a diagram that looks like,

$$\begin{array}{cccc}
X \hookrightarrow & X^{(1)} & \hookrightarrow & X \times_S X \times_S X \\
\| & q_{12} \downarrow q_{23} \downarrow q_{13} \downarrow & & p_{12} \downarrow p_{23} \downarrow p_{13} \downarrow \\
X \hookrightarrow & X^{(1)} & \hookrightarrow & X \times_S X \\
 & q_1 \downarrow q_2 \downarrow & & p_1 \downarrow p_2 \downarrow \\
 & X & = & X
\end{array}$$

where p_i or p_{ij} indicates projection onto the i^{th} or ij^{th}factor and q_i or q_{ij} is the restriction to the first order neighborhood of the respective projection maps. The reader should think of this diagram as a patching, possibly with a cocycle condition, or a "descent interpretation".

Keeping this in the back of our minds, let us translate the bottom part into algebra language. Let $S = Spec\,(A)\,, X = Spec\,(B)$ so that B is an A algebra. Then the middle row may be interpreted by the exact sequence (where, for simplicity, all $\otimes$ are taken over A unless specified,)

$$0 \to \mathcal{I}/\mathcal{I}^2 \to B \otimes B/\mathcal{I}^2 \to B \to 0.$$

Define $\Omega^1_{B/A} := \mathcal{I}/\mathcal{I}^2$ and $P^1_{B/A} := B \otimes B/\mathcal{I}^2$. ($P^n_{B/A} = B \otimes B/\mathcal{I}^{n+1}$ and is called the module of n^{th} principal parts.) Now

$$\begin{aligned}
\mathcal{I} &= \left\{\sum a_i \otimes b_i \mid \sum a_i b_i = 0\right\} \\
&= \left\{\sum a_i \left(1 \otimes b_i - b_i \otimes 1\right) \mid \sum a_i b_i = 0\right\},
\end{aligned}$$

and so, as a B module, it is generated by $\left\{d_{B/A} b \mid b \in B\right\}$ where we define $d_{B/A} b := 1 \otimes b - b \otimes 1$. We note however that P has two distinct B-algebra structures given by the maps q_1 and q_2 respectively. We will always use the left B action (which comes from q_1) and show the right B action by the immediately verified formulas

$$b \cdot 1 = b \otimes 1 \text{ and } b + db = 1 \otimes b.$$

Thus $x \cdot_r b = (b + db)x$. Note that $(db) \cdot (dc) = 0$. So $P = B \oplus \Omega$ with multiplication given by

$$\begin{aligned}(a + b\,(dc))\,(e + f\,(dg)) &= a\,(e + f\,(dg)) + b\,(dc)\,(e + f\,(dg)) \\ &= ae + (af\,(dg) + be\,(dc))\end{aligned}$$

in our 'differential' notation.

Ultimately we will be dealing with a non-commutative case and so we introduce here some additional definitions in the setting of a possibly non-commutative A algebra B and a B bimodule. We denote this setting by (B, M). Recall that a B bimodule M has a left and right B action such that $(b_1 m)\, b_2 = b_1\,(m b_2)$ for all $b_1, b_2 \in B$ and $m \in M$.

Definition 1. *Let B be a possibly non-commutative A algebra. An A derivation of B in a B bimodule M is an A linear map $\delta : B \to M$ such that $\delta\,(b_1 b_2) = b_1 \delta\,(b_2) + \delta\,(b_1)\, b_2$.*

Definition 2. *An A derivation of B in a B bimodule M is said to be inner if there is an element $m \in M$ such that $\delta\,(b) = bm - mb$. The set of inner A derivations of B in M is an additive subgroup of $Der_A\,(M)$ denoted by $InnDer_A\,(M)$. The center of $M = Z\,(M)$ and consists of $\{m \in M / bm = mb \text{ for all } b \in B\}$.*

Definition 3. *A square zero deformation of a possibly non-commutative A algebra B by a B bimodule M is a flat A algebra B' and a sequence of homomorphisms*

$$0 \to I \xrightarrow{i} B' \xrightarrow{\pi} B \to 1$$

such that π is an A algebra homomorphism, $I^2 = 0$,and $I \approxeq M$ as a B bimodule with the action induced by $I^2 = 0$. The collection of all such deformations is denoted $Def_A\,(B, M)$. If $B' \approxeq B \oplus I$ as A modules, i.e. π admitsd an A module splitting homomorphism, the square zero extension is called a Hochschild deformation.

When working with a Hochschild deformation, we will always identify I with M and B' with $B \oplus I$ by choosing an A splitting for π. A word of warning. In the usual deformation theory setting, deformations are not required to split but merely to be flat over A. Extensions that are split as A modules are called Hochschild extensions in [Wei94, 9.3] which accounts for our definition. If B is a projective A module, then the splitting follows immediately. Also an inner derivation is an A derivation in M since $\delta\,(b_1 b_2) = b_1\,(b_2 m - m b_2) + (b_1 m - m b_1)\, b_2$.

In the (B, M) setting there is always a trivial deformation of B given by the sequence

$$0 \to M \to B \oplus M \to B \to 1$$

where $M^2 = 0$ so that multiplication is given by $(b, m)\,(b', m') := (bb', (bm' + m'b))$.

Lemma 1. *Let B be a possibly non-commutative A algebra, M a B bimodule, and $B' = B \oplus M$ the trivial square zero deformation of B. Then giving an A algebra homomorphism $\phi : B \to B'$ is equivalent to giving $[\delta] \in Der_A\,(M) / Z\,(M)$*

Proof. Let $\phi(b) = (b, \delta_\phi\,(b))$. Then $\delta_\phi\,(b_1 b_2) = \delta_\phi\,(b_1)\, \delta_\phi\,(b_2)$ translates into

$$(b_1 b_2, \delta_\phi\,(b_1 b_2)) = (b_1, \delta_\phi b_1)\,(b_2, \delta_\phi\,(b_2))$$

which means δ_ϕ is an A derivation of B in M. Conversely any such derivation defines an A algebra homomorphism by the above rule for ϕ. □

Proposition 1. *Let B be a commutative A-algebra. Then $d_{B/A} : B \to \Omega^1_{B/A}$ is an A-derivation and the pair $\left(\Omega^1_{B/A}, d_{B/A}\right)$ is universal for all A-derivations of B into any B module M.*

Proof. We drop, as before, the sub and superscripts. Then

$$\begin{aligned} d(b_1 b_2) &= 1 \otimes b_1 b_2 - b_1 b_2 \otimes 1 = b_1 (1 \otimes b_2 - b_2 \otimes 1) + (1 \otimes b_1 - b_1 \otimes 1) \cdot_r b_2 \\ &= b_1 db_2 + (b_2 + db_2)\, db_1 = b_1 db_2 + b_2 db_1. \end{aligned}$$

We note that if M is any B module and we consider $C := B \oplus M$ as a ring with $(b, m)(b', m') = (bb', (mb' + bm'))$, then C is a B algebra via $b'(b, m) = (b'b, b'm)$ and should be considered as a square zero deformation of B since M is an ideal of square zero. Any other B algebra structure will be given by $\phi : B \to C$ where

$$\phi(b, m) = (b, D_\phi(b))$$

and $D_\phi : B \to M$ is a derivation, and any derivation can be used to create a B homomorphism. (Check multiplicativity!). Thus C is specified with two different B algebra structures by specifying the derivations $D = 0$ and $D = D_\phi$. But this information is exactly what we need to define a map $(i, \phi)\, Spec(C) \to Spec(B) \times Spec(B)$ where i is the first structure map and ϕ the second. Since $Spec(C)$ is a square zero deformation, the map (i, ϕ) factors, over B, uniquely through $Spec(B \oplus \Omega)$. So the induced map $Spec(C) \hookrightarrow Spec(B \oplus \Omega)$ is equivalent to specifying a derivation D_ϕ and, as is immediately checked, D_ϕ is uniquely determined by the B algebra homomorphism. □

Corollary 1. *The isomorphism $\Phi : B \oplus \Omega \to B \oplus \Omega$ given by $\Phi(b, \omega) = (b, db + w)$ is a B algebra isomorphism between the left and right B algebra structures on $B \oplus \Omega$. Φ is universal for isomorphisms between a pair of B algebra structures on $B \oplus M$ for any square zero deformation of B by M.*

Proof. Isomorphisms between two B algebra structures on $B \oplus M$ are given by derivations. □

This argument is the basis for Grothendieck's approach. There is always the trivial first order deformation and any other one is determined by a derivation. Consequently the first order deformation of the diagonal embedding is universal for such pairs. This naturally leads to the notion of a connection on a B module M.

Grothendieck views a connection on a B module M as an isomorphism $\Phi : q_1^*(M) \to q_2^*(M)$ which covers the identity $1_M : M \to M$ when restricted to X. Note that this is like determining patching data for the first order neighborhood instead of patching data $p_1^*(M) \to p_2^*(M)$ on the product $X \times_S X$. In algebra terms we are looking for an isomorphism (that I will also call Φ)

$$\Phi : M \otimes_B (B \oplus \Omega) \to (B \oplus \Omega) \otimes_B M$$

where we are using the second B action on the image of Φ and so I have written it as $(B \oplus \Omega) \otimes M$ whereas we have the standard B action on the domain and so I have written it as $M \otimes (B \oplus \Omega)$. By adjointness this map is defined uniquely once we know what the map Φ does on $M \otimes_B B = M$ since Φ must be a P^1-module isomorphism. This isomorphism must cover $1_M : M \to M$ after factoring out by the square zero ideal Ω. Thus Φ is determined by the B module homomorphism $\Phi(m) = (m, \nabla_\Phi(m)) \in (B \oplus \Omega) \otimes_B M$ where

$$\Phi(bm) = (bm, \nabla_\Phi(bm)) = (b + db)(m, \nabla_\Phi(m)) = (bm, db \otimes m + b\nabla_\Phi(m)).$$

Note that since Φ covers 1_M, Φ is a P^1 isomorphism by Nakayama's Lemma as long as M is finitely generated. This gives the usual connection formula and shows that giving a connection on M is equivalent to giving a P^1 isomorphism $\Phi : q_1^*(M) \to q_2^*(M)$.

Finally the same kind of analysis shows that the connection ∇ is integrable, i.e has zero curvature, if and only if $q_{23}^*(\Phi)\, q_{12}^*(\Phi) = q_{13}^*(\Phi)$, that is, if and only if the patching data Φ satisfies the first order neighborhood cocycle condition. Note that if $M = N \otimes B$, then N consists of horizontal sections with respect to the natural integrable connection.

2. Azumaya algebras have connections

2.1. **Azumaya algebras.** In this section A denotes a commutative ring and all unadorned tensor products are taken over A. Recall that an Azumaya A-algebra Λ is an A algebra that is a projective, finitely generated A module such that $\Lambda \otimes A/\mathfrak{m}$ is a central simple division algebra over the field $A/\mathfrak{m}$ for all maximal ideals $\mathfrak{m} \subset A$. Thus $\Lambda \otimes A/\mathfrak{m} \approxeq M_n(D_\mathfrak{m})$ for some integer n and division algebra $D_\mathfrak{m}$ with center $A/\mathfrak{m}$. Let $\Lambda^e := \Lambda \otimes \Lambda^{op}$ where Λ^{op} is the opposite algebra which is Λ as an A module but with the opposite multiplication, i.e. $a \cdot_{op} b = ba$. Note that giving a Λ bimodule is the same as giving a left Λ^e module. The fundamental sequence we are interested in is

$$0 \to \mathcal{J} \to \Lambda^e \stackrel{m}{\to} \Lambda \to 0$$

where m is the multiplication map and $\mathcal{J}$ is the kernel. Since Λ is non commutative, this is only a $\Lambda - \Lambda$ bimodule sequence. However since Λ is a projective A module, we know that the sequence remains exact when reduced modulo any maximal ideal $\mathfrak{m}$. But the sequence

$$0 \to \mathcal{J} \otimes A/\mathfrak{m} \to (\Lambda \otimes A/\mathfrak{m})^e \stackrel{m}{\to} \Lambda \otimes A/\mathfrak{m} \to 0$$

being exact means that it splits since $\Lambda \otimes A/\mathfrak{m} \approxeq M_n(D_\mathfrak{m})$ and so we conclude that Λ is a faithful, projective Λ^e module. (Cue the cheers! This means that there are no non-trivial deformations of Λ as an A algebra!!)

2.2. **Hochschild cohomology.** Next we need an apparent digression on Hochschild cohomology of algebras. (The general reference:is [Wei94]) Let R be a commutative ring and let A be a not necessarily commutative R algebra. Then if M and N are finitely generated, left A modules, $Ext^n_{A/R}(M, N) = H^n(Hom_A(P^*, N))$ where P^* is a projective resolution of M by finitely generated, left projective A modules such that the complex

$$\cdots \to P^1 \to P^0 \to M \to 0$$

is split exact as R modules. The Hochschild cohomology of A/R with coefficients in the A bimodule M is defined to be $H^*(A/R, M) := H^*(Hom_R(P^*, M))$ where P^* is any R split A projective resolution of R. The bar resolution for A/R, which can be constructed from a cotriple ([Wei94, 8.6]) is usually used. We are interested in the case where $A = \Lambda$ and $R = A$. In this case the cochain complex $Hom_A\left(\Lambda^{\otimes n}, M\right) = \left\{A - \text{multilinear maps } f : \Lambda^{\otimes n} \to M\right\}$has a differential given by [Wei94, 9,1]

$$\partial^i f\left(\lambda_0, \ldots, \lambda_n\right) = \begin{cases} \lambda_0 f\left(\lambda_1, \ldots, \lambda_n\right) & \text{if } i = 0 \\ f\left(\lambda_0, \ldots, \lambda_{i-1}\lambda_i, \ldots\right) & \text{if } 0 < i < n \\ f\left(\lambda_0, \ldots, \lambda_{n-1}\right)\lambda_n & \text{if } i = n \end{cases}$$

and so $H^n\left(\Lambda, M\right)$ is the n^{th} homology of the cochain complex

$$0 \to M \to Hom_A\left(\Lambda, M\right) \to Hom_A\left(\Lambda \otimes \Lambda, M\right) \to Hom_A\left(\Lambda \otimes \Lambda \otimes \Lambda, M\right) \to \cdots$$

This may be interpreted in terms of the Ext functor ([Wei94, Lemma 9.1.3]) as

$$H^n\left(\Lambda, M\right) = Ext^n_{\Lambda^e/A}\left(\Lambda, M\right).$$

In particular, when Λ is an Azumaya A algebra, we conclude that $H^n\left(\Lambda . M\right) = 0$ if $n > 0$ and $H^0\left(\Lambda, M\right) = \left\{m/\partial^0 m = \partial^1\left(m\right)\right\} = \left\{m/\lambda m = m\lambda\right\} = Z\left(M\right)$. Recall that M being a Λ bimodule is equivalent to M being a left Λ^e module.

With this in mind, our final result will show that Λ admits an integrable connection.

Theorem 1 ([Wei94, 9.3.1])**.** *Let Λ be a finitely generated, projective A module with a possibly non-commutative A algebra structure, and let M be a finitely generated Λ bimodule. Then*

- $Def_A\left(\Lambda, M\right) = H^2\left(\Lambda, M\right)$,
- $H^1\left(\Lambda, M\right) = Aut_A\left(\Lambda'\right) = Der\left(\Lambda, M\right)/InnDer\left(\Lambda, M\right)$. *where Λ' is the trivial deformation of Λ by M.*

Proof. The hypotheses guarantee that any deformation sequence will split. So choose a splitting and let

$$0 \to M \to \Lambda \oplus M \to \Lambda \to 1$$

be the resulting square zero deformation of Λ by M. Since $m_1 \cdot m_2 = 0$ and $\lambda_1 \cdot \lambda_2 = \lambda_1\lambda_2$ for all $m_1, m_2 \in M$ and $\lambda_1, \lambda_2 \in \Lambda'$, a multiplication map $\mu : \left(M \oplus B\right) \otimes \left(M \oplus B\right) \to M \oplus B$ is completely determined by $f : \Lambda \otimes \Lambda \to M$ where

$$\left(\lambda_1, m_1\right)\left(\lambda_2, m_2\right) = \left(\lambda_1\lambda_2, \lambda_1 m_2 + m_1\lambda_2 + f\left(\lambda_1, \lambda_2\right)\right).$$

In order for this multiplication to be associative, the product $\left(\lambda_0, 0\right)\left(\lambda_1, 0\right)\left(\lambda_2, 0\right)$ shows that f must satisfy

$$\lambda_0 f\left(\lambda_1, \lambda_2\right) - f\left(\lambda_0\lambda_1, \lambda_2\right) + f\left(\lambda_0, \lambda_1\lambda_2\right) - f\left(\lambda_0, \lambda_1\right)\lambda_2 = 0.$$

Consequently $f \in Z^2\left(\Lambda, M\right)$ must be a cocycle. A different choice of splitting yields an $f' \in Z^2\left(\Lambda, M\right)$ which is cohomologous to f. Coinsequently the square zero deformations of Λ by M are precisely given by the cohomology classes in $H^2\left(\Lambda, M\right)$. □

Corollary 2. *Let B be an A algebra, and let Λ be an Azumaya algebra over B. Then Λ has a connection $\triangledown : \Lambda \to \Omega^1_{B/A} \otimes \Lambda$ satisfying*

$$\triangledown\left(\lambda_1\lambda_2\right) = \lambda_1\nabla\left(\lambda_2\right) + \nabla\left(\lambda_1\right)\lambda_2 \textit{ where } \nabla\left(b\right) = d_{B/A}b \otimes 1.$$

∇_1 is another such connection if and only if $\nabla_1 - \nabla = [\lambda, -]$ for a unique $\lambda \in \Omega \otimes_B \Lambda^+$ where Λ^+ is the additive group $\Lambda - B \cdot 1$.

Proof. Since Λ is a projective Λ^e module, $H^2(\Lambda, M) = H^1(\Lambda, M) = 0$ for any Λ bimodule. Setting $M = \Omega \otimes_B \Lambda$ shows that there are no non-trivial square zero deformations. Thus $(B \oplus \Omega) \otimes_B \Lambda \approxeq \Lambda \otimes_B (B \oplus \Lambda)$ and there is a connection on Λ which is determined by $\nabla : B \otimes_B \Lambda \to \Omega \otimes_B \Lambda$ satisfying the identity since it is part of an algebra isomorphism. This identity shows that any other such connection is of the form $\nabla_1 = \nabla + D$ where $D : \Lambda \to \Omega \otimes_B \Lambda$ is a B derivation of Λ in $\Omega \otimes_B \Lambda$. But $H^1(\Lambda, \Omega \otimes_B \Lambda) = Der_B(\Lambda, \Omega \otimes_B \Lambda) / InnDer_B(\Lambda, \Omega \otimes_B \Lambda)$ vanishes and so any B derivation of Λ in $\Omega \otimes_B \Lambda$ is of the form $[\alpha, -]$ for some $\alpha \in \Omega \otimes_B \Lambda$ which is uniquely determined up to adding an element in $Z(\Omega \otimes_B \Lambda)$. Since $Z(\Lambda) = B \cdot 1 \subset \Lambda$ the desired result follows. □

References

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