FPQC DESCENT AND GROTHENDIECK TOPOLOGIES IN A DIFFERENTIAL SETTING

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1. Grothendieck Topologies

All rings are noetherian, all rings are commutative with 1 unless specified at the beginning of the section; ring extensions are of finite type unless otherwise specified; and the notation $((-))$ stands for the category described by $-$.

1.1. **Sites.** We begin with the underlying idea for a Grothendieck topology. Consider your favorite topological space X. Its topology, don't use the Zariski topology, is determined by the partially ordered category $\mathcal{U}_X$ consisting of all the open subsets of X ordered with respect to inclusion. In order to verify that a presheaf is a sheaf on X, we must first construct all possible coverings $\mathcal{U} = \left\{(U_i \subset U)_{i \in I}\right\}$ of open sets U in X and then verify the two sheaf conditions hold for each covering $(U_i \subset U)_{i \in I}$. But we also want to compare coverings in the following two ways. If $(U_i)_{i \in I}$ is a covering of U and for each i, $\left(V_j^i\right)_{j \in J_i}$ is a covering of U_i, then $\left(V_j^i\right)_{\cup J_i}$ is also a covering of U. If $(U_i)_{i \in I}$ is a covering of U and $V \subset U$ is an open set, then $(U_i \cap V)_{i \in I}$ is a covering of V. We are just stating that refinements of each open in a covering yields a covering and intersecting a covering with a smaller open set still yields a covering. Keep this example in mind as we go through some category theoretical definitions. Also remember that $U_i \cap U_j = U_i \times_X U_j$.

We will only consider 'local' topologies. Let $\mathcal{C}/X$ be a category with finite products, i.e. a terminal object X and products of pairs of objects.

Definition 1 (following Artin)**.** *A pretopology $\mathcal{T}$ on $\mathcal{C}/X$ is a set $Cov(\mathcal{T})$ consisting of families $(\pi_i : U_i \to U)_{i \in I}$ for all $U \to X \in \mathcal{C}/X$ such that*

1. *If $\pi : V \to U$ is an isomorphism, then $(\pi) \in Cov(\mathcal{T})$.*
2. *If $(U_i \to U)_{i \in I} \in Cov(\mathcal{T})$ and $\left(V_j^i \to U_i\right)_{j \in I_i} \in Cov(\mathcal{T})$, then $\left(V_j^i \to U\right)_{\cup I_j} \in Cov(\mathcal{T})$.*
3. *If $(U_i \to U)_{i \in I} \in Cov(\mathcal{T})$ and $V \to U \in Mor(\mathcal{C}/X)$, then $U_i \times_U V$ exists and $(U_i \times_U V \to V)_{i \in I} \in Cov(\mathcal{T})$.*

Technically this is called a pretopology on $\mathcal{C}/X$ but it generates a unique Grothendieck topology and the sheaves for this topology are characterized by the pretopology. Such a category with its coverings is called a site.

So here are some examples where X is a scheme.

Example 1. *X_{Zar}, the Zariski site on X, has $\mathcal{C}/X$ equal to the category of open subschemes $U \subset X$.*

$$Cov\{X_{Zar}\} = \left\{(U_i \subset U) \mid {}^{1)}U_i \text{ is open and } {}^{2)}U = \bigcup U_i\right\}$$

Example 2. *X_{et}, the étale site on X, has $\mathcal{C}/X$ equal to the category of all étale schemes $U \to X$.*

$$Cov\{X_{et}\} = \left\{(p_i : U_i \to U) \mid {}^{1)}p_i \text{ is étale and } {}^{2)}U = \bigcup p_i(U_i)\right\}$$

Example 3. *X_{pl}, the flat site on X, has $\mathcal{C}/X$ equal to the category of all schemes $U \to X$ which are locally of finite type and flat over X.*

$$Cov\{X_{pl}\} = \left\{(p_i : U_i \to U) \mid {}^{1)}p_i \text{ is flat, locally of finite type and } {}^{2)}U = \bigcup p_i(U_i)\right\}$$

Some comments:

1. My definition varies somewhat from Milne. I have defined what he calls the small sites while his notation (as above) refers to the big sites.

(2) A good characterization of étale is ([4, I, Theorem 3.14]). This states that $f : X \to Y$ is étale iff for all $x \in X$, there are affine open neighborhoods $V = Spec\,(B)$ of x and $U = Spec\,(A)$ of $f\,(x)$ such that B can be described as follows:
   There is a monic polynomial $P\,(T) \in A\,[T]$ such that
$$B \approxeq \left(\frac{A\,[T]}{\langle P\,(T)\rangle}\right)_g$$
for some $g \in \frac{A[T]}{\langle P(T)\rangle}$ with $P'\,(t)$ a unit in B (t is the image of T in B). Note that this characterization of étale means f is flat, locally of finite type, and unramified, i.e. $f^a\left(\mathfrak{p}_{f(x)}\right) = \mathfrak{p}_x \subseteq B_{\mathfrak{p}_x}$.

(3) If R is a differential ring with derivation δ and S is étale over R, then there is a unique extension of δ to S since locally on $Spec\,(S)$, $S \approxeq \left(\frac{R[T]}{\langle P(T)\rangle}\right)_g$ as above. Hence, if $P\,(T) = a_0 + \cdots + T^n$ and $t = \overline{T}$,
$$\delta\,(t) = \left(\sum \delta\,(a_i)\,t^i\right)/P'\,(t)$$
which is uniquely defined since $P'\,(t)$ is a unit in S.

(4) Flat morphisms that are locally of finite type are open. Thus coverings of U in the flat or étale topology determine a covering of U in the Zariski topology.

(5) In order to define a site by a class E of morphism as above the class of morphisms, e.g. open immersions, étale or flat and locally of finite presentation, must satisfy the sorités:
   - all isomorphisms are in E,
   - if $\alpha : X \to Y$ and $\beta : Y \to Z$ are in E, so is $\beta\alpha$, i.e. E is closed under composition of morphisms, and
   - if $\alpha : X \to Y$ is in E and $Z \to Y$, then $X \times_Y Z \to Z$ is in E, i.e. E is closed under base change.

For the purposes of this discussion let $D = Spec\,(R)$ where R is a differential ring containing $\mathbb{Q}$ with derivation δ. We introduce two new topologies, $D_{\delta-et}$ and $D_{\delta-pl}$ as follows. Let $\mathcal{C}/D_\delta$ be the category of schemes $\pi : X \to D$ such that X is a scheme with a derivation δ_X and π preserves the differentiation.

Example 4. *$D_{\delta-et}$, the $\delta-$étale site on D, has $\mathcal{C}/D$ equal to the full sub-category of $\mathcal{C}/D_\delta$ consisting of all étale schemes $\pi : U \to D$.*
$$Cov\,\{U_{\delta-et}\} = \left\{(p_i : U_i \to U) \mid \begin{array}{c} {}^{1)}p_i \textit{ is étale, } {}^{2)}U = \bigcup p_i\,(U_i)\,, \\ \textit{and } {}^{3)}p_i \textit{ is a differential morphism}\end{array}\right\}$$

Example 5. *$D_{\delta-pl}$, the $\delta-$flat site on D, has $\mathcal{C}/D$ equal to the full sub-category of $\mathcal{C}/D_\delta$ consisting of all schemes $U \to D$ which are locally of finite type and flat over D.*
$$Cov\,\{U_{\delta-pl}\} = \left\{(p_i : U_i \to U) \mid \begin{array}{c} {}^{1)}p_i \textit{ is flat and locally of finite type, } {}^{2)}U = \bigcup p_i\,(U_i)\,, \\ \textit{and } {}^{3)}p_i \textit{ is a differential morphism}\end{array}\right\}$$

If $X \to Y$ and $X' \to Y$ are differential morphisms of differential schemes, then $X \times_Y X'$ is a differential scheme and the projection maps are differential homomorphisms. This follows from the usual covering argument and the fact that, in the affine case, if $\iota_1 : R \to S_1$ and $\iota_2 : R \to S_2$ are differential ring homomorphisms of commutative differential rings, then $S_1 \otimes_R S_2$ is a differential ring with $\delta_{12}\,(s_1 \otimes s_2) = \delta_1\,(s_1) \otimes s_2 + s_1 \otimes \delta_2\,(s_2)$ as derivation and $\varepsilon_i : S_i \to S_1 \otimes_R S_2$ is a sum in the category of commutative differential rings. Furthermore the composite of differential morphisms is a differential morphism. These are the differential sites of interest.

1.2. **FPQC descent and sheaves.** We now (hopefully!) have some idea of what a site is, but we still have no examples of sheaves. This is where descent theory comes in. The current approach is very stacky and categorically tedious. Instead we will follow the original order and discuss descent first. If we look at our model and ask how to construct a (real) vector bundle on our topological space X, we use an equivalence relation on a trivial vector bundle on the disjoint union of the open sets in a covering of X as follows..First pick a covering $(U_i \subset X)$. For each i, we fix a trivial vector bundle $E_i = \mathbb{R}^n \times U_i$. Then on $U_i \cap U_j$, we construct a patching isomorphism

$$\alpha_{ij} : (\mathbb{R}^n \times U_i) \mid_{U_i \cap U_j} \overset{\approx}{\to} (\mathbb{R}^n \times U_j) \mid .$$

Finally transitivity requires that the cocycle condition be satisfied

$$\alpha_{jk} \mid_{U_{ijk}} \alpha_{ij} \mid_{U_{ijk}} = \alpha_{ik} \mid_{U_{ijk}} : (\mathbb{R}^n \times U_i) \mid_{U_i \cap U_j \cap U_k} \to (\mathbb{R}^n \times U_k) \mid_{U_i \cap U_j \cap U_k}$$

where U_{ijk} is notation for $U_i \cap U_j \cap U_k$. (See separate Descent Diagram at end.) Fpqc descent (= fidèlement plat et quasi-compact) transfers this construction to a Grothendieck topology. Doing it correctly requires keeping track of innumerable canonical identifications. Instead we will assume that all canonical isomorphisms may be replaced with equality, and refer the dedicated but masochistic reader to a recent updating of the categorical foundations ([1]).

But this is getting ahead of the story. First we need some sheaves.

Definition 2. *Let $\mathcal{C}/X$ be a site.*

(1) *A presheaf is a contravariant functor $\mathcal{F} : (\mathcal{C}/X)^{op} \to Sets$.*
(2) *A presheaf $\mathcal{F}$ is a sheaf of sets if, for all $\left(U_i \overset{p_i}{\to} U\right) \in Cov(\mathcal{T})$, the following conditions are satisfied:*
S1 *$\mathcal{F}(U) \to \prod \mathcal{F}(U_i)$ is one-to-one*
S2 *If $(s_i) \in \prod \mathcal{F}(U_i)$ has $pr_1^*(s_i) = pr_2^*(s_j) \in \prod_{(i,j)\in I\times I} \mathcal{F}(U_i \times_U U_j)$, then there is $s \in \mathcal{F}(U)$ such that $p_i^*(s) = s_i$ for all $i \in I$.*

An equivalent formulation of the two sheaf conditions is that $\mathcal{F}(U) \to \prod \mathcal{F}(U_i)$ is the difference kernel of

$$\prod_{i\in I} \mathcal{F}(U_i) \underset{pr_2^*}{\overset{pr_1^*}{\rightrightarrows}} \prod_{(i,j)\in I\times I} \mathcal{F}(U_i \times_U U_j)$$

where the two maps pr_i^* are projections onto the i^{th} factor. Consequently we can (and will) use this definition to define sheaves with values in any category $\mathcal{D}$ that has difference kernels. A presheaf satisfying $S1$ is said to be separated.

With this in mind, observe that each of the categories underlying the sites $X_{Zar}, X_{et}, X_{\delta-et}$, and $X_{\delta-pl}$ is a (not necessarily full) subcategory of the category defining X_{pl}. Moreover any covering in $Cov(X_{pl})$ is a covering for the site defined by the subcategory if it's maps are in the subcategory. Consequently any sheaf (or presheaf) for X_{pl} defines a sheaf for one of the other four sites. We make use of this be just producing sheaves for X_{pl} and then recognizing them as sheaves for one of the other sites. A little thought shows that a presheaf on X_{pl} is a sheaf iff it is a Zariski sheaf for any $U \in X_{pl}$ and satisfies the sheaf conditions $S1$ and $S2$, for coverings of the form $\{U_i \to U\}$ where U and U_i are affine. (See [4, II, Proposition 1.5].) Since we already understand Zariski sheaves, this reduces us to algebra.

A central result is flat descent whose statement comes from Milne ([4, I, Theorem 2.17 and Remark 2.19]).

Proposition 1. *If $f : A \to B$ is a faithfully flat ring homomorphism and M is an A module, then the sequence*

$$\begin{aligned} 0 \quad \to \quad & M \overset{1\otimes f}{\to} M \otimes_A B \overset{1\otimes d^1}{\to} M \otimes_A B^{\otimes 2} \overset{1\otimes d^2}{\to} \cdots \\ & \overset{1\otimes d^{r-2}}{\to} M \otimes_A B^{\otimes r} \overset{1\otimes d^r}{\to} M \otimes_A B^{\otimes r+1} \end{aligned}$$

is exact where $B^{\otimes r} = B \otimes_A B \otimes_A \cdots \otimes_A B$ r times, $d^r = \sum_{i=1}^{r+1} (-1)^i e_i$ and $e_i(b_1 \otimes b_2 \otimes \cdots \otimes b_r) = b_1 \otimes \cdots b_{i-1} \otimes 1 \otimes b_i \otimes \cdots \otimes b_r$ with a 1 in the i^{th} factor.

Idea of proof. Since B is faithfully flat, the sequence is exact iff it is exact after tensoring over A with B. Thus we may assume that we are in the situation $B \to B \otimes_A B$ where the map sends b to $b \otimes 1$. Now multiplication $B \otimes_A B \to B$ provides a 'section' that can be used to construct a contracting homotopy for the complex, and so it must be acyclic. □

Corollary 1. *Let $\mathcal{M}$ be a quasi-coherent sheaf on X. The presheaf*

$$\left(U \xrightarrow{p} X\right) \mapsto \Gamma\left(U, p^*\left(\mathcal{M}\right)\right)$$

is a sheaf written $W\left(\mathcal{M}\right)$on X_{pl} as is the presheaf

$$\left(U \xrightarrow{p} X\right) \mapsto Hom_{\mathcal{O}_U}\left(p * \left(\mathcal{M}\right), p^*\left(\mathcal{N}\right)\right)$$

if $\mathcal{M}, \mathcal{N}$ are coherent sheaves on X.

In commutative algebra terms this is the statement that for A modules M, N, with M of finite type, the presheaf

$$\left(U = Spec\left(B\right) \to Spec\left(A\right)\right) \mapsto M \otimes_A B$$

and

$$\left(U = Spec\left(B\right) \to Spec\left(A\right)\right) \mapsto Hom_B\left(M \otimes_A B, N \otimes_A B\right)$$

are both sheaves on $Spec\left(A\right)_{pl}$. Note that the latter presheaf uses the isomorphism

$$Hom_B\left(M, N\right) \otimes_A B \approxeq Hom_B\left(M \otimes_A B, N \otimes_A B\right). \tag{1.1}$$

Clearly the above notation is too cumbersome. From now on presheaves and sheaves G will have their values on U indicated by $\Gamma\left(U, G\right)$. Given a site X_*, $Sh\left(X_*\right)$ will indicate the category of sheaves of abelian groups on X_*.

Examples of sheaves with notation include:

- $W\left(\mathcal{O}_X\right) \in Sh\left(X_{pl}\right)$ and $W_\delta\left(\mathcal{O}_X\right) \in Sh\left(X_{d-pl}\right)$
- $\mathbb{G}_m \in Sh\left(X_{pl}\right)$ and $\mathbb{G}_{m,\delta} \in Sh\left(X_{d-pl}\right)$ where $\Gamma\left(U, \mathbb{G}_{m,\delta}\right) =$ units in $\Gamma\left(U, \mathcal{O}_U\right)$
- If G is a constant group, $\Gamma\left(U, G\right) := G^{\pi_0(U)}$ defines a sheaf where $\pi_0\left(U\right)$ is the set of connected components of the scheme U.
- $Gl_n \in Sh\left(X_{pl}\right)$ and $Gl_{n,\delta} \in Sh\left(X_{\delta-pl}\right)$ where we observe that $End\left(\mathcal{O}_x^n\right) \in Sh\left(X_{pl}\right)$ is a sheaf with the usual values on U and the exactness of (1) shows that an endomorphism on A that is an automorphism on B must have been an automorphism to begin.
- Presheaves defined from any commutative group scheme ([4, II, Corollary 1.7]) with a subscript δ if they are regarded as sheaves on the site $X_{\delta-et}$ of $X_{\delta-pl}$.

This gives us a reasonable collection of sheaves. General theorems ([4, III, Proposition 1.1]) then show that $S\left(X_{pl}\right) :=$ ((sheaves of abelian groups on X_{pl})) is an abelian category with enough injectives so that the usual games can begin!

But Proposition 1 has another important application as mentioned at the beginning of this section, the construction of torsors or principal homogeneous spaces.

Here the question is the following:

Suppose we are given a faithfully flat ring homomorphism $A \to B$ and a B module M. What information do we need to canonically write $M = N \otimes_A B$? (You should think of $B = \prod B_i$ where $\left(U_i \to U\right)$ is a covering, $U_i = Spec\left(B_i\right)$, and $U = Spec\left(A\right)$.)

I am going to use the following notation which is different from Milne, but, I hope, more informative. We must construct patching isomorphisms on intersections that agree on 'triple' intersections under the restriction map. In commutative algebra terms this means a map from $B \otimes_A B \to B \otimes_A B \otimes_A B$ where we will start with a $B \otimes_A B$ module and extend the base with this ring homomorphism. I will write $e_{ij} : B \otimes_A B \to B \otimes_A B \otimes_A B$ to indicate the map which uses the i^{th} and j^{th} factors of $B^{\otimes 3}$ as the images of the first and second factors of $B \otimes_A B$ and places a 1 into the omitted factor, e.g. $e_{13}\left(b \otimes b'\right) = b \otimes 1 \otimes b'$. In order to simplify the notation, all tensor products are taken over A and a B module, M, if it appears in a tensor product, is always understood to have the factor B in that position acting on it. For a morphism, a subscript i or ij indicates that the identity map acts on the missing factor.

Definition 3. *Let M be a B module. Descent data for M consists of a $B \otimes B$ isomorphism $\phi : M \otimes B \to B \otimes M$.*

Descent data (M,ϕ) *satisfies the cocycle condition if the diagram of isomorphisms*

$$\begin{array}{ccc} & & B\otimes M\otimes B \\ & \nearrow \phi_{12} & \\ M\otimes B\otimes B & & \downarrow \phi_{23} \\ & \searrow \phi_{13} & \\ & & B\otimes B\otimes M \end{array}$$

commutes, i.e. $\phi_{23}\phi_{12}=\phi_{13}:M\otimes B\otimes B\to B\otimes B\otimes M$.

Definition 4. *Given* $A\to B$, *the category of descent data for* B *over* A *consists of pairs* (M,ϕ) *where* M *is a finitely generated* B *module and* ϕ *is descent data for* M. $Hom\left((M,\phi),(M',\phi')\right)$ *consists of* B *module homomorphisms* $f:M\to M'$ *such that the diagram*

$$\begin{array}{ccc} M\otimes B & \stackrel{\phi}{\to} & B\otimes M \\ \downarrow f\otimes B & & \downarrow B\otimes f \\ M'\otimes B & \stackrel{\phi'}{\to} & B\otimes M' \end{array}$$

commutes. ((*Descent data for* B*-modules + cocycle*)) *stands for the full subcategory of* ((*Descent data for* B*-modules*)) *consisting of the descent data that satisfy the cocycle condition.*

The statement of the next theorem should be compared with the Descent Diagram in the Appendix.

Theorem 1. *Let* $A\to B$ *be a faithfully flat ring homomorphism. Then the functor*

$$-\otimes B:((\textit{Finitely generated } A\textit{-modules}))\to((\textit{Descent data for } B\textit{-modules + cocycle}))$$

is an equivalence of categories.

Idea behind proof. Suppose (M,ϕ) is descent data satisfying the cocycle condition. Define

$$N:=Ker\left[M\stackrel{\phi(1\otimes e_2)-1\otimes_B e_1}{\to}B\otimes M\right].$$

This kernel is not a B module, only an A submodule of M. But there is a natural map $\alpha:N\otimes B\to M$ which turns out to be an isomorphism, and then $\phi=(B\otimes\alpha)\circ\left(\alpha^{-1}\otimes B\right):M\otimes_A B\to B\otimes_A M$. For more details see the diagram. □

For descent questions we are only interested in the descended module, not in the descent data. Fix a module over A that will act as a 'model' for descent. Let's call it F. Let $Aut(F)$ be the presheaf given by $(Aut(F))(B)=Aut(F\otimes B)$. Our goal is to identify A modules N such that there is a B isomorphism

$$N\otimes B\to F\otimes B.$$

The set of such modules is 'almost' the same as the set of descent data for $F\otimes B$ that satisfies the cocycle condition. However we are not interested in the homomorphism part of the descent data, and so we must introduce an equivalence relation.

Definition 5. *Given a faithfully flat ring extension* $A\to B$, *a 'model'* A *module* F, *define*

$$Z^1(B/A,Aut(F)):=\{\phi\in Aut(F\otimes B\otimes B)\mid\ \phi_{23}\phi_{12}=\phi_{13}\}$$

Let $\phi,\sigma\in Z^1(B/A,Aut(F))$. *Then* $\phi\sim\sigma$ *if there is* $f:F\otimes B\to F\otimes B\in Aut(F\otimes B)$ *such that*

$$\sigma=f_2^{-1}\phi f_1.$$

It is easy to check that this is an equivalence relation and, in fact, removes the descent data from the picture for if we have descent data ϕ, then it comes, by the theorem, from an isomorphism $\beta:F\otimes B\to N\otimes B$ as $\phi=\beta_2^{-1}\beta_1$. (Caution: $\beta=\alpha^{-1}$.) If $\sigma:F\otimes B\to B\otimes F$ is different descent data for $F\otimes B$ resulting in an isomorphic A module N, then $\sigma=\gamma_2^{-1}\gamma_1$ where $\gamma:F\otimes B\to N\otimes B$ defines the descent data in the second case. But then $\gamma=\beta f$ where $f:F\otimes B\to F\otimes B$ is simply $\beta^{-1}\gamma$. Hence $\sigma=(\beta f)_2^{-1}(\beta f)_1=\left(f^{-1}\beta^{-1}\right)_2\beta_1 f_1=f_2^{-1}\phi f_1$ which is exactly the equivalence relation introduced in the definition.

Definition 6. *Given a faithfully flat ring extension $A \to B$, an A module F, define*

$$H^1(B/A, Aut(F)) := Z^1(B/A, Aut(F)) / \sim .$$

The following result is then a direct consequence of Theorem 1.

Theorem 2. *Let B be a faithfully flat A algebra, F an A module. Then there is a natural isomorphism of pointed sets*

$$H^1(B/A, Aut(F)) \approxeq \left\{ N \mid \begin{array}{c} N \text{ is an } A \text{ module and there is} \\ a\ B \text{ module isomorphism } F \otimes B \approxeq N \otimes B \end{array} \right\}.$$

If F happens to have additional structure given by a tensor such as multiplication, then $N \in H^1(B/A, Aut(F))$ will have such a structure also since it is described as a kernel.

We get rid of the dependence on a single covering in the site X_* by observing that for a given $U = Spec(B) \in \mathcal{C}/X$, $Cov(U)$ becomes a directed set and so we define

$$H^1(B_*, Aut(F)) = \lim_{(U_i \to U) \in Cov(U)} H^1(B_i/B, Aut(F))$$

where $U_i = Spec(B_i)$. (The general definition applies, of course, to arbitrary schemes in X_* and coverings of $U \in X_*$.) This results in a better version of Theorem 2 as follows.

Theorem 3. *Let $Spec(A)_*$ be a site for one of our Grothendieck topologies, and let F be an A module. Then there is a natural isomorphism of pointed sets*

$$H^1(A_*, Aut(F)) \approxeq \left\{ N \mid \begin{array}{c} N \text{ is an } A \text{ module and there is a covering} \\ Spec(B) \to Spec(A) \in Cov(Spec(A)) \text{ and} \\ a\ B \text{ module isomorphism } F \otimes B \approxeq N \otimes B \end{array} \right\}.$$

Thus attention is focussed on the set of objects that are 'locally' isomorphic to F in the $* = Zar$, et, pl, $\delta - et$, or $\delta - pl$ topology.

Example 6. (1) *$F = A$, $Aut(F) = G_m$, and*

$$\begin{aligned} H^1(A, Aut(F)) &= H^1(A_*, G_m) = \left\{ L \mid \begin{array}{c} L \otimes B \approxeq B \text{ for some faithfully flat} \\ A \text{ algebra } B \text{ of finite type over } A \end{array} \right\} \\ &= Pic(A) \end{aligned}$$

(2) *$F = A$, $Aut(F) = G_m^\delta$, and*

$$\begin{aligned} H^1(A_{\delta-*}, Aut(F)) &= H^1\left(A_{\delta-*}, G_m^\delta\right) \\ &= \left\{ L \mid \begin{array}{c} L \otimes B \approxeq B \text{ as differential modules} \\ \text{where } B \text{ is a covering in the } \delta - * \text{ topology} \end{array} \right\} \end{aligned}$$

(3) *$F = A^{\oplus n}$, $Aut(F) = Gl_n$, and*

$$H^1(A, Aut(F)) = H^1(A_*, Gl_n) = \{P/\ P \otimes B \text{ is free of rank } n\}$$

(4) *$F = A^{\oplus n}$, $Aut(F) = Gl_n^\delta$, and*

$$\begin{aligned} H^1(A, Aut(F)) &= H^1\left(A_{\delta-*}, Gl_n^\delta\right) \\ &= \left\{ P \mid \begin{array}{c} P \otimes B \text{ is free of rank } n \text{ as a differential module} \\ \text{with a basis defining } \delta \text{ where } B \text{ is a covering in the } \delta - * \text{ topology} \end{array} \right\} \end{aligned}$$

Here we note that automorphisms of a free differential module of rank n are elements of Gl_n that commute with the derivation and so are denoted Gl_n^δ.

(5) *$F = M_n(A)$, $Aut(F) := PGl_n$, and*

$$\begin{aligned} H^1(A, Aut(F)) &= H^1(A_*, PGl_n) \\ &= \left\{ \Lambda \mid \begin{array}{c} \Lambda \otimes B \approxeq M_n(B) \text{ as algebras} \\ \text{for some covering } A \to B \text{ in the } * \text{ topology} \end{array} \right\} \end{aligned}$$

In this case, the isomorphism is a B algebra isomorphism since it is easy to see that the descended module, which is Λ, is closed under multiplication.

In examples 1 and 3 $*$ can be *Zar*, *et*, or *pl* without affecting the result since any locally free module becomes free on a Zariski covering. We will see that this is not the case in examples 2 and 4 since we will need a Picard-Vesiot extension to make a δ locally free module differentially isomorphic to a free module with standard differentiation. Example 5 is more interesting yet since $H^1(A_{Zar}, PGl_n)$ is a one point set if A is a regular local ring while $H^1(A_{et}, PGl_n) = H^1(A_{pl}, PGl_n)$ consists of all Azumaya algebras over A of rank n^2 in general.

1.3. **Application to Azumaya algebras.**

Definition 7. *Let A be a ring. An Azumaya algebra Λ is a finite A algebra such that the center of Λ is A, Λ is locally free as an A module, and for all maximal ideals $\mathfrak{m}$, $\Lambda/\mathfrak{m}\Lambda$ is a central, simple algebra over $A/\mathfrak{m}$.*

The following facts are well known.

- A finite, flat A algebra is an Azumaya A algebra iff the natural map
$$\Phi : \Lambda \otimes_A \Lambda^{op} \to End_A(\Lambda)$$
given by $\Phi(\lambda_1 \otimes \lambda_2)(\lambda) = \lambda_1 \lambda \lambda_2$ is an isomorphism.
- An A algebra Λ is an Azumaya algebra if and only if $\Lambda \otimes_A B$ is an Azumaya B algebra for a faithfully flat A algebra B.
- There is a one-to-one correspondence between two sided ideals of an Azumaya A algebra and ideals in A given by $I \subset A$ corresponds to $I\Lambda$. In particular any Azumaya algebra over a field is simple. (This can be show by descent theory since it holds for matrix rings.)
- For any Azumaya algebra Λ over a local ring, there is a Galois ring extension $A \to B$ of the local ring A and a B algebra isomorphism $\Lambda \otimes_A B \approxeq End_B(B^n)$. In particular the rank of Λ is always a square. Moreover given an Azumaya A algebra, there is an étale covering $(U_i = Spec(B_i) \to Spec(A))_{i \in I}$ such that $\Lambda \otimes_A B_i \approxeq M_n(B_i)$ for all $i \in I$. In fact since $Spec(A)$ is quasi compact, we may assume that the index set is finite and then that there is an étale ring extension $A \to B$ which is faithfully flat and splits Λ by setting $B = \prod B_i$.

Let's collect some facts about these algebras. Rieffel (1964) has given a Proof from the Book of Wedderburn's theorem over a field. This version is taken from Lang's Algebra.

Theorem 4 (Wedderburn's Theorem)**.** *Let Λ be a ring with unit element whose only two sided ideals are Λ and (0). Let L be a non-zero left ideal, $\Lambda' = End_\Lambda(L)$ and let $\Lambda'' = End_{\Lambda'}(L)$ so that Λ'' is the commutant of Λ' in $End(L)$. Then the natural ring homomorphism $\phi : \Lambda \to \Lambda''$ given by $\phi(\lambda)(\ell) = \lambda\ell$ is an isomorphism.*

Proof. $Ker(\phi)$ is a two-sided ideal in Λ and so ϕ must be injective. $L\Lambda = \Lambda$ since it is a two-sided ideal. Thus $\phi(L)\phi(\Lambda) = \phi(\Lambda)$. For any $x, y \in L$ and $f \in \Lambda''$, we must have $f(xy) = f(x)y$ since right multiplication by y is a left Λ endomorphism of L and so in Λ'.

Hence $\phi(L)$ is a left ideal in Λ'', and so
$$\Lambda'' = \Lambda''\phi(\Lambda) = \Lambda''\phi(L)\phi(\Lambda) = \phi(L)\phi(\Lambda) = \phi(\Lambda)$$
as desired. □

Corollary 2. *Let Λ be a central simple algebra over a field K. Then $\Lambda \approxeq M_n(D)$ for a unique division algebra D.*

Proof. $\Lambda \approxeq End_{\Lambda'}(L)$ where L is a non-zero left ideal of minimum dimension. Since L is a simple left module, $\Lambda' = End_\Lambda(L)$ is a division ring. □

As an illustration of the power of the descent technique, let's prove Hochschild's Theorem.

Theorem 5 (Hochschild)**.** *Let A be a ring with a derivation δ, and let Λ be an Azumaya A algebra. Then there is a derivation $D : \Lambda \to \Lambda$ which extends δ. If A is a semi-local ring with derivation, any other derivation D' extending D is of the form $D' = D + [\lambda, -]$ for some $\lambda \in \Lambda$ where $[\lambda, a] = (\lambda \cdot a) - (a \cdot \lambda)$ for any $a \in \Lambda$.*

Proof. Let Λ be an A module of rank n^2. We first find an étale A algebra B and a B algebra isomorphism $\beta : M_n(B) \to \Lambda \otimes B$. This gives us descent data $\phi : M_n(B \otimes B) \to M_n(B \otimes B)$ such that

$$\Lambda = Ker\left(M_n(B \otimes B) \overset{\phi e_1 - e_2}{\to} M_n(B \otimes B)\right).$$

Now there is a unique extension of δ to B which we continue to indicate by δ. We use β to transfer the coordinatewise derivation δ to $\Lambda \otimes B$. Note that this makes the descent datum ϕ a derivation preserving isomorphism. But e_1 and e_2 also preserve the coordinate derivation on $M_n(B)$ and so Λ, being the kernel of two derivation preserving maps, acquires a derivation such that β is derivation preserving.

Consider $D - D'$, the difference of derivations that each extend $'$ on A. This is a derivation of Λ extending the 0 derivation on A. So assume that A consists of constants and consider the set of derivations on Λ. Let δ be a derivation on Λ with A contained in the constants. The derivation δ defines an $A[\varepsilon]$ automorphism ϕ_δ of $\Lambda[\varepsilon]$=$\Lambda \otimes_A A[\varepsilon]$ where $\varepsilon^2 = 0$, by setting

$$\phi_\delta(\lambda_0 + \lambda_1\varepsilon) = \lambda_0 + (\lambda_1 + \delta(\lambda_0))\varepsilon.$$

By Morita theory such an automorphism is given by conjugation by a unit if $Pic(A[\varepsilon]) = 0$ (See Corollary 5). Such a unit is of the form $u = u_0 + u_1\varepsilon \in \Lambda[\varepsilon]$. An easy calculation shows that $u^{-1} = u_0^{-1} - u_0^{-1}u_1u_0^{-1}\varepsilon$ where u_0 is a unit and u_1 is arbitrary. Then if $\lambda = \lambda_0 + \lambda_1\varepsilon$,

$$\begin{aligned}\phi_\delta(\lambda) &= \lambda + \delta(\lambda_0)\varepsilon = u(\lambda_0 + \lambda_1\varepsilon)u^{-1} \\ &= u_0\lambda u_0^{-1} - u_0\lambda u_0^{-1}u_1u_0^{-1}\varepsilon + u_1\left(u_0^{-1}u_0\right)\lambda_0u_0^{-1}\varepsilon \\ &= \lambda + \left(u_1u_0^{-1}\right)\lambda_0 - \lambda_0\left(u_1u_0^{-1}\right).\end{aligned}$$

Hence $\delta(\lambda_0) = \left[\left(u_1u_0^{-1}\right), \lambda_0\right]$ as claimed. □

Corollary 3. *Let Λ be a central simple algebra over the differential field K such that $\Lambda \approxeq M_n(D)$. If Λ has a derivation δ, then $\delta : D \to D$ is a derivation.*

Proof. D is realized as $e\Lambda e$ where e is a primitive idempotent, i.e. $e = \begin{pmatrix} 1 & 0 & \cdots \\ 0 & \ddots & \vdots \\ \vdots & 0 & \cdots \end{pmatrix}$. Since $e^2 = e$, we have $\delta e = 0$. □

Corollary 4. *Let A be a semi-local ring, Λ a differential Azumaya algebra of rank n^2. Then there is a $\delta - pl$ covering $A \to B$ such that there is a differential B algebra isomorphism $\Lambda \otimes_A B \to M_n(B)$ where $M_n(B)$ has coordinatewise differentiation. In particular*

$$\begin{aligned}H^1(A_{\delta-pl}, Aut(M_n)) &= H^1(A_{\delta-pl}, PGl_n)) \\ &= \left\{\Lambda \mid \begin{array}{c}\Lambda \otimes B \approxeq M_n(B) \text{ as differential algebras where} \\ B \text{ is } \delta\text{-flat and } M_n(B) \text{ has coordinatewise differentiation}\end{array}\right\} \\ &= \left\{\Lambda \mid \Lambda \text{ is a differential Azumaya algebra of rank } n^2 \text{ over } A\right\}\end{aligned}$$

Proof. We can first find an etale extension $A \to A_1$ such that $\Lambda \otimes_A A_1 \to M_n(A_1)$ as Azumaya algebras. We use this isomorphism to endow $M_n(A_1)$ with a derivation δ_1 coming from $\Lambda \otimes_A A_1$ that extends the derivation on A_1. We denote coordinatewise differentiation by $'$. Then the theorem shows that $\delta_1 = {}' + [v, -]$ for some $v \in M_n(A_1)$. Now the argument in Juan and Magid [2, Proposition 2] applies directly since the only fact they need is that derivations of Λ extending 0 are inner, but we give our own argument. The semi-local ring hypothesis together with the Skolem-Noether theorem characterize automorphisms of matrix rings as those algebra homomorphisms given by conjugation with a unit. Thus an algebra automorphism $\phi : (M_n, \delta_1) \to \left(M_n, {}'\right)$ that converts δ_1 into coordinatewise differentiation is given by $\phi(\lambda) = u\lambda u^{-1}$ where

$$\begin{aligned}\phi(\delta_1\lambda) &= \phi(\lambda' + [v, \lambda]) = u\lambda' u^{-1} + u[v, \lambda]u^{-1} \\ &= (\phi(\lambda))' = u'\lambda u^{-1} + u\left(\lambda' u^{-1} - \lambda u^{-1}u'u^{-1}\right).\end{aligned}$$

Consequently for a given $v \in M_n(A_1)$, we must find $u \in Gl_n(A_1)$ such that

$$\begin{aligned} uv\lambda u^{-1} - u\lambda vu^{-1} &= u'\lambda u^{-1} - u\lambda u^{-1}u'u^{-1} \text{ or} \\ v\lambda - \lambda v &= u^{-1}u'\lambda - \lambda u^{-1}u' \text{ for all } \lambda \in M_n(A_1). \end{aligned}$$

But the equation $u^{-1}u' = v$ can be solved in a $\delta - pl$ covering $A_1 \to B$. (Note that $u^{-1}u' = -\left(u^{-1}\right)'\left(u^{-1}\right)$. Then the equation becomes $w'w = -v$ or $w' + vw = 0$ if $w = u^{-1}$ if you prefer the normal order of matrix multiplication.) □

The usual argument as above to show that any derivation of an Azumaya algebra is given by $[\lambda, -]$ uses the Skolem-Noether theorem characterizing automorphisms of Azumaya algebras. In order to be complete we will include a proof of this based on the Morita theorems [3, p.53].

Theorem 6 (Morita)**.** *Let Λ be an Azumaya A algebra, and let P be a finitely generated faithful, projective right Λ module. Let $\Gamma := End_\Lambda(P)$ and define $Q := Hom_\Lambda(P, \Lambda)$ so that we are in the situation ${}_\Gamma P_\Lambda$ and ${}_\Lambda Q_\Gamma$.where the actions are defined by $(\lambda q\gamma)(p) = \lambda q(\gamma(p))$ and $\gamma p\lambda = \gamma(p\lambda)$ with the obvious notation. Then*

1. *The maps $f_P : P \otimes_\Lambda Q \to \Gamma$ via $(p \otimes q)(x) = p \cdot q(x)$ and $g_P : Q \otimes_\Gamma P \to \Lambda$ via $q \otimes p \mapsto q(p)$ are $\Gamma - \Gamma$ and $\Lambda - \Lambda$ bimodule isomorphisms respectively.*
2. *The functors*

$$\begin{aligned} P \otimes_\Lambda - &: (({}_\Lambda Mod)) \to (({}_\Gamma Mod)) \\ Q \otimes_\Gamma - &: (({}_\Gamma Mod)) \to (({}_\Lambda Mod)) \\ - \otimes_\Lambda Q &: ((Mod_\Lambda)) \to ((Mod_\Gamma)) \\ - \otimes_\Gamma P &: ((Mod_\Gamma)) \to ((Mod_\Lambda)) \end{aligned}$$

are equivalences of categories.

Corollary 5. *Let Λ be an Azumaya algebra over a ring A such that $Pic(A) = 0$, and let $\phi : \Lambda \to \Lambda$ be an A algebra automorphism. Then there is a unit $u \in \Lambda$ such that $\phi(\lambda) = u^{-1}\lambda u$. Moreover if Λ is a differential Azumaya A algebra with derivation δ, then ϕ is a differential automorphism if and only if $u^{-1}\delta u \in A$. In particular, $Aut_\delta(M_n) = Aut\left((M_n)^\delta\right)$*

Proof. First part later if there is enough interest.

Note that $\delta\left(u^{-1}u\right) = 0 = u^{-1}\delta(u) + \delta\left(u^{-1}\right)u$ and so $\delta\left(u^{-1}\right) = u^{-1}\delta(u)u^{-1}$. Now

$$\phi(\delta(\lambda)) = u^{-1}\delta(\lambda)u = \delta(\phi(\lambda)) = \delta\left(u^{-1}\right)\lambda u + u^{-1}\delta(\lambda)u + u^{-1}\lambda\delta(u)$$

and so $\delta\left(u^{-1}\right)\lambda u + u^{-1}\lambda\delta(u)$ must vanish if ϕ is a differential automorphism. But the above calculation shows that this is equivalent to the derivation $\left[u^{-1}\delta(u), -\right]$ vanishing for all λ. Hence we must have $u^{-1}\delta(u) \in A$. In the $\delta - pl$ topology we can solve the linear equation $T^{-1}\delta T = -a$ locally by $B \to C$ and then, if t is a solution, $(tu)^{-1}\delta(tu) = 0$ and so conjugation by $tu \in Aut\left(M_n\left(C^\delta\right)\right)$. □

Note that this allows us to calculate, for F a free A module of rank n, the sheaf of automorphisms of $End(F)$ and the obstruction to ϕ being a δ algebra automorphism as $Aut(End(F)) = Gl_n/G_m$ if A is a local ring and $Ob(\phi) = \left[u^{-1}\delta(u), -\right] \in Der(\Lambda)$.

1.4. **Summary.** We will only consider 'local' topologies. Let $\mathcal{C}/X$ be a category with finite products, i.e. a terminal object X and products of pairs of objects.

Definition 8 (following Artin)**.** *A pretopology $\mathcal{T}$ on $\mathcal{C}/X$ is a set $Cov(\mathcal{T})$ consisting of families $(\pi_i : U_i \to U)_{i \in I}$ for all $U \to X \in \mathcal{C}/X$ such that*

1. *If $\pi : V \to U$ is an isomorphism, then $(\pi) \in Cov(\mathcal{T})$.*
2. *If $(U_i \to U)_{i \in I} \in Cov(\mathcal{T})$ and $\left(V_j^i \to U_i\right)_{j \in I_i} \in Cov(\mathcal{T})$, then $\left(V_j^i \to U\right)_{\cup I_j} \in Cov(\mathcal{T})$.*
3. *If $(U_i \to U)_{i \in I} \in Cov(\mathcal{T})$ and $V \to U \in Mor(\mathcal{C}/X)$, then $U_i \times_U V$ exists and $(U_i \times_U V \to V)_{i \in I} \in Cov(\mathcal{T})$.*

Such a category with its coverings is called a site. Here are some examples where X is a scheme.

Example 7. *X_{Zar}, the Zariski site on X, has $\mathcal{C}/X$ equal to the category of open subschemes $U \subset X$.*

$$Cov\{X_{Zar}\} = \left\{(U_i \subset U) \,/\, {}^{1)}U_i \text{ is open and } {}^{2)}U = \bigcup U_i\right\}$$

Example 8. *X_{et}, the étale site on X, has $\mathcal{C}/X$ equal to the category of all étale schemes $U \to X$.*

$$Cov\{X_{et}\} = \left\{(p_i : U_i \to U) \,/\, {}^{1)}p_i \text{ is étale and } {}^{2)}U = \bigcup p_i(U_i)\right\}$$

Example 9. *X_{pl}, the flat site on X, has $\mathcal{C}/X$ equal to the category of all schemes $U \to X$ which are locally of finite type and flat over X.*

$$Cov\{X_{pl}\} = \left\{(p_i : U_i \to U) \,/\, {}^{1)}p_i \text{ is flat, locally of finite type and } {}^{2)}U = \bigcup p_i(U_i)\right\}$$

Let $D = Spec(R)$ where R is a differential ring containing $\mathbb{Q}$ with derivation δ. We introduce two new topologies, $D_{\delta-et}$ and $D_{\delta-pl}$ as follows. Let $\mathcal{C}/D_\delta$ be the category of schemes $\pi : X \to D$ such that X is a scheme with a derivation δ_X and π is a differentiation preserving morphism.

Example 10. *$D_{\delta-et}$, the $\delta-$étale site on D, has $\mathcal{C}/D$ equal to the full sub-category of $\mathcal{C}/D_\delta$ consisting of all étale schemes $\pi : U \to D$.*

$$Cov\{U_{\delta-et}\} = \left\{(p_i : U_i \to U) \mid \begin{array}{c} {}^{1)}p_i \text{ is étale, } {}^{2)}U = \bigcup p_i(U_i), \\ \text{and } {}^{3)}p_i \text{ is a differential morphism} \end{array}\right\}$$

Example 11. *$D_{\delta-pl}$, the $\delta-$flat site on D, has $\mathcal{C}/D$ equal to the full sub-category of $\mathcal{C}/D_\delta$ consisting of all schemes $U \to D$ which are locally of finite type and flat over D.*

$$Cov\{U_{\delta-pl}\} = \begin{array}{c} (p_i : U_i \to U) \mid^{1)} p_i \text{ is flat and locally of finite type,} \\ {}^{2)}U = \bigcup p_i(U_i), \text{ and } {}^{3)}p_i \text{ is a differential morphism} \end{array}$$

Examples of sheaves with notation include:

- $W(\mathcal{O}_X) \in Sh(X_{pl})$ and $W_\delta(\mathcal{O}_X) \in Sh(X_{d-pl})$
- $\mathbb{G}_m \in Sh(X_{pl})$ and $\mathbb{G}_{m,\delta} \in Sh(X_{d-pl})$ where $\Gamma(U, \mathbb{G}_{m,\delta}) =$ units in $\Gamma(U, \mathcal{O}_U)$
- If G is a constant group, $\Gamma(U, G) := G^{\pi_0(U)}$ defines a sheaf where $\pi_0(U)$ is the set of connected components of the scheme U.
- $Gl_n \in Sh(X_{pl})$ and $Gl_{n,\delta} \in Sh(X_{\delta-pl})$ where we observe that $End(\mathcal{O}_x^n) \in Sh(X_{pl})$ is a sheaf with the usual values on U and the exactness of (1) shows that an endomorphism on A that is an automorphism on B must have been an automorphism to begin.
- Presheaves defined from any commutative group scheme ([4, II, Corollary 1.7]) with a subscript δ if they are regarded as sheaves on the site $X_{\delta-et}$ of $X_{\delta-pl}$.
- $S(X_{pl}) :=$ ((sheaves of abelian groups on X_{pl})) is an abelian category with enough injectives.

Definition 9. *Let M be a B module. Descent data for M consists of a $B \otimes B$ isomorphism $\phi : M \otimes B \to B \otimes M$.*

Descent data (M, ϕ) satisfies the cocycle condition if the diagram of isomorphisms

$$\begin{array}{ccc} & & B \otimes M \otimes B \\ & \nearrow \phi_{12} & \\ M \otimes B \otimes B & & \downarrow \phi_{23} \\ & \searrow \phi_{13} & \\ & & B \otimes B \otimes M \end{array}$$

commutes, i.e. $\phi_{23}\phi_{12} = \phi_{13} : M \otimes B \otimes B \to B \otimes B \otimes M$.

Given $A \to B$, the category of descent data for B over A consists of pairs (M, ϕ) where M is a finitely generated B module and ϕ is descent data for M. $Hom((M, \phi), (M', \phi'))$ consists of B module homomorphisms $f : M \to M'$ such that the diagram

$$\begin{array}{ccc} M \otimes B & \xrightarrow{\phi} & B \otimes M \\ \downarrow f \otimes B & & \downarrow B \otimes f \\ M' \otimes B & \xrightarrow{\phi'} & B \otimes M' \end{array}$$

commutes.

Theorem 7. *Let $A \to B$ be a faithfully flat ring homomorphism. Then the functor*

$$- \otimes B : ((\text{Finitely generated } A\text{-modules})) \to ((\text{Descent data for } B\text{-modules + cocycle}))$$

is an equivalence of categories.

Definition 10. *Given a faithfully flat ring extension $A \to B$ and an A module F, define*

$$Z^1(B/A, Aut(F)) := \{\phi \in Aut(F \otimes B \otimes B) \mid \ \phi_{23}\phi_{12} = \phi_{13}\}$$

Let $\phi, \sigma \in Z^1(B/A, Aut(F))$. Then $\phi \sim \sigma$ if there is $f : F \otimes B \to F \otimes B \in Aut(F \otimes B)$ such that

$$\sigma = f_2^{-1}\phi f_1.$$

Definition 11.

$$H^1(B/A, Aut(F)) := Z^1(B/A, Aut(F)) / \sim .$$

The following result is then a direct consequence of 1.

Theorem 8. *Let $Spec(A)_*$ be in a site for one of our Grothendieck topologies, and let F be an A module. Then there is a natural isomorphism of pointed sets*

$$H^1(A_*, Aut(F)) \approxeq \left\{ N \mid \begin{array}{c} N \text{ is an } A \text{ module and there is a covering} \\ Spec(B) \to Spec(A) \in Cov(Spec(A)) \text{ and a} \\ B \text{ module isomorphism } F \otimes B \approxeq N \otimes B \end{array} \right\}.$$

Thus attention is focussed on the set of objects that are 'locally' isomorphic to F in the $* = Zar$, et, pl, $\delta - et$, or $\delta - pl$ topology.

Example 12. (1) *$F = A$, $Aut(F) = G_m$, and*

$$\begin{aligned} H^1(A, Aut(F)) &= H^1(A_*, G_m) \\ &= \left\{ L \mid \begin{array}{c} L \otimes B \approxeq B \text{ for some faithfully flat } A \text{ algebra} \\ B \text{ of finite type over } A \text{ in the } * \text{ topology} \end{array} \right\} \\ &= Pic(A) \end{aligned}$$

(2) *$F = A$, $Aut(F) = G_m^\delta$, and*

$$\begin{aligned} H^1(A_{\delta-*}, Aut(F)) &= H^1\left(A_{\delta-*}, G_m^\delta\right) \\ &= \left\{ L \mid \begin{array}{c} L \otimes B \approxeq B \text{ as differential modules where} \\ B \text{ is a covering in the } \delta - * \text{ topology} \end{array} \right\} \end{aligned}$$

(3) *$F = A^{\oplus n}$, $Aut(F) = Gl_n$, and*

$$\begin{aligned} H^1(A, Aut(F)) &= H^1(A_*, Gl_n) \\ &= \left\{ P \mid \begin{array}{c} P \otimes B \text{ is free of rank } n \text{ where} \\ B \text{ is a covering in the } * \text{ topology} \end{array} \right\} \end{aligned}$$

(4) *$F = A^{\oplus n}$, $Aut(F) = Gl_n^\delta$, and*

$$\begin{aligned} H^1(A, Aut(F)) &= H^1\left(A_{\delta-*}, Gl_n^\delta\right) = \\ & \left\{ P \mid \begin{array}{c} P \otimes B \text{ is free of rank } n \text{ as differential modules where} \\ B \text{ is a covering in the } \delta - * \text{ topology} \end{array} \right\} \end{aligned}$$

Here we note that automorphisms of a free differential module of rank n are elements of Gl_n that commute with the derivation and so are denoted Gl_n^δ.

(5) *$F = M_n(A)$, $Aut(F) := PGl_n$, and*

$$\begin{aligned} H^1(A, Aut(F)) &= H^1(A_*, PGl_n) = \\ & \left\{ \Lambda \mid \begin{array}{c} \Lambda \otimes B \approxeq M_n(B) \text{ as algebras where} \\ B \text{ is a covering in the } * \text{ topology} \end{array} \right\} \end{aligned}$$

In this case, the isomorphism is a B algebra isomorphism since it is easy to see that the descended module, which is Λ, is closed under multiplication.

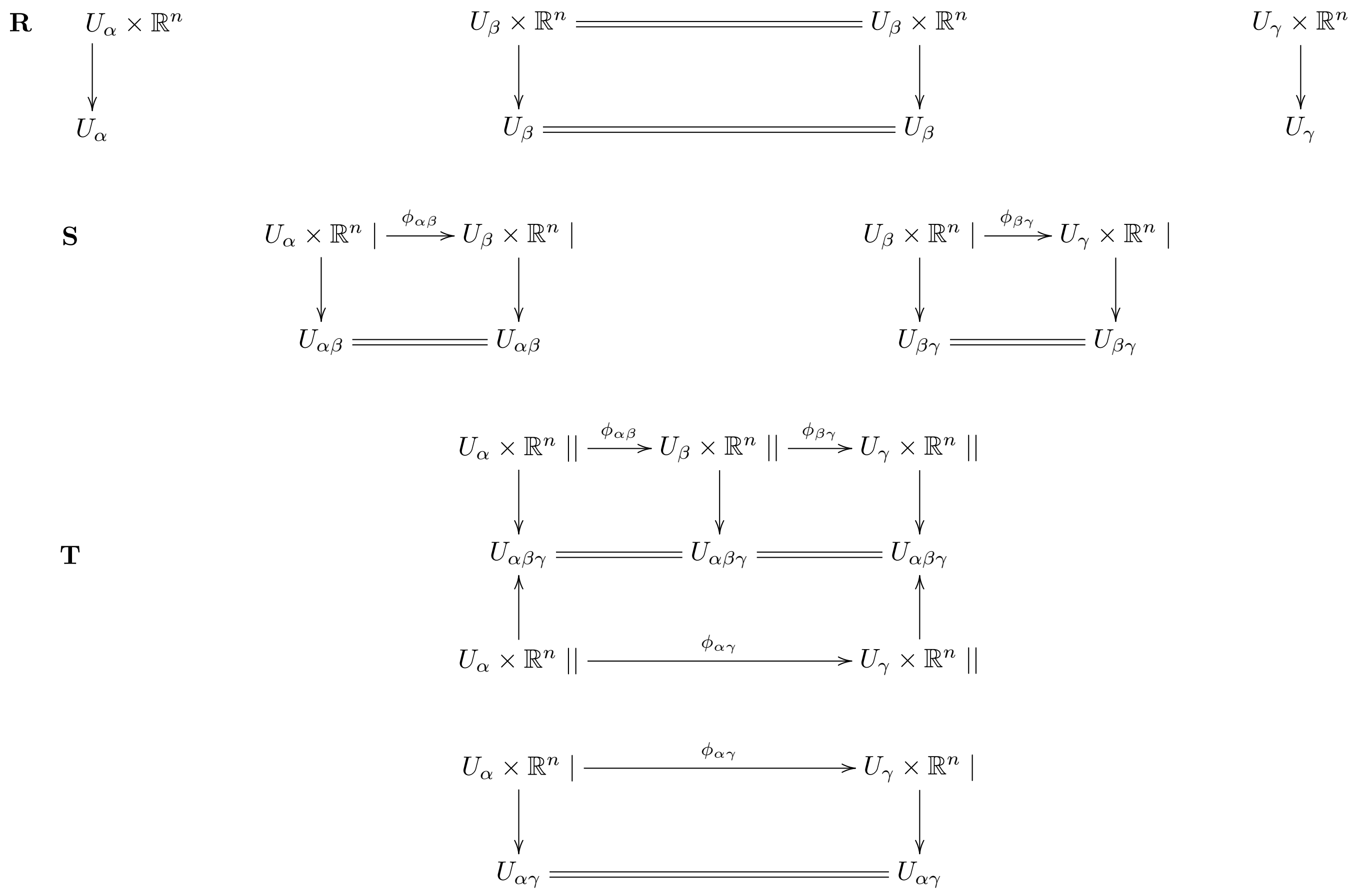
(6) $F = M_{n,\delta}(A), Aut(F) := PGl_{n,\delta},$ *and*

$$\begin{aligned} H^1(A_{\delta-pl}, Aut(M_{n,\delta}))(&= H^1(A_{\delta-pl}, PGl_{n,\delta})) \\ &= \left\{ \Lambda \mid \begin{array}{c} \Lambda \otimes B \approxeq M_n(B) \text{ as differential algebras where} \\ B \text{ is } \delta\text{-flat and } M_n(B) \text{ has coordinatewise differentiation} \end{array} \right\} \\ &= \left\{ \Lambda \mid \Lambda \text{ is a differential Azumaya algebra of rank } n^2 \text{ over } A \right\} \end{aligned}$$

Appendix A. Descent Diagram

The first diagram describes the construction of a real vector bundle of rank n on X by establishing an equivalence relation on $\bigcup_{i\in I}(U_i \times \mathbb{R}^n)$. The second diagram translates this into the language of algebra. Note that in the first diagram every other line indicates the open sets being used in the construction. These are omitted from the second diagram. Notation: $\{U_\alpha\}$ is a covering of X by "open" sets. $U_{\alpha\beta} = U_\alpha \cap U_\beta$, etc.

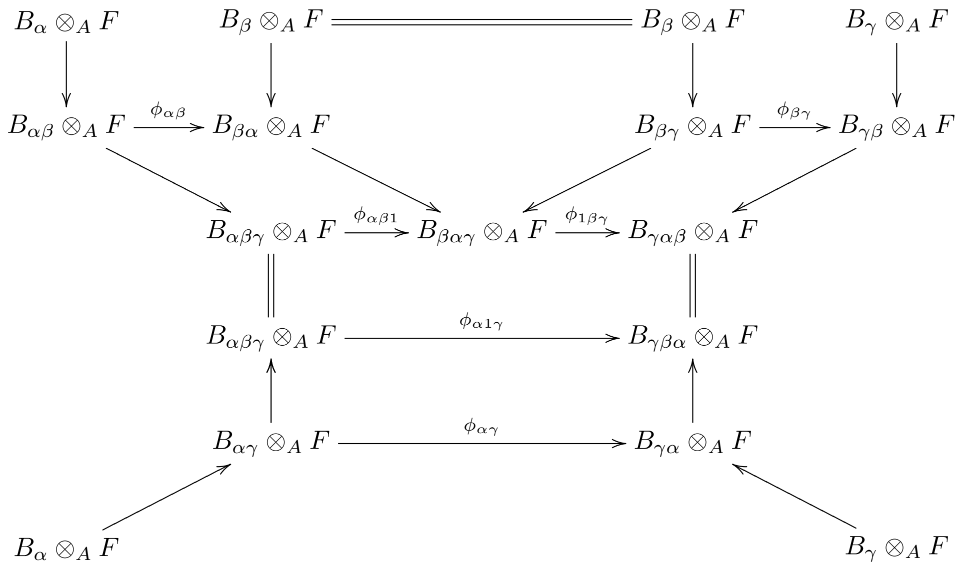
(A.1)



In algebraic terms, let $U_\alpha = Spec(B_\alpha)$ and $B = \prod B_\alpha$ so that $B \otimes_A B = \prod B_\alpha \otimes_A B_\beta$ etc. We assume the product is over a finite index set so that $(\prod B_\alpha) \otimes_A F = \prod(B_\alpha \otimes_A F)$. We let $B_{\alpha\beta}$ stand for $B_\alpha \otimes_A B_\beta$, etc. (A "1" in a subscript on ϕ indicates that the identity map is used on the factor corresponding to the missing Greek letter. Canonical isomorphisms such as $B_{\gamma\alpha\beta} \approxeq B_{\alpha\beta\gamma}$ are not indicated.)

The analogous diagram for the "model" A module F looks like:

(A.2)



The identification requirement is that the upper rectangle in the middle, the one with the vertical = signs, must be commutative, i.e. the cocycle condition must be satisfied. Fpqc descent says that if B is a faithfully flat A algebra, then such a diagram defines a module M over A such that $M \otimes_A B \approxeq F \otimes_A B$ and the $\phi_{\alpha\beta}$ is defined by this isomorphism.

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