The Differential Brauer Group

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Outline

1. Review of Brauer groups of fields, rings, and \( \Delta \)-rings

2. Cohomology

3. Cohomological interpretation of \( \Delta \)-Brauer groups (with connections to Hodge theory)
Brauer Groups of Fields

Finite dimensional division algebras $\Lambda$ over a field $K$ are classified by $Br(K)$, the Brauer group of $K$.

- If $\Lambda$ is a central, simple algebra over $K$, then it is isomorphic to $M_n(D)$ for some division algebra $D$.
- Given two such algebras $\Lambda$ and $\Gamma$, $\Lambda \otimes_K \Gamma$ is again a central simple $K$ algebra.
- They are said to be Brauer equivalent if there are vector spaces $V$, $W$ and a $K$—algebra isomorphism $\Lambda \otimes_K End(V) \cong \Gamma \otimes_K End(W)$. This is an equivalence relation, and $Br(K)$ is then defined to be the group formed from the equivalence classes with $\otimes$ as product. $Br(K)$ classifies division algebras over $K$.
- For any such algebra $\Lambda$ over $K$, there is a Galois extension $L/K$ such that $\Lambda \otimes_K L \cong End_L(V)$.
- Galois cohomology is then used to classify all such equivalence classes using an isomorphism $Br(K) \cong H^2\left(G_{\overline{K}/K}, \mathbb{K}^*\right)$. 

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Azumaya algebras over a commutative ring $R$

- A finitely generated central $R$ algebra $\Lambda$ is an Azumaya algebra algebra if $\Lambda \otimes_R L$ is a central simple algebra over $L$ for any homomorphism from $R$ to a field $L$.
- An Azumaya algebra $\Lambda$ is a central, finitely generated $R$ algebra which is a projective $\Lambda \otimes_R \Lambda^{op}$ algebra.

Two such Azumaya algebras $\Lambda$ and $\Gamma$ are Brauer equivalent if there are faithful, projective $R$ modules $P$, $Q$ and an $R$–algebra isomorphism $\Lambda \otimes_R \text{End} (P) \cong \Gamma \otimes_R \text{End} (Q)$.

If $R$ is a local ring, there is an etale extension $S/R$ such that $\Lambda \otimes_R S \cong \text{End}_S (P)$ for some projective $S$ module $P$.

Etale cohomology is then used to classify all such equivalence classes using an isomorphism $\partial : Br (R) \rightarrow H^2 (R_{et}, \mathbb{G}_m)$. 

Brauer Groups of $\Delta$-rings

Let $\Delta = \{\delta_1, ..., \delta_n\}$ be a set of $n$ commuting derivations on $R$, a ring containing $\mathbb{Q}$.

- A $\Delta$ Azumaya algebra over $R$ is an Azumaya algebra $\Lambda$ over $R$ equipped with derivations extending the action of $\Delta$ on $R$.
- Two such $\Delta$ Azumaya algebras $\Lambda$ and $\Gamma$ are $\Delta$ Brauer equivalent if there are faithful, projective $\Delta - R$ modules $P, Q$ and a $\Delta - R$ algebra isomorphism $\Lambda \otimes_R End(P) \cong \Gamma \otimes_R End(Q)$. This is an equivalence relation, and $Br_\Delta(R)$ is the resulting group on the set of equivalence classes with $\otimes_R$ as the product.
- If $R$ is local, there is an etale extension $S$ and a $\Delta - S$ isomorphism $\Lambda \otimes_R S \cong End_S(P)$ for some $\Delta - S$ projective module $P$. 

Let $C$ be a category with fibred products. A pretopology on $C$ consists of specifying for all $X \in \text{ob}(C)$, a set $\text{Cov}(X)$ whose members are collections $\{f_{\alpha} : U_{\alpha} \to X | \alpha \in A\} \in \text{Cov}(X)$ satisfying

1. If $f : X \to X$ is an isomorphism, $\{f\} \in \text{Cov}(X)$.
2. If $\{f_{\alpha} : U_{\alpha} \to X\} \in \text{Cov}(X)$ and $\{g_{i}^{\alpha} : V_{i}^{\alpha} \to U_{\alpha}\} \in \text{Cov}(U_{i})$ for all $i$, then $\{f_{\alpha}g_{i}^{\alpha} : V_{i}^{\alpha} \to X\} \in \text{Cov}(X)$.
3. If $\{f_{\alpha} : U_{\alpha} \to X\} \in \text{Cov}(X)$ and $Y \to X \in C$, then $\{f_{\alpha} \times_{X} Y : U_{\alpha} \times_{X} Y \to Y\} \in \text{Cov}(Y)$.

A presheaf $F : C^{op} \to ((\text{Sets}))$ is a sheaf if for all $X \in C$ and $\{U_{\alpha} \to X\} \in \text{Cov}(X)$,

$$F(X) \leftarrow \prod F(U_{\alpha}) \Rightarrow \prod F(U_{\alpha} \times_{X} U_{\beta})$$

is exact.
Example

1. $X_{et}$ has $\{\{f_\alpha : V_a \to U \mid f_\alpha \text{ is an etale map and } U = \bigcup f_\alpha (V_a)\}\} = Cov_{et} (U)$.

2. $X_{\Delta-fl}$ has $\{\{g_\alpha : V_a \to U \mid g_\alpha \text{ is a flat } \Delta \text{ map of finite type and } U = \bigcup g_\alpha (V_a)\}\} = Cov_{\Delta-fl} (U)$.

If $G$ is a scheme, then its functor of points defines a sheaf in either of these topologies. Moreover there is a map of sites $\tau : X_{\Delta-fl} \to X_{et}$ since any etale map is a flat $\Delta$ map. Thus $\tau^{-1} (\{f_\alpha\}) \in Cov_{\Delta-fl} (U)$. Moreover $H^* (X_{et}, G) \cong H^* (X_{\Delta-fl}, G)$ for sheaves $G$ defined by smooth, quasi-projective group schemes over $X$ like the sheaf of units, $\mathbb{G}_m$. 

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In particular on $X_{\Delta-fl}$ we have the exact sequence

$$0 \to \mathbb{G}_m^\Delta \to \mathbb{G}_m \xrightarrow{d \ln} Z_X^1 \to 0$$

whose cohomology sequence contains

$$H^0 \left( X_{\Delta-fl}, Z_X^1 \right) \to H^1 \left( X_{\Delta-fl}, \mathbb{G}_m^\Delta \right) \to \text{Pic} (X) \xrightarrow{c_1} H^1 \left( X_{\Delta-fl}, Z_X^1 \right)$$

$$\to H^2 \left( X_{\Delta-fl}, \mathbb{G}_m^\Delta \right) \to \text{Br} (X) \to 0$$

if $X$ is smooth since then $H^2 \left( X_{et}, \mathbb{G}_m \right)$ is torsion unlike the vector space $H^2 \left( X_{\Delta-fl}, Z_X^1 \right)$!

How do we interpret this??
Cohomological Interpretation

**Theorem**

Let \( X \) be a quasi-projective variety of finite type over a field \( K \) of characteristic 0. If \( x \in H^2(X, \mu_N) \), then there is an Azumaya algebra \( \Lambda \) equipped with an integrable connection constructed from \( x \) such that

\[
\partial ([\Lambda]) = i_N (x) \in H^2(X_{et}, G_m)
\]

where \( i_N : \mu_N \to G_m \) is inclusion.

For simplicity, let’s consider the case where \( X = \text{Spec}(R) \) is a local ring and \( R \) contains a primitive \( N^{th} \) root of unity. Then there is an etale extension \( R \to S \in \text{Cov}(\text{Spec}(R)) \), i.e. \( U = \text{Spec}(S) \to \text{Spec}(R) \), and a Cech 2 cocycle \( \zeta \in \mu_N(S^3) \) such that \([\zeta] = x \in \check{H}^2((R \to S), \mu_N)\).

Now by refining \( S \) we may assume that it is in the standard form

\( S = (R[T]/(p(T)))_{g(t)} \)

where \( p(T) \) is a monic polynomial of degree \( D \).

So we approximate \( S \) by \( R[t] := R[T]/(p(T)) = \bigoplus_1^D R \).
Now our cocycle $\zeta \in \mu_N (S \otimes R S \otimes R S)$ is constant on each connected component of $S \otimes^3$ but may vary from one component to another. So we must use an index set that accounts for this. We let

$$J = \{ \text{connected components of } S^3 \}$$

Of course $\mathcal{F} := \left( \prod_{\alpha \in J} R [t]_\alpha \right) = \left( \bigoplus_{\alpha \in J} \left( \bigoplus_D R \right) \right)$ is not usually connected but for each connected component $\alpha$ of $S \otimes^3$ there is an $R [t]_\alpha$ which admits multiplication by the value $\zeta_\alpha$ of $\zeta$ on that connected component and, as an $R$ module, $\mathcal{F}$ is free of rank $M = D \cdot (\# (J))$. Then we define an $R$ module isomorphism

$$\ell_\zeta = \bigoplus_J \zeta_\alpha : \mathcal{F} \otimes_R S \otimes_R S \to \mathcal{F} \otimes_R S \otimes_R S$$

by multiplying the $\alpha^{th}$ factor in $\mathcal{F}$ by $\zeta_\alpha$. Note that this amounts to a diagonal block matrix where the $\alpha^{th}$ block is $\zeta_\alpha I_D$. 

Cohomological Interpretation

We let $c (\bigoplus J \zeta_\alpha) : \text{End}_R (\mathcal{F}) \otimes_R S \otimes_R S \to \text{End}_R (\mathcal{F}) \otimes_R S \otimes_R S$ be the algebra isomorphism given by conjugation by $\ell \zeta$. Then we get descent data from the diagram

$\Lambda \to \text{End} (\mathcal{F}) \otimes_R S$ $\downarrow c (\bigoplus J \zeta_\alpha)$

$\text{End} (\mathcal{F}) \otimes_R S \otimes_R S$ $e_1$ $\downarrow$

$\text{End} (\mathcal{F}) \otimes_R S \otimes_R S$ $e_2$ $\uparrow$

where $e_i$ means insert $1_S$ into the $i^{th}$ copy of $S$. Note that $\text{End} (\mathcal{F})$ is the algebra of $M \times M$ matrices with $\delta (e_{ij}) = 0$ for all $\delta \in \Delta$. Here $c (\bigoplus J \zeta_\alpha) = c (\ell \zeta)$ is the patching data used to define $\Lambda$ and it preserves the action of $\Delta$ since $c (\ell \zeta)$ is given by conjugation by an $N^{th}$ root of unity on each block in $\text{End} (\mathcal{F})$. 
Cohomological Interpretation

It satisfies the cocycle condition

\[
\text{End}(\mathcal{F}) \otimes_R S^\otimes 3
\]

\[
e_3 (c (\ell_\zeta)) \uparrow
\]

\[
\text{End}(\mathcal{F}) \otimes_R S^\otimes 3
\]

\[
\downarrow e_1 (c (\ell_\zeta))
\]

\[
e_2 (c (\ell_\zeta)) \downarrow
\]

\[
\text{End}(\mathcal{F}) \otimes_R S^\otimes 3
\]

This commutes because

\[
(e_2 (c (\bigoplus_j \zeta_\alpha)))^{-1} (e_1 (c (\bigoplus_j \zeta_\alpha))) (e_3 (c (\bigoplus_j \zeta_\alpha)))
\]

is conjugation on

\[
\text{End}(\mathcal{F}) \otimes_R S \otimes_R S \otimes_R S
\]

by \(1 \otimes \zeta\) since \(\zeta\) is a 2 cocycle. But this is

\[
1_{\text{End}(\mathcal{F}) \otimes S^\otimes 3}
\]

which is the cocycle condition for descent.
Thus the Cech cocycle provides the needed descent data and we immediately see that

$$\partial ([c (\oplus j \zeta_\alpha)]) = [\zeta] \in \check{H}^2 (X, \mu_n)$$

where $\Lambda = [c (\oplus j \zeta_\alpha)]$ is the desired $\Delta$ Azumaya algebra.