Rational and Algebraic Invariants of a Group Action

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Synopsis

Propose algebraic constructions of a **generating set of invariants** that comes with a simple **rewriting algorithm**.

E. Hubert and I. Kogan, *Rational Invariants of a Group Action. Construction and Rewriting.* Journal of Symbolic Computation 42:1-2, p 203-217 (2007).

Original motivations:

- Differential elimination/completion for symmetric differential systems [Mansfield 2001]
- Avoid the implicit function theorem in [Fels & Olver 1999]

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1 Motivation: Symmetric Differential Systems

Differential Elimination

$$\mathcal{S}\left\{\begin{array}{rcl} s\,(\phi_{xx}+\phi_{yy})+s_x\,\phi_x+s_y\,\phi_y+\phi &=& 0\\ s\,(\psi_{xx}+\psi_{yy})+s_x\,\psi_x+s_y\,\psi_y+\psi &=& 0\\ \psi_x\,\phi_x+\psi_y\,\phi_y &=& 0\end{array}\right.$$

What are the conditions on s for $\mathcal{S}$ to have a solution?

(2003) Hubert, *Notes on Triangular Sets and Triangulation - Decomposition Algorithms.* LNCS 2630 .

I: Polynomial systems (1-39). *II: Differential Systems* (40-87).

(2000) Hubert, *Factorisation Free Decomposition Algorithms in Differential Algebra*, Journal of Symbolic Computation 29:4-5 (641-662).

(1997-) diffalg `inria.fr/cafe/Evelyne.Hubert/diffalg`

Differential Elimination

$$\mathcal{S}\left\{\begin{array}{rcl} s\,(\phi_{xx}+\phi_{yy})+s_x\,\phi_x+s_y\,\phi_y+\phi &=& 0\\ s\,(\psi_{xx}+\psi_{yy})+s_x\,\psi_x+s_y\,\psi_y+\psi &=& 0\\ \psi_x\,\phi_x+\psi_y\,\phi_y &=& 0\end{array}\right.$$

What are the conditions on s for $\mathcal{S}$ to have a solution?

It is computationally challenging.

Idea: reduce by the symmetry

[Lisle & Reid 1992], [Mansfield 2001]

Guideline: moving frame construction

[Cartan 1935],..., [Fels & Olver 1999]

Still is.

Symmetry

$$\mathcal{S}\left\{\begin{array}{rcl} s\,(\phi_{xx}+\phi_{yy})+s_x\,\phi_x+s_y\,\phi_y+\phi &=& 0\\ s\,(\psi_{xx}+\psi_{yy})+s_x\,\psi_x+s_y\,\psi_y+\psi &=& 0\\ \psi_x\,\phi_x+\psi_y\,\phi_y &=& 0\end{array}\right.$$

[Desolv,Vessiot]

$$x\mapsto\frac{\alpha}{\rho}x-\frac{\beta}{\rho}y+\frac{a}{\rho},\qquad y\mapsto\frac{\alpha}{\rho}x+\frac{\beta}{\rho}y+\frac{b}{\rho}$$

$$s\ \mapsto\ \frac{s}{\rho^2\tau},\qquad \phi\ \mapsto\ \frac{\phi}{\mu},\qquad \psi\ \mapsto\ \frac{\psi}{\nu}$$

$$\begin{array}{lllllll} s_x & \mapsto & \frac{\beta}{\rho}\, s_x - \frac{\alpha}{\rho}\, s_y, & & s_y & \mapsto & \frac{\alpha}{\rho}\, s_y + \frac{\beta}{\rho}\, s_x, \\ \phi_x & \mapsto & \frac{\rho\,\alpha}{\mu}\, \phi_x - \frac{\rho\,\beta}{\mu}\, \phi_y, & & \phi_y & \mapsto & \frac{\rho\,\alpha}{\mu}\, \phi_y + \frac{\rho\,\beta}{\mu}\, \phi_x \\ & & & & \ldots & & \end{array}$$

$$\rho, \mu, \nu \in \mathbb{R}^*, \quad a, b \in \mathbb{R}, \quad \beta^2 + \alpha^2 = 1$$

Fundamental Invariants

All differential invariants can be written in terms of:

$$s_1^2 = \frac{s_x^2+s_y^2}{4\,s} \quad s_2 = \frac{s_{xy}(s_y^2 - s_x^2) + s_x s_y (s_{xx} - s_{yy})}{8\,s\,s_1^3}$$

$$s_3 = \frac{s_x^2 s_{yy} + s_y^2 s_{xx} - 2\, s_x s_y s_{xy}}{8\,s\,s_1^3} - s_1$$

$$\phi_1 := \frac{s_y \phi_x - s_x \phi_y}{2\,s_1\,\phi} \quad \psi_1 := \frac{s_y \psi_x - s_x \psi_y}{2\,s_1\,\psi} \quad \psi_2 := \frac{s_x \psi_x + s_y \psi_y}{2\,s_1\,\psi}$$

$$\phi_2 := \frac{s_x \phi_x + s_y \phi_y}{2\,s_1\,\phi}$$

and their derivatives with respect to the invariant derivations:

$$\begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = \frac{\pm\sqrt{s\,(s_y^2 + s_x^2)}}{s_x^2 - s_y^2} \begin{pmatrix} -s_y & s_x \\ s_x & -s_y \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}$$

Algebra of differential invariants

The invariant derivations satisfy non trivial commutation rules

[aida]

$$\delta_1\delta_2 - \delta_2\delta_1 = s_3\,\delta_1 + s_2\,\delta_2$$

and the *differential syzygies* are [aida]

$$\mathcal{Z} \left\{ \begin{array}{rcl} \delta_1(s_1) & = & s_1 s_2 \\ \delta_1(s_2) - \delta_2(s_3) & = & s_3^2 + s_2^2 + s_1\,(s_2 + s_3) \\ \delta_1(\phi_2) - \delta_2(\phi_1) & = & \phi_1\, s_3 + \phi_2\, s_2, \\ \delta_1(\psi_2) - \delta_2(\psi_1) & = & \psi_1\, s_3 + \psi_2\, s_2. \end{array} \right.$$

Reduced problem

We can write $\mathcal{S}$ as [aida]

$$\mathcal{S} \left\{ \begin{array}{c} \delta_1(\phi_1) + \delta_2(\phi_2) + \phi_1^2 + \phi_2^2 - s_2\phi_1 + (2\,s_1 + s_3)\phi_2 + 1 = 0, \\ \delta_1(\psi_1) + \delta_2(\psi_2) + \psi_1^2 + \psi_2^2 - s_2\psi_1 + (2\,s_1 + s_3)\psi_2 + 1 = 0, \\ \phi_1\,\psi_1 + \phi_2\,\psi_2 = 0. \end{array} \right.$$

Perform differential elimination on $\mathcal{Z} \cup \mathcal{S}$ with derivations that satisfy non trivial commutation rules.

Note: s_1, s_2, s_3 and δ_1, δ_2 depend only on s and derivatives

The additional relationships on s_1, s_2, s_3 and their derivatives provide the answer to the original problem.

Progress on the Project

- Computing the fundamental invariants:

  Hubert & Kogan, *Rational Invariants of a Group Action. Construction and Rewriting.* J. of Symbolic Computation (2007).

- ◦ Algebra of (differential) invariants

  Hubert & Kogan. *Smooth and Algebraic Invariants of a Group Action. Local and Global Constructions.* (Submitted 2006).

  Algebra of Differential Invariants for Finite Dimensional group. In preparation.

- Differential elimination with non commuting derivations

  Hubert. *Differential Polynomial Algebra with non Commuting Derivations.* Journal of Pure and Applied Algebra (2005).

Software (MAPLE)
`inria.fr/cafe/Evelyne.Hubert/aida`
`inria.fr/cafe/Evelyne.Hubert/diffalg`

2 Rational Invariants of a Group Action

2.1 Definitions

Algebraic Group $\mathcal{G}$

$\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

$\mathcal{G} \subset \mathbb{K}^l$ an algebraic variety $\qquad G \subset \mathbb{K}[\lambda_1, \ldots, \lambda_l]$ its ideal

$$\begin{array}{rccc} m: & \mathcal{G}\times\mathcal{G} & \to & \mathcal{G} \\ & (\lambda,\mu) & \mapsto & \lambda\cdot\mu \end{array} \qquad \text{and} \qquad \begin{array}{rccc} i: & \mathcal{G} & \to & \mathcal{G} \\ & \lambda & \mapsto & \lambda^{-1} \end{array}$$

$$\lambda\cdot\mu \in \mathbb{K}[\lambda,\mu] \qquad \text{and} \qquad \lambda^{-1}\in\mathbb{K}[\lambda]$$

$$e\in\mathcal{G} \quad e\cdot\lambda = \lambda\cdot e = \lambda$$

$\mathcal{G}$	$\mathbb{K}^*$	$\mathbb{K}\times\{-1,1\}$	$SO(2)$
G	$(\lambda_1\,\lambda_2 - 1)$	$(\lambda_2^2 - 1)$	$(\lambda_1^2+\lambda_2^2-1)$
$\lambda\cdot\mu$	$(\lambda_1\mu_1,\, \lambda_2\mu_1)$	$(\lambda_1+\mu_1, \lambda_2\mu_2)$	$(\lambda_1\mu_1-\lambda_2\mu_2, \lambda_1\mu_2+\lambda_2\mu_1)$
e	$(1,1)$	$(0,1)$	$(1,0)$
λ^{-1}	$(\lambda_2,\, \lambda_1)$	$(-\lambda_1,\lambda_2)$	$(\lambda_1,-\lambda_2)$

Rational Action on $\mathcal{Z} = \mathbb{K}^n$

$$\begin{array}{rccc} g: & \mathcal{G}\times\mathcal{Z} & \to & \mathcal{Z} \\ & (\lambda,z) & \mapsto & \lambda\star z = \left(\dfrac{g_1(\lambda,z)}{h(\lambda,z)},\ldots,\dfrac{g_n(\lambda,z)}{h(\lambda,z)}\right) \end{array} \qquad (\lambda\cdot\mu)\star z = \lambda\star(\mu\star z)$$

Orbit of $z\in\mathcal{Z}$

$$\mathcal{O}_z = \{\lambda\star z \mid \lambda\in\mathcal{G}\}$$

$$h, g_1, \ldots, g_n \in \mathbb{K}[\lambda_1, \ldots, \lambda_l, z_1, \ldots, z_n]$$

$\mathcal{G}$	$\mathbb{K}^*$	$\mathbb{K} \times \{-1, 1\}$	$SO(2)$
G	$(\lambda_1\, \lambda_2 - 1)$	$(\lambda_2^2 - 1)$	$(\lambda_1^2 + \lambda_2^2 - 1)$
$\lambda \star z$	$\begin{pmatrix} \lambda_1 z_1 \\ \lambda_1 z_2 \end{pmatrix}$	$\begin{pmatrix} z_1 + \lambda_1 \\ \lambda_2 z_2 \end{pmatrix}$	$\begin{pmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$
	scaling	translation+reflection	rotation

scaling translation+reflection rotation

Field of Rational Invariants $\mathbb{K}(z)^G$

Rational invariant: $\frac{p}{q} \in \mathbb{K}(z)$ $\qquad \frac{p(\lambda \star z)}{q(\lambda \star z)} = \frac{p(z)}{q(z)} \mod G$

Field of rational invariants: $\mathbb{K}(z)^G$

$\mathcal{G}$	$\mathbb{K}^*$	$\mathbb{K} \times \{-1, 1\}$	$SO(2)$
$\mathbb{K}(z)^G$	$\mathbb{K}\left(\frac{z_1}{z_2}\right)$	$\mathbb{K}\left(z_2^2\right)$	$\mathbb{K}\left(z_1^2 + z_2^2\right)$

2.2 Results

Rational Invariants

ALGORITHM

In : $G, \quad (g_1(\lambda, z), \ldots, g_n(\lambda, z), h(\lambda, z)) \in \mathbb{K}[\lambda, z]$

Out : $\{r_1, \ldots, r_\kappa\} \subset \mathbb{K}(z)^G$

$$\longrightarrow_Q : \begin{array}{rcl} \mathbb{K}(z)^G & \to & \mathbb{K}(y_1, \ldots, y_\kappa) \\ r & \mapsto & R \end{array} \qquad r = R(r_1, \ldots, r_\kappa)$$

So : $\mathbb{K}(z)^G = \mathbb{K}(r_1, \ldots, r_\kappa)$

I : Gröbner basis of an unmixed dimensional ideal of dimension s

II : Gröbner basis of a zero dimensional ideal
Relies on the choice of a generic linear space of codimension s

Code: `www.inria.fr/cafe/Evelyne.Hubert/aida`

Algebraic Moving Frame

We introduce replacement invariants

$$\xi = (\xi_1, \ldots, \xi_n), \quad \xi_i \in \overline{\mathbb{K}(z)}^G$$

with property

$$r(z_1, \ldots, z_n) = r(\xi_1, \ldots, \xi_n) \quad \forall r \in \mathbb{K}(z)^G$$

ξ is the algebraic counterpart of the Cartan normalized invariants.

3 Intermezzo : Gröbner bases

Gröbner bases in $\mathbb{K}[z_1, \ldots, z_n]$

I a (radical) ideal in $\mathbb{K}[z] = \mathbb{K}[z_1, \ldots, z_n]$

Hilbert: $I = (q_1, \ldots, q_l)$

Reduc^o: $z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}$

$q = z^\alpha - \sum_{\beta<\alpha} c_\beta \, z^\beta$ $\qquad z^{\alpha+\gamma} \quad \longrightarrow_q z^\gamma \quad \sum_{\beta<\alpha} c_\beta \, z^\beta$

Gröbner: $\{q_1, \ldots, q_l\}$ a Gröbner basis if $\qquad p \in I \quad \Leftrightarrow \quad p \longrightarrow_Q^* 0$

Prop: Reduced Gröbner bases are canonical representative for ideals

Algo: Input: $p_1, \ldots, p_m$ a generating set of I
Output: $Q = \{q_1, \ldots, q_l\}$ a reduced Gröbner basis of I

$\rightsquigarrow$ $\mathbb{K}$-basis for $\mathbb{K}[z]/I$, its Hilbert polynomial, resolution.

Observe: $p_1, \ldots, p_m \in k[z] \;\Rightarrow\; Q \subset k[z]$ $\qquad k \subset \mathbb{K}$

$\rightsquigarrow$ The coefficients of Q give the field of definition of I.

4 Construction and rewriting of rational invariants

4.1 Graph ideal $\rightsquigarrow$ Rosenlicht (1956),..

Graph of the action & its ideal O

- Graph of the action

$$\mathcal{O} = \{(z, z') \in \mathcal{Z} \times \mathcal{Z} \mid \exists \lambda \in \mathcal{G} \text{ s.t. } z' = \lambda \star z\}$$

- Its ideal: $O = (G + (\, Z - \lambda \star z\,)\,) \cap \mathbb{K}[z, Z]$

$Z = (Z_1, \ldots, Z_n)$ new set of variables

$$(Z - \lambda \star z) = (\, h\, Z_i - g_i \mid 1 \leq i \leq n\,) : h^\infty$$

- O^e the extension of O to $\mathbb{K}(z)[Z]$.
$\rightsquigarrow$ the ideal of a generic orbit

Construction of rational invariants

Invariance: $(z, z') \in \mathcal{O} \;\Rightarrow\; (\lambda \star z, z') \in \mathcal{O}$

Thm: The reduced Gröbner basis of O^e is contained in $\mathbb{K}(z)^G[Z]$.

Examples

$\mathcal{G}$	$\mathbb{K}^*$	$\mathbb{K} \times \{-1, 1\}$	$SO(2)$

Q	$Z_2 - \frac{z_2}{z_1} Z_1$	$Z_2^2 - z_2^2$	$Z_2^2 + Z_1^2 - (z_1^2 + z_2^2)$

Rewriting & Generation

Q reduced Gröbner basis of O^e

$\{r_1, \ldots, r_\kappa\}$ the coefficients of Q

Theorem:

$$\mathbb{K}(z)^G = \mathbb{K}(r_1, \ldots, r_\kappa)$$

¡2-¿

Rewriting $\quad \frac{p}{q} \in \mathbb{K}(z)^G$

- $y_1, \ldots, y_\kappa$ a new indeterminates
- $Q_y := Q(r_i \leftarrow y_i)$
- $p(Z) \longrightarrow^*_{Q_y} \sum_\alpha a_\alpha(y)\, Z^\alpha$
- $q(Z) \longrightarrow^*_{Q_y} \sum_\alpha b_\alpha(y)\, Z^\alpha$
- $\frac{p(z)}{q(z)} = \frac{a_\alpha(r)}{b_\alpha(r)}$

Example of rewriting for the scaling

$$Q = \{Z_2 - \frac{z_2}{z_1} Z_1\} \qquad r = \frac{z_2}{z_1} \qquad Q_y = \{Z_2 - y Z_1\}$$

$$\frac{p}{q} = \frac{z_1^2 + 4z_1z_2 + z_2^2}{z_1^2 - 3z_2^2}$$

$$p(Z) = Z_1^2 + 4Z_1Z_2 + Z_2^2 \quad \longrightarrow_{Q_y} \quad (y^2 + 4y + 1)Z_2^2$$

$$q(Z) = Z_1^2 - 3Z_2^2 \quad \longrightarrow_{Q_y} \quad (y^2 - 3)Z_2^2$$

$$q(z)p(Z) \equiv p(z)q(Z) \operatorname{mod} O^e \;\Rightarrow\; q(z)(r^2-3)Z_2^2 = p(z)(r^2+4r+1)Z_2^2$$

$$\frac{z_1^2 + 4z_1z_2 + z_2^2}{z_1^2 - 3z_2^2} = \frac{r^2 + 4r + 1}{r^2 - 3} \text{ where } r = \frac{z_1}{z_2}$$

Previously

Müller-Quade & Beth 99

- Case of linear group actions.
- Proof: $(Q) = (Z - z) \cap \mathbb{K}(z)^G[Z]$

Vinberg & Popov 89

- There exists a generating set Q of O^e the coefficients $\{r_1, \ldots, r_\kappa\}$ of which are in $\mathbb{K}(z)^G$
- $\{r_1, \ldots, r_\kappa\}$ separate orbits
- A set of rational invariant that separate orbits is a generating set for $\mathbb{K}(z)^G$

Rosenlicht 56

- The coefficients of the Chow form of O^e are rational invariants and separate orbits
- A set of rational invariant that separate orbits is a generating set for $\mathbb{K}(z)^G$

4.2 Graph-section ideal $\quad\rightsquigarrow$ Fels & Olver (1999)

Cross-section of degree d

A variety $\mathcal{P}$ that intersects generic orbits in d simple points.

$$O^e = (G + (Z - \lambda \star z)) \cap \mathbb{K}(z)[Z].$$

s = dimension of O^e = dimension of generic orbits

The ideal P defines a cross-section $\mathcal{P}$ of degree d:

- $P \subset \mathbb{K}[Z]$ prime ideal of codimension s
- $I^e = O^e + P$ radical and zero-dimensional
- $\dim_{\mathbb{K}(z)} \mathbb{K}(z)[Z]/I^e = d$

$$P = (a_{i1}Z_1 + \ldots + a_{in}Z_n - b_i,\ 1 \le i \le s)$$

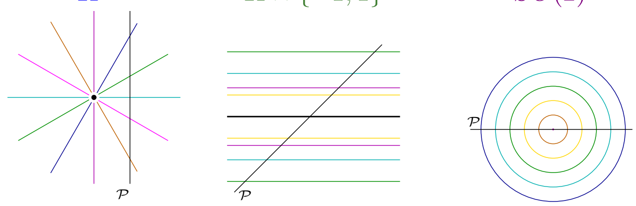
Rational Invariants 2

$$I^e = P + O^e = (\ P + G + (Z - \lambda \star z)\ )\ \cap\ \mathbb{K}(z)[Z]$$

Q a reduced Gröbner basis of I^e $\qquad$ $\{r_1, \ldots, r_\kappa\}$ its coefficients

Theorem: $\mathbb{K}(z)^G = \mathbb{K}(r_1, \ldots, r_\kappa)$ + rewriting

$\mathcal{G}$ $\quad$ $\mathbb{K}^*$ $\quad$ $\mathbb{K} \times \{-1, 1\}$ $\quad$ $SO(2)$



Q $\quad$ $Z_1 - 1,\ Z_2 - \frac{z_2}{z_1}$ $\quad$ $Z_1 - Z_2,\ Z_2^2 - z_2^2$ $\quad$ $Z_2,\ Z_1^2 - (z_1^2 + z_2^2)$

5 Algebraic Moving frame

Replacement Invariant ξ

- $\mathcal{P}$ a cross-section of degree $d = 1$
  $I^e = (\ Z_1 - r_1(z), \ldots, Z_n - r_n(z))$

$$r(z_1, \ldots, z_n) = r(r_1, \ldots, r_n) \qquad \forall r \in \mathbb{K}(z)^G$$

- $\mathcal{P}$ a cross-section of degree $d > 1$
  $I^G = I^e \cap \mathbb{K}(z)^G[Z] = (Q)$ has d distinct $\overline{\mathbb{K}(z)}^G$-zeros

Thm: $\xi = (\xi_1, \ldots, \xi_n)$ a $\overline{\mathbb{K}(z)}^G$-zero of I^G.

$$r(z) = r(\xi), \quad \forall r \in \mathbb{K}(z)^G$$

Replacement Invariant ξ. Examples

$\mathcal{P}$ a cross-section of degree d

$I^G = I^e \cap \mathbb{K}(z)^G[Z] = (Q)$ $\qquad$ has d distinct $\overline{\mathbb{K}(z)}^G$-zeros

Thm: $\xi = (\xi_1, \ldots, \xi_n)$ a $\overline{\mathbb{K}(z)}^G$-zero of I^G. $\qquad$ $r(z) = r(\xi),\ r \in \mathbb{K}(z)^G$

$\mathcal{G}$	$\mathbb{K}^*$	$\mathbb{K} \times \{-1, 1\}$	$SO(2)$
Q	$Z_1 - 1,\ Z_2 - \frac{z_2}{z_1}$	$Z_1 - Z_2,\ Z_2^2 - z_2^2$	$Z_2,\ Z_1^2 - (z_1^2 + z_2^2)$
ξ	$(1,\ \frac{z_2}{z_1})$	$(\pm z_2,\ \pm z_2)$	$\left(\pm\sqrt{z_1^2 + z_2^2},\ 0\right)$

Algebraic moving frame - preview

$\mathcal{P}$ an algebraic cross-section $\Rightarrow$ $\mathcal{P} \cap \mathcal{U}$ a local cross-section.

Replacement invariant : $\overline{\mathbb{K}(z)}^G$-zero of I^G

Thm: The normalized invariants $(\bar{\iota}z_1, \ldots, \bar{\iota}z_n)$ form the smooth zero of I^G that agrees with the coordinate functions on $\mathcal{P} \cap \mathcal{U}$.

As a result, the components of this replacement invariant actually generates, functionally, the smooth invariants locally.

E. Hubert and I. Kogan. *Smooth and Algebraic Invariants of a Group Action. Local and Global Constructions.* (Preprint).