1. General Theory

We fix the following setting. $A$ will be a $\Delta$-ring, $C$ the subring of constants, $E$ a $\Delta$ module, locally free if necessary, $T_n = \oplus_{i=1}^n A t_i$ where $\partial_i(t_i) = 0$ for all $i, j$. We let $((A - \Delta \text{ alg}))$ denote the category of differential $A$ algebras. If $M$ is any $\Delta$ module, let $\mathcal{M}:((A - \Delta \text{ alg})) \to ((A \text{ mod}))$ be the associated covariant functor given by $\mathcal{M}(R) = M \otimes_A R$. Note that $\mathcal{HOM}(F, E)$ is the Hom functor consisting of homomorphisms that are not necessarily $\Delta$ maps. Moreover

$$\mathcal{HOM}(F, E)(R) = Hom_{A \text{ mod}}(F, E)_R \to Hom_{R \text{ mod}}(F_R, E_R)$$

where $M_R := M \otimes_A R$.

**Lemma 1.** $\mathcal{M}$ is represented by the $\Delta$ algebra $S(M^\wedge)$ if $M$ is locally free as an $A$ module. $S(M^\wedge)$ is an $A - \Delta$ algebra.

**Proof.** $\mathcal{M}(B) = M_B \xrightarrow{\alpha} Hom_{A \text{ mod}}(M^\wedge, A)_B \to Hom_{A \text{ alg}}(S(M^\wedge), B)$ which is a $\Delta$ isomorphism since $\alpha(\delta m)(f) = f(\delta m)$ and $\delta (\alpha(m))(f) = \delta(f(m)) - (\delta f)(m) = \delta(f(m)) - [\delta(f(m)) - f(\delta m)]$. $Hom(M^\wedge, A)_B \cong Hom_{A \text{ alg}}(S(M^\wedge), B)$ is a $\Delta$ isomorphism which is easier to check as $Hom_{A \text{ mod}}(M, A)_B \cong Hom_{A \text{ alg}}(S(M), B)$. 

Note that representation does not mean with $\Delta$ maps, i.e. not $\Delta$ representable. However we view the various functors on $((A - \Delta \text{ alg}))$.

**Corollary 1.** If $M$ is a locally free $A - \Delta$ module, $\mathcal{M}^\Delta$ is represented by $\Delta$ homomorphisms from $S(M^\wedge)$. Thus $\mathcal{M}^\Delta(B) := (M_B)^\Delta = Hom_{A - \Delta \text{ alg}}(S(M^\wedge), B)$.

**Proof.** $(M_B)^\Delta \cong Hom(M^\wedge, A)_B^\Delta = Hom_{A - \Delta \text{ alg}}(S(M^\wedge), B)$.

**Corollary 2.** $\mathcal{M} \otimes N$ is represented by $S(M^\wedge) \otimes_A S(N^\wedge)$ if $M, N$ are locally free $A$ modules.

**Corollary 3.** If $M$ is locally free and $A \to B$ is any $\Delta$ ring homomorphism, then $S(M^\wedge_B)$ represents the functor $R \mapsto M_B \otimes_B R$ on the category of $((B - \Delta \text{ alg}))$.

This construction of the symmetric algebra of $M^\wedge$ to represent the functor of the module $M$ is clearly functorial.

**Corollary 4.** If $M \to N$ is one-to-one, resp. onto, then $S(N^\wedge) \to S(M^\wedge)$ is onto, resp. one-to-one.
2. PV Theory

2.1. Classical case: C is a field. We apply the lemma in the following setting where $E$ is an $A - \Delta$ module of rank $n$:

$$\text{ISOM} (T_n, E)^\Delta \hookrightarrow \text{ISOM} (T_n, E) \cap \text{HOM} (T_n, E)^\Delta \hookrightarrow \text{HOM} (T_n, E)$$

where $T_n = \oplus_{i=1}^n A t_i$ as above with $\partial_j (t_i) = 0$ for all $i, j$ and $\text{ISOM} (T_n, E)$ is the open subfunctor consisting of isomorphisms, i.e. $\det$ is a unit. Thus, in view of the corollary,

$$\text{ISOM} (T_n, E)^\Delta (B) = \text{Isom}_\Delta (T_n, B, E_B) = \{ \mathbb{B}/\mathbb{B} \text{ is an ordered basis of } E_B \text{ consisting of constant elements} \}
\subseteq \text{Hom}_{A - \Delta \text{ alg}} (\mathbb{B} (\text{Hom} (F^\vee, T_n^\vee)), B).$$

Thus we see that there is an initial differential $A$ algebra, which, for simplicity, we write as $CB_E$, such that for all differential $A$ algebras $B$,

$$\text{Hom}_{A - \Delta \text{ alg}} (CB_E, B) = \{(e_1, \ldots, e_n) / \{e_i\} \text{ is a basis of } E_B \text{ and } \partial_j (e_i) = 0 \text{ for all } i, j\}.$$

For simplicity, we let $CB_E$ stand for the functor $\text{ISOM} (T_n, E)^\Delta$. Stated slightly differently, $CB_E$ is an initial object in the category of all pairs consisting of an $A - \Delta$ algebra $B$ and a constant ordered basis $\mathbb{B}$ for $E_B$. Such a pair means, of course, that $(E_B)^\Delta$ is a free $B^\Delta$ module with an isomorphism $(E_B)^\Delta \otimes B \rightarrow E_B$.

This gives us one of the conditions for a Picard-Vessiot extension.

The category $((A - \Delta \text{ alg}))$ possesses sums which means that

$$\text{Hom}_{A - \Delta \text{ alg}} (B, -) \oplus \text{Hom}_{A - \Delta \text{ alg}} (B', -) = \text{Hom}_{A - \Delta \text{ alg}} (B \otimes_A B', -).$$

This is well known for commutative $A$ algebras and is immediately checked for $A - \Delta$ algebras. Thus $CB_E \otimes_A CB_F$ represents the functor that we can think of as

$$B \mapsto \{(e_1, \ldots, e_m) / \{e_i\} \text{ is a constant basis of } E_B \} \oplus \{(f_1, \ldots, f_m) / \{f_i\} \text{ is a constant basis of } F_B\}$$

But, when we consider the case $E = F$, we can also describe this as

$$B \mapsto \{(e_1, \ldots, e_m) / \{e_i\} \text{ is a constant basis of } E_B \} \oplus \{\sigma \in \text{Aut}_{B, \Delta} (E_B)\}$$

where $\sigma (e_i) = \sum b_{ij} f_j$ which is the $\Delta$ module homomorphism sending the ordered basis $(e_i)$ to the ordered basis $(f_i)$. Note that $\text{Aut}_{B, \Delta} (E_B) = \text{Gl}_E (B^\Delta)$. Of course if $E$ is not decomposed by $B$, then there will be restrictions on possible automorphisms reflecting the differential structure on $E_B$. Thus the action

$$CB_E (B) \times \text{Aut}_\Delta (E) (B) \rightarrow CB_E (B) \times CB_E (B)$$

is an isomorphism for any $B$ over which $E_B$ has a constant basis. But this can always be achieved by a faithfully flat extension and so $\tau$ is an isomorphism of sheaves. We will need to use this action so we introduce the notation $(\mathbb{B}, g) \in CB_E (B) \times \text{Aut}_\Delta (E) (B) \mapsto (\mathbb{B}, g \ast \mathbb{B})$. This establishes the next result.

**Proposition 1.** The natural action of $\text{Aut}_\Delta (E)$ on $CB_E$ makes $CB_E$ into a principal homogeneous space for $\text{Aut}_\Delta (E)$. In particular $CB_E$ is a principal homogeneous space for the algebraic group $\text{Aut}_\Delta (E)$.

This has the following very nice Corollary.
Corollary 5. Let $\sigma_i : CB_E \to B$, $i = 1, 2$, be two $A - \Delta$ homomorphisms to an arbitrary $A - \Delta$ algebra. Then there is $\tau \in \text{Aut}_\Delta (E)(B)$ such that $\sigma_2 = \tau \circ \sigma_1 : CB_E \to B$.

Here are some useful examples.

Example 1. Suppose $E = T_{n}$. Then we have a given ordered basis of constants, namely $(t_1, t_2, \ldots, t_n)$ and so an ordered basis of constants for $T_{n,B}$ is determined by specifying $\sigma \in \text{GL}_n (B^\Delta)$. Consequently $CB_{T_n} = A \left[ g_{ij}, \det^{-1} \right]$ where $1 \leq i, j \leq n$ and $\det (g_{ij})$ is the determinant of the corresponding matrix.

Example 2. Suppose $E = T_k \oplus E'$. Then $CB_E \cong CB_{T_k} \oplus CB_{E'}$ and so $CB_E \to A \left[ g_{ij}, \det^{-1} \right] \otimes_A CB_{E'}$ where $g_{ij}$ are constant, $1 \leq i, j \leq k$.

There are other ways that constants can appear in $CB_E$. For instance, instead of a repeated $T_1$, $E$ might contain a repeated rank $\Delta$ submodule. Thus if $E = T_k \otimes_A E'$, $CB_E$ would also contain new constants coming from $A \left[ g_{ab}, \det^{-1} \right]$ where $1 \leq a, b \leq k$ that arise from letting $\text{GL}_k (A^\Delta)$ act on the first tensor factor in $E$.

Next we turn to the question of constants and simplicity. So suppose $A$ is a simple $\Delta$ domain with quotient field $K$ and field of constants $C$. If $B$ is a $A - \Delta$ algebra that is a domain, then its quotient field $Q (B)$ contains a transcendental new constant only if $B$ contains a non-zero $\Delta$ ideal. Thus if $B$ is simple, all new constants must be algebraic. So up to algebraic extensions, we need only deal with simplicity.

Now if $I \subseteq CB_E$ is any differential ideal, $I \cap A = (0)$. Thus if we wanted to, we could pass to $K = Q (A)$ to understand the differential ideal structure of $CB_E$, but let’s not do this. Replacing $CB_E$ with $PV_E := CB_E / \mathfrak{m}$ where $\mathfrak{m}$ is a maximal differential ideal in $CB_E$ now gives a simple $\Delta$ ring such that $E_{PV_E}$ has a constant basis and will represent the functor $CB_E$ when restricted to the category of simple $A - \Delta$ algebras. Of course, we must first show that $PV_E$ is independent of the choice of $\mathfrak{m}$.

Theorem 1. Let $A$ be a $\Delta$ simple ring with $C = A^\Delta$ an algebraically closed field, and let $E$ be an $A - \Delta$ module.

1. If $PV_E = CB_E / \mathfrak{m}$ and $PV'_E = CB_E / \mathfrak{m}'$ are two simple extensions defined by maximal $\Delta$ ideals $\mathfrak{m}$ and $\mathfrak{m}'$ respectively, then there is an $A - \Delta$ algebra isomorphism $PV_E \cong PV'_E$.

2. $\text{Aut}_{A-\Delta_{alg}} (PV_E) = \text{Stab}_{\text{Aut} (E_{CB_E})} (\mathfrak{m}) \subseteq \text{Aut}_{A-\Delta} (E_{CB_E})$.

3. $PV_E$ is a principal homogeneous space for $G (E) := \text{Aut}_{A-\Delta_{alg}} (PV_E)$.

Proof. Choose a maximal $\Delta$ ideal $\mathcal{N}$ in $PV_E \otimes_A PV'_E$, and let $B = (PV_E \otimes_A PV'_E) / \mathcal{N}$. We first observe that $PV_E$ and $PV'_E$, being simple, both embed into $B$. Moreover $B^\Delta = C$ since $B$ is a simple $\Delta$ $PV_E$ algebra and $(PV_E)^\Delta$ is also $C$. Thus $B$ is a simple $\Delta$ extension containing both $PV_E$ and $PV'_E$ over which $E$ has a basis of constants, one basis coming from $PV_E$ while the other basis comes from $PV'_E$. Thus there is $\sigma \in \text{Aut}_{A} (E_B) = \text{GL}_n (C)$ such that $\sigma (PV_E) = PV'_E$ since we may apply the $CB_E$ algebra automorphism defined from $\sigma \in \text{Aut}_{A-\Delta} (E)$ (which is $\text{GL}_n (C)$ if $E$ is free) to $CB_E$ before passing to the residue rings.

$\Delta$ automorphisms of $E_{CB_E}$ define $A - \Delta$ algebra automorphisms of $CB_E$ and those that preserve $\mathfrak{m}$ clearly induce $A - \Delta$ algebra automorphisms of $PV_E$. This defines a map

$$\text{Stab}_{\text{Aut} (E_{CB_E})} (\mathfrak{m}) \to \text{Aut}_{A-\Delta_{alg}} (PV_E).$$
Since any differential automorphism of $PV_E$ will have to take a constant basis of $E_{PV_E}$ to a different constant basis, it will come from a differential automorphism of $E_{CB_E}$ and so this map is onto. Similarly if an automorphism of $CB_E$ stabilizes $m$ and induces the identity algebra automorphism on $PV_E$, then it must come from the identity automorphism of $E_{CB}$ and so be the identity algebra automorphism. Consequently

$$Aut_{A - \Delta \text{ alg}} (PV_E) = Stab_{Aut_{\Delta} (E_{CB_E})} (m).$$

$G(E)$ comes equipped with an embedding into $Aut_{\Delta} (E_{CB_E})$

The assertion that $PV_E$ is a principal homogeneous space for $Aut_{\Delta} (E)$ will be established by restricting and evaluating the principal homogeneous space construction for $CB_E$ (2). Let $PV_E$ denote the functor defined by $PV_E$. Then

$$PV_E (B) = Hom_{A - \Delta \text{ alg}} (PV_E, B) \hookrightarrow CB_E (B) = Hom_{A - \Delta \text{ alg}} (CB_E, B)$$

$$\{ \phi : CB_E \to B/\phi (m) = 0 \}$$

and the functor $G$ stabilizing $PV_E$ is characterized as

$$G (B) \hookrightarrow Aut_{\Delta} (E_B)$$

$$\{ g \in Aut_{\Delta} (E_B) / g * \phi \in PV_E (B) \text{ for all } \phi \in PV_E (B) \}$$

$Aut_{\Delta} (E) (B)$ acts transitively on $CB_E (B)$ and so if $\phi, \phi' \in PV_E (B)$, there is a unique $g \in Aut_{\Delta} (E_B)$ such that $g * \phi = \phi'$. But both $\phi (m) = \phi' (m) = 0$ and so $g \in G (B)$. Now we restrict the isomorphism $\tau$ to get the diagram

$$CB_E (B) \times Aut_{\Delta} (E) (B) \cong CB_E (B) \times CB_E (B)$$

$$\cup$$

$$\cup$$

$$PV_E (B) \times G (B) \cong PV_E (B) \times PV_E (B)$$

where the top map is an isomorphism. Clearly the bottom map is 1-to-1. On the other hand $G (B)$ acts transitively on $PV_E (B)$ and so its restriction to the bottom is also an isomorphism.

Finally $G (E)$ is clearly $G (PV_E)$ which finishes the argument. \qed

Note however that the constants have had to be algebraically extended to an algebraically closed field to realize this isomorphism. Consequently, in general, we need an algebraic extension $B$ of $A$ before realizing $PV_{E,B}$.

The key point in the proof is that $\Delta$ module automorphisms of $E_{CB_E}$ are equivalent to $\Delta$ algebra automorphisms of $CB_E$ and those that preserve $m$ clearly induce $A - \Delta$ algebra automorphisms of $PV_E$. Now once we know that $PV_E$ is simple, then we know that $(PV_E)^{\Delta}$ is an algebraic extension of $C$. So if $C$ is an algebraically closed field, we can conclude that $PV_E$ is our usual Picard-Vessiot extension. Thus it is critical that $B - \Delta$ module automorphisms of $E_B$ translate into $A - \Delta$ algebra automorphisms of $CB_E$ when $B$ is simple with algebraically closed constants.

**Example 3.** Suppose $E = T_n$. Then $PV_E \cong A$.

**Example 4.** Suppose $E = T_k \oplus E'$. Then $PV_E \cong PV_{E'}$. This follows from the second example since $PV_E$ is constructed by choosing a maximal $\Delta$ ideal in $CB_E$. But $CB_E \to CB_{E'} \otimes_A A[x_1, \ldots, x_k]$ is onto so we may choose a maximal ideal in $CB_{E'}$ to construct $PV_E$. 
The original question that gave rise to this sequence of ideas was how to calculate $PV_E \otimes_A PV_F$ for two $A - \Delta$ modules $E$ and $F$. While we cannot completely answer the question we can observe that $PV_E \otimes_A PV_F \cong PV_E \otimes_{PV_F} PV_F := PV$ where $F'$ is a $\Delta - PV_E$ module satisfying $F_{PV_E} = T_{k, PV_E} \oplus F'$. If we could eliminate the constants from the $PV_E$ algebra $PV$, we could complete the description, but this requires further analysis of the possible $PV_E$ submodules of $F'$.

Note that we have not assumed that $A$ is a $\Delta$ field, only that $A^\Delta$ is an algebraically closed field. This is of particular interest when $A$ is the local ring of a closed point on a smooth variety $X$ over a field $k$ with derivations $\Delta = \{\partial/\partial x_i\}$ where $m_A = (x_i)$.

2.2. **The case $C$ is a domain.** Here there should be a nascent theory comparable to ramification theory for a domain $R$ with quotient field $K$ and a Galois extension $L/K$. Following Kovacic we consider the case where $C = A^\Delta \to A$ is an almost constant extension or, equivalently, where the set of prime ideals of $C$ equals the set of prime differential ideals of $A$ ...