DIFFSPEC AND OTHER MATTERS

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Dedicated to Jerry Kovacic on his 67th birthday

0.1. **Sheaf Theory.** These are notes of a talk devoted to interpreting results in Benoist's article [Ben08] (He credits Kovacic ([Kov02b] [Kov02a]) for most of them.) in terms of sheaf theory. The goal is to better see where the various conditions imposed on $Spec_\delta$ come from.

Let A be a differential ring containing $\mathbb{Q}$. In order to construct a differential scheme we would hope that A determines a pair $\left(Spec_\delta(A), \mathcal{O}_A^\delta\right)$ where, as a set,

$$Spec_\delta(A) = \{x \mid x \text{ is a differential prime ideal in } A\}.$$

We make it into a topological space by considering the embedding

$$j : Spec_\delta(A) \to Spec(A)$$

and using the induced topology. Alternatively we observe that, while elements $f \in A$ are not functions on $Spec_\delta(A)$, it does make sense to say that f vanishes at $x \in Spec_\delta(A)$ if $f \in \mathfrak{p}_x$, the prime ideal corresponding to x. Then closed sets are of the form $\cap V(f_a) = V([\{f_\alpha\}])$ where $[\{f_\alpha\}]$ indicates the differential ideal generated by $\{f_\alpha\}$ and a basis for the topology is $D_\delta(f)$, $f \in A$.

This takes care of the topological space but not the differential structure sheaf. Here Benoist and Kovacic define $\mathcal{O}_A^\delta$ to be the sheaf associated to the presheaf on $Spec_\delta(A)$, that we indicate by $\underline{A}$, given by $\underline{A}(D_\delta(f)) = A_f$. (It is worth observing that $\mathcal{O}_A^\delta$ can also be described as $j^*(\mathcal{O}_A)$ where j is the contiuous map $Spec_\delta(A) \to Spec(A)$.) Observe that $\mathcal{O}_{A,x}^\delta = A_x$ for any prime differential ideal $x \in Spec_\delta(A)$. But then there are some problems caused by the existence of differential zeros; that is, elements $a \in A$ such that $1 \in [Ann(a)]$. These are elements whose support lies in $Spec(A) - Spec_\delta(A)$ and so cannot be seen using only differential primes. Dealing with such elements caused Kovacic to introduce conditions on A such as AAD (Annihilators Are Differential) or, more generally, reduced ([Kov02b],[Kov02a]). Benoist ([Ben08]) calls the AAD condition 'well-mixed'. Conditions like these are then used to show that $A \to \widehat{A} := \Gamma\left(Spec_\delta(A), \mathcal{O}_A^\delta\right)$ is 'almost surjective' and then everything flows nicely.

We start with the definition of almost surjective and list the consequences.

Definition 1. *A differential ring homomorphism $\phi : A \to B$ is said to be almost surjective if for any differential prime ideal $\mathfrak{p} \subset A$ and any element $b \in B$, there are elements $a_1, a_2 \in A$ with $a_1 \neq \mathfrak{p}$ such that $\phi(a_1) b = \phi(a_2)$.*

Let $\Gamma_\delta : ((\delta - schemes)) \to ((\delta - rings))$ be the global section functor. Let $\widehat{}$ be the composition $\Gamma_\delta \circ Spec_\delta(-)$. Then there is a natural transformation $\iota_A : A \to \widehat{A}$ given by $a \mapsto \frac{a}{1} \in A_x$ for all $x \in Spec_\delta(A)$. This map need not be injective (there

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exist differential zeros) nor surjective (any differentially simple ring that is not a field). We wish to understand the conditions as above that have been introduced to deal with these difficulties in terms of the sheaves involved.

Definition 2. *Let F be a presheaf on X. $+F$ is the presheaf defined by*

$$+F(U) = \lim_{\substack{\{U_\alpha\}_{\alpha\in I} \\ \{U_\alpha\} \text{ is a covering of } U}} Ker\left[\prod_{\alpha\in I} F(U_\alpha) \overset{\rho^{U_\alpha}_{U_\alpha\cap U_\beta} - \rho^{U_\beta}_{U_\alpha\cap U_\beta}}{\rightarrow} \prod_{(\alpha,\beta)\in I\times I} F(U_\alpha\cap U_\beta)\right].$$

Definition 3. *A presheaf F on X is said to be separated if for all open sets $U \subseteq X$ and any covering $\{U_\alpha\}_{\alpha\in I}$ of U, $F(U) \subseteq \prod_{\alpha\in I} F(U_\alpha)$.*

The following Proposition is straightforward and can be found as an exercise in the last chapter of [Ten75], a useful introduction to sheaves.

Proposition 1. *Let F be a presheaf on X.*

(1) *$+F$ is a separated presheaf.*
(2) *If F is a separated presheaf, then $+F$ is a sheaf and $F \hookrightarrow +F$.*

Corollary 1. *If F is a presheaf, the associated sheaf can be defined as $+(+F)$.*

The concepts of almost surjective and lack of differential zeros are critical for understanding $\mathcal{O}^\delta_A$. Our main result identifies the corresponding sheaf properties.

Proposition 2. *Let A be a $\delta-$ring, $X = Spec_\delta(A)$. Then $\iota_A : A \to \widehat{A}$ is almost surjective if and only if $(+\underline{A})(X) \to \widehat{A}$ is surjective.*

Proof. Assume ι_A is almost surjective. Let $s \in \widehat{A}$ be given by $s_x \in A_x$ for all $x \in X$. Then there is a finite covering $\{U_i := D_\delta(b_i)\}_{1\le i\le n}$ of X and a representation of the section s by elements $\frac{a_i}{b_i} \in A_{b_i}$ such that $\left(\frac{a_i}{b_i}\right)_x = s_x \in A_x = \mathcal{O}^\delta_{A,x}$ for every $x \in D_\delta(b_i)$. For each $x \in X$, let $U_x := D^x_\delta(b^x)$ and let $a_x, b_x \in A$ with $a_x(x) \neq 0$ such that

$$\iota_A(a_x)s = \iota_A(b_x).$$

Let $U_x = D_\delta(a_x)$ so that we may assume that

$$s\mid_{U_x} = \iota_{A_{a_x}}\left(\frac{b_x}{a_x}\right) \in \Gamma\left(U_x, \mathcal{O}^\delta_A\right).$$

Since X is quasi-compact, a finite number of U_x, say $U_1, \cdots, U_n$ containing $x_1, \ldots, x_n$ respectively suffice to cover X. But then the element

$$\left(\frac{b_{x_i}}{a_{x_i}}\right) \in \prod A_{fx_i}$$

is in $+\underline{A}(X)$ since

$$\frac{b_{x_i}}{a_{x_i}}\mid_{U_i\cap U_j} = \frac{b_{x_j}}{a_{x_j}}\mid_{U_i\cap U_j}$$

and clearly maps onto $s \in \widehat{A}$. Hence $(+\underline{A})(X) \to \widehat{A}$ is surjective.

Conversely, if $(+\underline{A})(X) \to \widehat{A}$ is surjective and we are given $x \in X$ and $s \in \widehat{A}$ then there is a covering that we may assume is finite by basic opens $U_i := D_\delta(a_i)$, $0 \le i \le n$, and elements $\frac{b_i}{a_i} \in A_{a_i}$ such that $\iota_{A_{a_i}}\left(\frac{b_i}{a_i}\right) = s\mid_{U_i}$. Let U_0 be the open that contains x. For each j, $U_j \cap U_0 = D_\delta(a_0 a_j)$ and on $U_j \cap U_0$ we have $\frac{b_j}{a_j} = \frac{b_0}{a_0} \in A_{a_0 a_j}$ by the definition of $+\underline{A}$. Consequently $(a_0 a_j) \in \sqrt{Ann(a_0 b_j - b_0 a_j)}$. Choose n sufficiently large so that $(a_0 a_j)^n (b_0 a_j - a_0 b_j) = 0$ for all j. Then $a_0^{n+1} \notin \mathfrak{p}_x$ and $\iota_A(a_0^{n+1}) s = \iota_A(b_0 a_0^n)$ since, on U_j,

$$\left(a_0^{n+1}\right)_y s_y = \frac{a_j^n}{a_j^n} \frac{a_0^n a_0 b_j}{a_j} = \frac{a_j^n}{a_j^n} \frac{a_0^n b_0 a_j}{a_j} = (a_0^n b_0).$$

□

Corollary 2. *If $\iota_A : A \to \widehat{A}$ is almost surjective and $\mathfrak{q} \subset \widehat{A}$ is any differential prime ideal with $\mathfrak{p} = i_A^{-1}(\mathfrak{q})$, then $A_\mathfrak{p} \to \widehat{A}_\mathfrak{q}$ is onto.*

Proof. Copying Benoist [Ben08, Corollary 5, 3⟹2], we show that if $\phi : A \to B$ is almost surjective and $\mathfrak{q} \subset B$ with $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$, then $A_\mathfrak{p} \to B_\mathfrak{q}$ is onto. Let $\left(\frac{f}{g}\right) \in B_\mathfrak{q}$ and find $a, b, c, d \in A$ with $b, d \notin \mathfrak{p}$ such that $\phi(b) f = \phi(a)$ and $\phi(d) g = \phi(c)$. Observe that $c \notin \mathfrak{p}$ since $\phi(c) = \phi(d) g \notin \mathfrak{q}$. Then

$$\left(\frac{f}{g}\right)_\mathfrak{q} = \left(\frac{\phi(ad)}{\phi(bc)}\right)_\mathfrak{q} = \phi\left(\left(\frac{ad}{bc}\right)_\mathfrak{p}\right)$$

because $\phi(bc) f = \phi(ac) = \phi(ad) g$. □

Corollary 3. *$\underline{A} \to \mathcal{O}_A^\delta$ is almost surjective when evaluated on any basic open $U = D_\delta(f)$ if and only if $+\underline{A} = \mathcal{O}_A^\delta$.*

Proof. $+\underline{A}$ is separated and so is a subpresheaf in its associated sheaf. □

The condition that $\iota_A : A \to \widehat{A}$ is injective yields even stronger results since then $\underline{A}$ is a separated presheaf ($\widehat{A} \subset \prod_{x \in X} \mathcal{O}_{A,x}^\delta$). This applies if the ring A satisfies AAD (Annihilators Are Differential) or, equivalently, is well-mixed.

Proposition 3. *Let A be a $\delta-$ring, $X = Spec_\delta(A)$. Then $\iota_A : A \to \widehat{A}$ is injective if and only if $\underline{A}$ is a separated presheaf. In this case $(+\underline{A}) \to \mathcal{O}_A^\delta$ is an isomorphism, and $A_f \subset \Gamma\left(D_\delta(f), \mathcal{O}_A^\delta\right)$ for all $f \in A$.*

Corollary 4. *If $\iota_A : A \to \widehat{A}$ is injective, then $\underline{A} \to \mathcal{O}_A^\delta$ is almost surjective. In particular $\iota_A : A \to \widehat{A}$ is almost surjective.*

Thus either of these conditions on the presheaf $\underline{A}$ forces $+\underline{A}$ to be the associated sheaf but the AAD/well mixed condition is slightly preferable since ι_A almost surjective is probably not equivalent to $\underline{A} \to \mathcal{O}_A^\delta$ being almost surjective.

Proposition 4. *Let A be a $\delta-$ring, $X = Spec_\delta(A)$. If $\iota_A : A \to \widehat{A}$ is almost surjective, $\iota_{\widehat{A}} : \widehat{A} \to \widehat{\widehat{A}}$ is an isomorphism.*

Proof. By Corollary 2 $A_\mathfrak{p} \to \left(\widehat{A}\right)_\mathfrak{q}$ is onto if $\mathfrak{p} = i_A^{-1}(\widehat{\mathfrak{p}})$. But we also know that $A_\mathfrak{p} \cong \mathcal{O}_{A,\mathfrak{p}}^\delta$ and this isomorphism factors through the surjection. Hence $A_\mathfrak{p} \to \left(\widehat{A}\right)_{\widehat{\mathfrak{p}}}$ is an isomorphism. But this means that on stalks, $\mathcal{O}_{A,\mathfrak{q}}^\delta \to \mathcal{O}_{\widehat{A},\widehat{\mathfrak{q}}}^\delta$ is an isomorphism if $\iota_{\widehat{A}}^{-1}(\widehat{\mathfrak{q}}) = \mathfrak{q}$. Hence the associated sheaves are isomorphic and $\iota_{\widehat{A}} : \widehat{A} \to \widehat{\widehat{A}}$ is also an isomorphism. □

0.2. **Tangent space.** Tangent vectors to a scheme X at a point $j_0 : x \to X$ are defined using the scheme of dual numbers, $Spec\,(k\,(x)\,[\varepsilon])$ where $\varepsilon^2 = 0$, i.e.$k\,(x)\,[\varepsilon] = k\,(x)\,[T] \,/\, (T^2)$. Thus $t_x = \left\{ j_1 : Spec\,(k\,(x)\,[\varepsilon]) \to X \mid j_1 \mid_{k(x)} = j_0 \right\}$. If we try to do this for differential schemes, we need differential dual numbers. Fix K a differential field. Then $K_\delta\,[\varepsilon]$ should be a differential ring that sees zero divisors but is a 'square zero' extension. Hence we should try

$$K_\delta\,[\varepsilon] = K\,\{T\} \,/\, \left[T^2\right]^+$$

where $I = [T]^+$ denotes the well-mixed differential ideal determined by T and, of course, $\varepsilon = \overline{T}$. This ideal is differentially generated by T^2 and has the property that if $ab \in \left[T^2\right]^+$, then so is ab'. We immediately see that $TT' \in I$, and then $TT'', TT''', \ldots, TT^{(i)}, \ldots \in I$. Applying the condition in the other order then shows that $T^{(i)}T^{(j)} \in I$ for any positive integers i, j. Hence as a K vector space

$$K_\delta\,[\varepsilon] = K + K\varepsilon + K\varepsilon' + \ldots + K\varepsilon^{(i)} + \ldots$$

with all products of basis vectors in $\left\{\varepsilon, \ldots, \varepsilon^{(i)}, \ldots\right\}$ being 0. Thus only the differentiation operator remains to characterize the differential ideal generated by ε. Note that $K \to K_\delta\,[\varepsilon]$ is a $\delta-$homomorphism as is $K_\delta\,[\varepsilon] \to K$. With this definition we could try defining the tangent space at a $\delta-$point $j_0 : Spec\,(K) \to X$ by

$$t_\delta\,(j_0) = \left\{ j_1 : Spec\,(K_\delta\,[\varepsilon]) \to X \mid j_0 \text{ is the composite } Spec\,(K) \to Spec\,(K\,[\varepsilon]) \stackrel{j_1}{\to} X \right\}.$$

In terms of the differential rings this amounts to fixing a differential local ring A_x with residue field K and considering all differential maps $A \to K_\delta\,[\varepsilon] \to K$. Such maps are determined by a differential map $\mathfrak{m}/\mathfrak{m}^2 \to \varepsilon \cdot K_\delta\,[\varepsilon]$ since $\mathfrak{m}$ is a differential ideal and the corresponding products in $K_\delta\,[\varepsilon]$ vanish. This raises the following question:

Question: Is $\dim_K \left(Hom_\delta \left(\mathfrak{m}/\mathfrak{m}^2, \varepsilon \cdot K_\delta\,[\varepsilon]\right)\right)$ finite and, if so, can we define a smooth point as one where it is the same as the dimension given by the differential dimension polynomial of the differential local ring A?

References

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