Why coalgebras?

Generalizations of results on Frobenius algebras, Hopf algebras and compact groups via co-representation theory, and applications

Miodrag C Iovanov
University of Southern California, LA

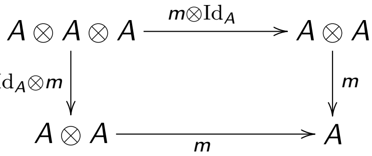
CUNY Graduate Center
April 2010

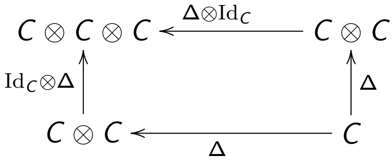
K-algebra A

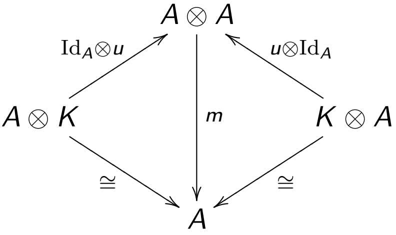
$m : A \otimes A \to A$ & $u : K \to A$

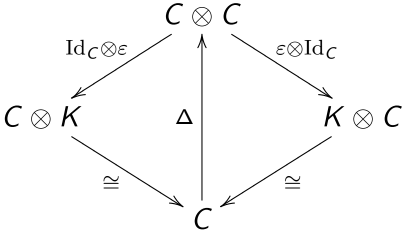
K-coalgebra C

$\Delta : C \to C \otimes C$ & $\varepsilon : C \to K$









Modules and Comodules (actions and coactions)

A-**module**: M with an A - action $A \otimes M \to M$, $(a, m) \mapsto a \cdot m$
C-**comodule**: M with a C - coaction $\rho : M \to M \otimes C$, $\rho(m) = \sum_i m_i \otimes c_i$

Modules and Comodules (actions and coactions)

A-**module**: M with an A - action $A \otimes M \to M$, $(a, m) \mapsto a \cdot m$
C-**comodule**: M with a C - coaction $\rho : M \to M \otimes C$, $\rho(m) = \sum_i m_i \otimes c_i$

and, of course, some compatibility conditions: associative and unital for modules, "coassociative and counital" for comodules

One defines morphisms of comodules, by duality with the definition of morphisms of modules.

Representation Theory

Let $\eta : A \to \mathrm{End}(V)$ a finite dimensional representation, v_i a basis of V. Then $\eta(a) = (a_{ij})$. Denote $\eta_{ij}(a) = a_{ij}$ and $\eta(ab) = \eta(a)\eta(b)$ reads

$$\eta_{ij}(ab) = \sum_k \eta_{ik}(a)\eta_{kj}(b).$$

Representation Theory

Let $\eta : A \to \mathrm{End}(V)$ a finite dimensional representation, v_i a basis of V. Then $\eta(a) = (a_{ij})$. Denote $\eta_{ij}(a) = a_{ij}$ and $\eta(ab) = \eta(a)\eta(b)$ reads

$$\eta_{ij}(ab) = \sum_k \eta_{ik}(a)\eta_{kj}(b).$$

$$R(A) = \{f : A \to K \mid f(ab) = \sum_i g_i(a)h_i(b) \,\mathrm{for\,some}\, g_i, h_i : A \to K\} = A^0$$

We have $m^* : A^* \to (A \otimes A)^* \supseteq A^* \otimes A^*$, and $R(A) = (m^*)^{-1}(A^* \otimes A^*)$

Representation Theory

Let $\eta : A \to \mathrm{End}(V)$ a finite dimensional representation, v_i a basis of V. Then $\eta(a) = (a_{ij})$. Denote $\eta_{ij}(a) = a_{ij}$ and $\eta(ab) = \eta(a)\eta(b)$ reads

$$\eta_{ij}(ab) = \sum_k \eta_{ik}(a)\eta_{kj}(b).$$

$$R(A) = \{f : A \to K \mid f(ab) = \sum_i g_i(a)h_i(b) \,\mathrm{for\,some}\, g_i, h_i : A \to K\} = A^0$$

We have $m^* : A^* \to (A \otimes A)^* \supseteq A^* \otimes A^*$, and $R(A) = (m^*)^{-1}(A^* \otimes A^*)$

Closely related situation: G - a (topological) group and $\eta : G \to Gl_n(V)$ a (continuous) representation over $\mathbb{C}$.

So for $f \in R(A)$, well determined $\sum_i g_i \otimes h_i \in A^* \otimes A^*$; by standard linear algebra, in fact $\sum_i g_i \otimes h_i \in R(A) \otimes R(A)$, giving a **comultiplication** of $R(A)$.
counit: $\varepsilon(f) = f(1)$.

So for $f \in R(A)$, well determined $\sum_i g_i \otimes h_i \in A^* \otimes A^*$; by standard linear algebra, in fact $\sum_i g_i \otimes h_i \in R(A) \otimes R(A)$, giving a **comultiplication** of $R(A)$.

counit: $\varepsilon(f) = f(1)$.

Coalgebra of representative functions or **Finite dual coalgebra** $R(A)$

So for $f \in R(A)$, well determined $\sum_i g_i \otimes h_i \in A^* \otimes A^*$; by standard linear algebra, in fact $\sum_i g_i \otimes h_i \in R(A) \otimes R(A)$, giving a **comultiplication** of $R(A)$.
counit: $\varepsilon(f) = f(1)$.

Coalgebra of representative functions or **Finite dual coalgebra** $R(A)$

Proposition

$R(A)$ is spanned by **all** η_{ij}, $\eta : A \to \mathrm{End}(V)$, v_i *basis; also* $f \in R(A) \Leftrightarrow \ker(f)$ *contains a two-sided ideal of finite codimension.*

Representation Theory

To any $\eta : A \to \mathrm{End}(V)$ representation (f.d. left A-module) associate a right $R(A)$-comodule V

$$v_i \longmapsto \sum_j v_j \otimes \eta_{ji}$$

Conversely, to a right $R(A)$-comodule $V, \rho : V \to V \otimes R(A)$, write $\rho(v_i) = \sum_j v_j \otimes f_{ji}$ associate the left A-action

$$a \cdot v_i = \sum_i f_{ji}(a) v_j$$

$f : V \to W$ morphism of A-modules **iff** $R(A)$-comodules.

To any $\eta : A \to \mathrm{End}(V)$ representation (f.d. left A-module) associate a right $R(A)$-comodule V

$$v_i \longmapsto \sum_j v_j \otimes \eta_{ji}$$

Conversely, to a right $R(A)$-comodule $V, \rho : V \to V \otimes R(A)$, write $\rho(v_i) = \sum_j v_j \otimes f_{ji}$ associate the left A-action

$$a \cdot v_i = \sum_i f_{ji}(a) v_j$$

$f : V \to W$ morphism of A-modules **iff** $R(A)$-comodules.

Theorem

The categories $f.d.A$-mod and comod-$R(A)$ are equivalent.

Coalgebras, comodules, rational modules

C-coalgebra $\Rightarrow$ C^* is an algebra with the **convolution product**:
$(f * g)(c) = \sum\limits_i f(a_i)g(b_i)$, where $\Delta(c) = \sum\limits_i a_i \otimes b_i$

Coalgebras, comodules, rational modules

C-coalgebra $\Rightarrow C^*$ is an algebra with the **convolution product**:
$(f * g)(c) = \sum_i f(a_i)g(b_i)$, where $\Delta(c) = \sum_i a_i \otimes b_i$

comod-$C \hookrightarrow C^*$-mod: for $m \in (M, \rho : M \to M \otimes C)$, $f \in C^*$ define
$f * m = \sum_i f(c_i)m_i$ where $\sum_i m_i \otimes c_i = \rho(m)$.
(note: above = the same for $A \to R(A)^*$ morphism of algebras)

Coalgebras, comodules, rational modules

C-coalgebra $\Rightarrow$ C^* is an algebra with the **convolution product**:
$(f * g)(c) = \sum_i f(a_i)g(b_i)$, where $\Delta(c) = \sum_i a_i \otimes b_i$

comod-$C \hookrightarrow C^*$-mod: for $m \in (M, \rho : M \to M \otimes C)$, $f \in C^*$ define
$f * m = \sum_i f(c_i)m_i$ where $\sum_i m_i \otimes c_i = \rho(m)$.
(note: above = the same for $A \to R(A)^*$ morphism of algebras)

C-comodules are called rational C^*-modules. Also, for any C^*-module M, define $Rat(M)$=the largest rational submodule of M.

The Fundamental Theorem of Coalgebras

Theorem

A f.g. rational C^*-(bi)module is finite dimensional.

The Fundamental Theorem of Coalgebras

Theorem

A f.g. rational C^*-(bi)module is finite dimensional.

So, any rational module is the sum of its finite dimensional submodules, and a coalgebra is the sum of finite dimensional subcoalgebras.

The Fundamental Theorem of Coalgebras

Theorem

A f.g. rational C^*-(bi)module is finite dimensional.

So, any rational module is the sum of its finite dimensional submodules, and a coalgebra is the sum of finite dimensional subcoalgebras.

So $C = \varinjlim C_i$, C_i-finite dimensional $\Rightarrow C^* = \varprojlim C_i^*$, a **profinite algebra**.
In close analogy to profinite groups:

Profinite Algebras; Pseudocompact Algebras

Profinite Algebras; Pseudocompact Algebras

Theorem

The following is equivalent for an algebra A:

• A is **profinite** ($A = \varprojlim A_i$, A_i f.d.)

• A is **pseudocompact**, i.e. it is a Hausdorff and complete topological algebra with a basis of nbhds of 0, consisting of ideals of finite codimension.

• $A = C^*$, for some coalgebra C.

Profinite Algebras; Pseudocompact Algebras

Theorem

The following is equivalent for an algebra A:

• A is **profinite** ($A = \varprojlim A_i$, A_i f.d.)

• A is **pseudocompact**, i.e. it is a Hausdorff and complete topological algebra with a basis of nbhds of 0, consisting of ideals of finite codimension.

• $A = C^*$, for some coalgebra C.

Moreover, in this situation, the category of right C-comodules is in duality with that of **pseudocompact** right A-modules.

Profinite Algebras; Pseudocompact Algebras

Theorem

The following is equivalent for an algebra A:

• A is **profinite** ($A = \varprojlim A_i$, A_i f.d.)

• A is **pseudocompact**, i.e. it is a Hausdorff and complete topological algebra with a basis of nbhds of 0, consisting of ideals of finite codimension.

• $A = C^*$, for some coalgebra C.

Moreover, in this situation, the category of right C-comodules is in duality with that of **pseudocompact** right A-modules.

C-coalgebra: $C = \bigoplus_i E(S_i)^{n_i}$ in mod-C^*;

$A = C^* = \prod_i E(S_i)^{*\,n_i}$ in C^*-mod;

$E(S_i)$-injective indecomposable with simple socle;

$E(S_i)^*$ (principal) projective indecomposable & local.

The development of the theory of infinite dimensional Frobenius algebras

The development of the theory of infinite dimensional Frobenius algebras

An algebra A called Frobenius if $A \simeq A^*$ as left A-modules. The definition comes from an old problem raised by Frobenius and who's solution leads to this equivalent characterization.

The development of the theory of infinite dimensional Frobenius algebras

An algebra A called Frobenius if $A \simeq A^*$ as left A-modules. The definition comes from an old problem raised by Frobenius and who's solution leads to this equivalent characterization.

Such algebras generalize the classical case: $A = KG$, G finite group.

- Maschke: $K = \mathbb{C}$ (or $\mathrm{char}(K) \nmid |G|$), $\mathbb{C}G$ semisimple.
- Otherwise not, but still KG Frobenius!

Frobenius Algebras

Frobenius' question arises naturally as follows: for a finite dimensional K-algebra A with basis $e_1, \ldots, e_n$

Frobenius Algebras

Frobenius' question arises naturally as follows: for a finite dimensional K-algebra A with basis $e_1, \ldots, e_n$

- $\varphi_r : A \longrightarrow \mathrm{End}_K(A)$; $\varphi_r(a) = x \mapsto ax$ - morphism of K-algebras. $a \cdot e_i = \sum\limits_{j=1}^{n} a_{ij} e_j$. Then $A \ni a \mapsto \alpha(a) = (a_{ij})_{i,j} \in M_n(K)$ is a morphism of algebras

Frobenius Algebras

Frobenius' question arises naturally as follows: for a finite dimensional K-algebra A with basis $e_1, \ldots, e_n$

- $\varphi_r : A \longrightarrow \mathrm{End}_K(A)$; $\varphi_r(a) = x \mapsto ax$ - morphism of K-algebras. $a \cdot e_i = \sum_{j=1}^{n} a_{ij} e_j$. Then $A \ni a \mapsto \alpha(a) = (a_{ij})_{i,j} \in M_n(K)$ is a morphism of algebras
- $\varphi_l : A \longrightarrow \mathrm{End}_K(A)$, $\varphi_l(a) = (x \mapsto xa)$ - antimorphism of algebras $e_i \cdot a = \sum_j b_{ji} e_j$; again $A \ni a \mapsto \beta(a) = (b_{ij})_{i,j} \in M_n(K)$ is a morphism of algebras.

Frobenius Algebras

Frobenius' question arises naturally as follows: for a finite dimensional K-algebra A with basis $e_1, \dots, e_n$

- $\varphi_r : A \longrightarrow \mathrm{End}_K(A)$; $\varphi_r(a) = x \mapsto ax$ - morphism of K-algebras. $a \cdot e_i = \sum\limits_{j=1}^{n} a_{ij} e_j$. Then
$A \ni a \mapsto \alpha(a) = (a_{ij})_{i,j} \in M_n(K)$ is a morphism of algebras
- $\varphi_l : A \longrightarrow \mathrm{End}_K(A)$, $\varphi_l(a) = (x \mapsto xa)$ - antimorphism of algebras
$e_i \cdot a = \sum\limits_j b_{ji} e_j$; again $A \ni a \mapsto \beta(a) = (b_{ij})_{i,j} \in M_n(K)$ is a morphism of algebras.

- Frobenius' question: when are the two representations α, β equivalent:
when $\exists\, S \in M_n(K)$ such that $\beta(a) = S^{-1}\alpha(a)S$, $\forall\, a \in A$?.

Co-Frobenius coalgebras

Definition

A coalgebra C is called right (left) co-Frobenius if there is a monomorphism $C \hookrightarrow C^*$ of right (left) C^*-modules. C is called co-Frobenius if it is both left and right co-Frobenius.

Co-Frobenius coalgebras

Definition

A coalgebra C is called right (left) co-Frobenius if there is a monomorphism $C \hookrightarrow C^*$ of right (left) C^*-modules. C is called co-Frobenius if it is both left and right co-Frobenius.

Definition

A coalgebra C is called right (left) quasi-co-Frobenius, or shortly right QcF coalgebra, if there is a monomorphism $C \hookrightarrow (C^*)^{(I)}$ of right (left) C^*-modules. C is called QcF coalgebra if it is both left and right QcF coalgebra.

(Co)Frobenius & Q(c)F algebras and coalgebras

(Co)Frobenius & Q(c)F algebras and coalgebras

- A Frobenius algebra is finite dimensional.

(Co)Frobenius & Q(c)F algebras and coalgebras

- A Frobenius algebra is finite dimensional.
- A "left" Frobenius (QF) algebra is also "right" Frobenius (QF).

(Co)Frobenius & Q(c)F algebras and coalgebras

- A Frobenius algebra is finite dimensional.
- A "left" Frobenius (QF) algebra is also "right" Frobenius (QF).
- A left co-Frobenius (QcF) coalgebra is not necesarity right co-Frobenius (QcF).

(Co)Frobenius & Q(c)F algebras and coalgebras

- A Frobenius algebra is finite dimensional.
- A "left" Frobenius (QF) algebra is also "right" Frobenius (QF).
- A left co-Frobenius (QcF) coalgebra is not necesarity right co-Frobenius (QcF).
- A **Hopf algebra** is left co-Frobenius **iff** it is right co-Frobenius.

(Co)Frobenius & Q(c)F algebras and coalgebras

- A Frobenius algebra is finite dimensional.
- A "left" Frobenius (QF) algebra is also "right" Frobenius (QF).
- A left co-Frobenius (QcF) coalgebra is not necesarity right co-Frobenius (QcF).
- A **Hopf algebra** is left co-Frobenius **iff** it is right co-Frobenius.
- C - finite dimensional coalgebra - is co-Frobenius if and only if C^* is Frobenius.

(Co)Frobenius & Q(c)F algebras and coalgebras

- A Frobenius algebra is finite dimensional.
- A "left" Frobenius (QF) algebra is also "right" Frobenius (QF).
- A left co-Frobenius (QcF) coalgebra is not necesarity right co-Frobenius (QcF).
- A **Hopf algebra** is left co-Frobenius **iff** it is right co-Frobenius.
- C - finite dimensional coalgebra - is co-Frobenius if and only if C^* is Frobenius.
- Also C finite dimensional coalgebra is QcF **iff** C^* is a QF ring, i.e. artinian and self-injective (& then also cogenerator) $\Leftrightarrow$ "injectives=projectives".

(Co)Frobenius & Q(c)F algebras and coalgebras

- A Frobenius algebra is finite dimensional.
- A "left" Frobenius (QF) algebra is also "right" Frobenius (QF).
- A left co-Frobenius (QcF) coalgebra is not necesarity right co-Frobenius (QcF).
- A **Hopf algebra** is left co-Frobenius **iff** it is right co-Frobenius.
- C - finite dimensional coalgebra - is co-Frobenius if and only if C^* is Frobenius.
- Also C finite dimensional coalgebra is QcF **iff** C^* is a QF ring, i.e. artinian and self-injective (& then also cogenerator) $\Leftrightarrow$ "injectives=projectives".
- C coalgebra is left QcF **iff** C is projective as left C^*-module. In this case, C is also a generator for rational C-comodules & C^* is right self-injective!

(Co)Frobenius & Q(c)F algebras and coalgebras

• A Frobenius algebra is finite dimensional.
• A "left" Frobenius (QF) algebra is also "right" Frobenius (QF).
• A left co-Frobenius (QcF) coalgebra is not necesarity right co-Frobenius (QcF).
• A **Hopf algebra** is left co-Frobenius **iff** it is right co-Frobenius.
• C - finite dimensional coalgebra - is co-Frobenius if and only if C^* is Frobenius.
• Also C finite dimensional coalgebra is QcF **iff** C^* is a QF ring, i.e. artinian and self-injective (& then also cogenerator) $\Leftrightarrow$ "injectives=projectives".
• C coalgebra is left QcF **iff** C is projective as left C^*-module. In this case, C is also a generator for rational C-comodules & C^* is right self-injective!
• C QcF $\Leftrightarrow$ C is a projective generator in comod-C (or C-comod).

The Weak Isomorphism

Definition

(i) Let $\mathcal{C}$ be a category having products. We say that $M, N \in \mathcal{C}$ are weakly π-isomorphic if and only if there are some sets I, J such that $M^I \simeq N^J$; we write $M \overset{\pi}{\sim} N$.

The Weak Isomorphism

Definition

(i) Let $\mathcal{C}$ be a category having products. We say that $M, N \in \mathcal{C}$ are weakly π-isomorphic if and only if there are some sets I, J such that $M^I \simeq N^J$; we write $M \stackrel{\pi}{\sim} N$.
(ii) Let $\mathcal{C}$ be a category having coproducts. We say that $M, N \in \mathcal{C}$ are weakly σ-isomorphic if and only if there are some sets I, J such that $M^{(I)} \simeq N^{(J)}$; we write $M \stackrel{\sigma}{\sim} N$.

Quasi-co-Frobenius Coalgebras: One Main Result

Theorem

Let C be a coalgebra. Then the following assertions are equivalent.

Quasi-co-Frobenius Coalgebras: One Main Result

Theorem

Let C be a coalgebra. Then the following assertions are equivalent.

(i) C is a QcF coalgebra.

Quasi-co-Frobenius Coalgebras: One Main Result

Theorem

Let C be a coalgebra. Then the following assertions are equivalent.

(i) C is a QcF coalgebra.

(ii) $C \overset{\sigma}{\sim} Rat(C^*)$ or $C \overset{\pi}{\sim} Rat(C^*)$ in ${}^C\mathcal{M}$

Quasi-co-Frobenius Coalgebras: One Main Result

Theorem

Let C be a coalgebra. Then the following assertions are equivalent.

(i) C is a QcF coalgebra.

(ii) $C \overset{\sigma}{\sim} Rat(C^*)$ or $C \overset{\pi}{\sim} Rat(C^*)$ in ${}^C\mathcal{M}$

(iii) $C^{(\mathbb{N})} \simeq (Rat(C^*))^{(\mathbb{N})}$ or $\prod\limits_{\mathbb{N}}^{C} C \simeq \prod\limits_{\mathbb{N}}^{C} Rat(C^*)$

Quasi-co-Frobenius Coalgebras: One Main Result

Theorem

Let C be a coalgebra. Then the following assertions are equivalent.

(i) C is a QcF coalgebra.

(ii) $C \overset{\sigma}{\sim} Rat(C^*)$ or $C \overset{\pi}{\sim} Rat(C^*)$ in ${}^C\mathcal{M}$

(iii) $C^{(\mathbb{N})} \simeq (Rat(C^*))^{(\mathbb{N})}$ or $\prod\limits_{\mathbb{N}}^{C} C \simeq \prod\limits_{\mathbb{N}}^{C} Rat(C^*)$

(iv, v) The left hand side version of (ii), (iii).

Applications: Co-Frobenius coalgebras

Theorem

A coalgebra C is co-Frobenius if and only if $C \cong Rat({}_{C^*}C^*)$ as left C^*-modules, if and only if $C \cong Rat(C^*_{C^*})$ as right C^*-modules.

Applications: Co-Frobenius coalgebras

Theorem

A coalgebra C is co-Frobenius if and only if $C \cong Rat({}_{C^*}C^*)$ as left C^*-modules, if and only if $C \cong Rat(C^*_{C^*})$ as right C^*-modules.

- A-finite dimensional is Frobenius $\Leftrightarrow A \cong A^*$.
- A-profinite, $A = C^*$ then C is co-Frobenius $\Leftrightarrow C \cong Rat(C^*)$. In this situation we have $A \cong A^\vee$ as left (or right) A-modules! Here $A^\vee$=topological completion of $\mathrm{Hom}_{\mathrm{cont}}(A, K)$.
- A-profinite, $A = C^*$ then C is Quasi-co-Frobenius $\Leftrightarrow C \overset{\sigma,\pi}{\sim} Rat(C^*)$. In this situation, $A \overset{\pi}{\sim} A^\vee$!

Categorical characterizations

Theorem

Let C be a coalgebra. Then the following assertions are equivalent:

Theorem

Let C be a coalgebra. Then the following assertions are equivalent:

(i) C is co-Frobenius.

Categorical characterizations

Theorem

Let C be a coalgebra. Then the following assertions are equivalent:

(i) C is co-Frobenius.

(ii) The functors $\mathrm{Hom}_{C^*}(-, C^*) : C$-comod$\to C^*$-mod and $\mathrm{Hom}(-, K) : C$-comod$\to C^*$-mod are isomorphic in the *category* of functors from C-comod to C^*-mod.

Categorical characterizations

Theorem

Let C be a coalgebra. Then the following assertions are equivalent:

(i) C is co-Frobenius.

(ii) The functors $\mathrm{Hom}_{C^*}(-, C^*) : C\text{-comod} \to C^*\text{-mod}$ and $\mathrm{Hom}(-, K) : C\text{-comod} \to C^*\text{-mod}$ are isomorphic in the *category* of functors from C-comod to C^*-mod.

Theorem

Let C be a coalgebra. Then the following assertions are equivalent:

Categorical characterizations

Theorem

Let C be a coalgebra. Then the following assertions are equivalent:

(i) C is co-Frobenius.

(ii) The functors $\mathrm{Hom}_{C^*}(-, C^*) : C$-comod$\to C^*$-mod and $\mathrm{Hom}(-, K) : C$-comod$\to C^*$-mod are isomorphic in the *category* of functors from C-comod to C^*-mod.

Theorem

Let C be a coalgebra. Then the following assertions are equivalent:

(i) C is QcF.

Categorical characterizations

Theorem

Let C be a coalgebra. Then the following assertions are equivalent:

(i) C is co-Frobenius.

(ii) The functors $\mathrm{Hom}_{C^*}(-, C^*) : C$-comod$\to C^*$-mod and $\mathrm{Hom}(-, K) : C$-comod$\to C^*$-mod are isomorphic in the *category* of functors from C-comod to C^*-mod.

Theorem

Let C be a coalgebra. Then the following assertions are equivalent:

(i) C is QcF.

(ii) The functors $\mathrm{Hom}_{C^*}(-, C^*) : C$-comod$\to C^*$-mod and $\mathrm{Hom}(-, K) : C$-comod$\to C^*$-mod are weakly isomorphic in the *category* of functors from C-comod to C^*-mod.

Hopf Algebras

Definition

H - Hopf algebra: an algebra (H, m, u) and a coalgebra (H, Δ, ε) + an antipode $S : H \to H$ s.t. $\Delta : H \to H \otimes H$ and $\varepsilon : H \to K$ are morphisms of algebras & S is convolution inverse to Id.

Hopf Algebras

Definition

H - Hopf algebra: an algebra (H, m, u) and a coalgebra (H, Δ, ε) + an antipode $S : H \to H$ s.t. $\Delta : H \to H \otimes H$ and $\varepsilon : H \to K$ are morphisms of algebras & S is convolution inverse to Id.

$\lambda \in H^*$ left (right) integral if $\lambda : H \to K$ is a morphism of left (right) H^*-modules (K = left H^*-module by $H^* \xrightarrow{u^*} K$). That is,

$$\lambda(h * g) = f(1)\lambda(g)$$

Hopf Algebras

Definition

H - Hopf algebra: an algebra (H, m, u) and a coalgebra (H, Δ, ε) + an antipode $S : H \to H$ s.t. $\Delta : H \to H \otimes H$ and $\varepsilon : H \to K$ are morphisms of algebras & S is convolution inverse to Id.

$\lambda \in H^*$ left (right) integral if $\lambda : H \to K$ is a morphism of left (right) H^*-modules (K = left H^*-module by $H^* \xrightarrow{u^*} K$). That is,

$$\lambda(h * g) = f(1)\lambda(g)$$

Example

G=compact group, $H = R(G)$ Hopf algebra, comultiplication as before, multiplication of complex functions, $S(f) = (x \mapsto f(x^{-1}))$.
$\int$ = integral of the left Haar measure then $\int |_{R(G)} : R(G) \to \mathbb{C}$ has $\int x \cdot f = \int f = u^*(x) \int f$ ($u : \mathbb{C} \to R(G)$, $G \to R(G)^* \xrightarrow{u^*} \mathbb{C}$).

Fundamental Results on Hopf Algebras

Theorem (Lin, Larson, Sweedler, Sullivan)

If H is a Hopf algebra, then the following assertions are equivalent.

Fundamental Results on Hopf Algebras

Theorem (Lin, Larson, Sweedler, Sullivan)

If H is a Hopf algebra, then the following assertions are equivalent.

(i) H is a right co-Frobenius coalgebra.

Fundamental Results on Hopf Algebras

Theorem (Lin, Larson, Sweedler, Sullivan)

If H is a Hopf algebra, then the following assertions are equivalent.

(i) H is a right co-Frobenius coalgebra.

(ii) H is a right QcF coalgebra.

Fundamental Results on Hopf Algebras

Theorem (Lin, Larson, Sweedler, Sullivan)

If H is a Hopf algebra, then the following assertions are equivalent.

(i) H is a right co-Frobenius coalgebra.

(ii) H is a right QcF coalgebra.

(iii) H is a right semiperfect coalgebra.

Fundamental Results on Hopf Algebras

Theorem (Lin, Larson, Sweedler, Sullivan)

If H is a Hopf algebra, then the following assertions are equivalent.

(i) H is a right co-Frobenius coalgebra.

(ii) H is a right QcF coalgebra.

(iii) H is a right semiperfect coalgebra.

(iv) $Rat({}_{H^*}H^*) \neq 0$.

Fundamental Results on Hopf Algebras

Theorem (Lin, Larson, Sweedler, Sullivan)

If H is a Hopf algebra, then the following assertions are equivalent.

(i) H is a right co-Frobenius coalgebra.

(ii) H is a right QcF coalgebra.

(iii) H is a right semiperfect coalgebra.

(iv) $Rat({}_{H^*}H^*) \neq 0$.

(v) $\int_l \neq 0$.

Fundamental Results on Hopf Algebras

Theorem (Lin, Larson, Sweedler, Sullivan)

If H is a Hopf algebra, then the following assertions are equivalent.

(i) H is a right co-Frobenius coalgebra.

(ii) H is a right QcF coalgebra.

(iii) H is a right semiperfect coalgebra.

(iv) $Rat({}_{H^*}H^*) \neq 0$.

(v) $\int_l \neq 0$.

(vi) $\dim \int_l = 1$.

Fundamental Results on Hopf Algebras

Theorem (Lin, Larson, Sweedler, Sullivan)

If H is a Hopf algebra, then the following assertions are equivalent.

(i) H is a right co-Frobenius coalgebra.

(ii) H is a right QcF coalgebra.

(iii) H is a right semiperfect coalgebra.

(iv) $Rat({}_{H^*}H^*) \neq 0$.

(v) $\int_l \neq 0$.

(vi) $\dim \int_l = 1$.

(vii) The left-right symmetric version of the above.

As a consequence of the techniques developed: a new proof of the bijectivity of the antipode.

Theory of Algebraic Integrals for Profinite Algebras

Let C - coalgebra; let M - right C-comodule. Define
$\int_{l,M} = \mathrm{Hom}_{\mathrm{comod}-C}(C, M)$.
For finite dimensional comodules:
$\int_{l,M} = \mathrm{Hom}^{C}(C, M) = \mathrm{Hom}_{C^*}(M^*, C^*)$.
Model: left integrals in a Hopf algebra, $\int_l = \mathrm{Hom}(H, K)$ (K right comodule as before by $K \to K \otimes H$, $1 \mapsto 1 \otimes 1_H$).

- Was considered before.
- In Hopf algebras, uniqueness of integrals reads $\dim(\int_l) \leq 1 = \dim(K)$; existence (in co-Frobenius Hopf algebras) $\dim(\int_l) \geq 1 = \dim(K)$.

Calling these "spaces integrals" has roots in...

Theory of Algebraic Integrals for Profinite Algebras

Let C - coalgebra; let M - right C-comodule. Define
$\int_{l,M} = \mathrm{Hom}_{\mathrm{comod}-C}(C, M)$.
For finite dimensional comodules:
$\int_{l,M} = \mathrm{Hom}^C(C, M) = \mathrm{Hom}_{C^*}(M^*, C^*)$.
Model: left integrals in a Hopf algebra, $\int_l = \mathrm{Hom}(H, K)$ (K right comodule as before by $K \to K \otimes H$, $1 \mapsto 1 \otimes 1_H$).

- Was considered before.
- In Hopf algebras, uniqueness of integrals reads $\dim(\int_l) \leq 1 = \dim(K)$; existence (in co-Frobenius Hopf algebras) $\dim(\int_l) \geq 1 = \dim(K)$.

Calling these "spaces integrals" has roots in... **compact groups**

Compact Groups and vector valued “quantum”-invariant Integrals

G - compact group.

Note

One could think of a measure which has the feature that translation of a set U by a has a certain effect on its measure $\mu(U)$ determined by a itself (we could think that the measure of the translation of U by a depends on the measure of U in a way that is "quantified" by a).

Compact Groups and vector valued "quantum"-invariant Integrals

G - compact group.

Note

One could think of a measure which has the feature that translation of a set U by a has a certain effect on its measure $\mu(U)$ determined by a itself (we could think that the measure of the translation of U by a depends on the measure of U in a way that is "quantified" by a).

Example

$d\mu_t(x) = e^{itx}dx$ on $G = (\mathbb{R}, +)$
$\int_{\mathbb{R}} f(x+a)d\mu_t(x) = \int_{\mathbb{R}} f(x+a)e^{itx}dx = \int_{\mathbb{R}}(f(x)e^{it(x-a)})dx =$
$e^{-ita}\int_{\mathbb{R}} f(x)d\mu_t(x)$

Compact Groups and vector valued "quantum"-invariant Integrals

For general G, one would need $\int a \cdot f = \eta(a) \int f$ for some $\eta(a) \in \mathbb{C}$.

More generally, we can consider vector valued integrals $\int : R(G) \to \mathbb{C}^n$, that is,

$$\int f d\mu = \begin{pmatrix} \int f d\mu_1 \\ \dots \\ \int f d\mu_n \end{pmatrix}$$

and the quantum invariance $\int a \cdot f d\mu = \eta(a) \cdot \int f d\mu$, where $\eta : G \to Gl_n(\mathbb{C})$.

Compact Groups and vector valued "quantum"-invariant Integrals

Note

Since $\eta(xy)\int f = \int xy \cdot f = \eta(x)\int y \cdot f = \eta(x)\eta(y)\int f$, we can see that η must be a (continuous!) representation of G.

Compact Groups and vector valued "quantum"-invariant Integrals

Note

Since $\eta(xy)\int f = \int xy \cdot f = \eta(x)\int y \cdot f = \eta(x)\eta(y)\int f$, we can see that η must be a (continuous!) representation of G.

$V = \mathbb{C}^n$ is a (left!) rep. of G with $\eta : G \to \mathrm{End}(V)$ **iff** V is a right $R(G)$-comodule. Moreover, a linear map $\varphi : V \to W$ is a morphism of G-modules **iff** it is a morphism of $R(G)$-comodules.

Compact Groups and vector valued "quantum"-invariant Integrals

Note

Since $\eta(xy)\int f = \int xy \cdot f = \eta(x)\int y \cdot f = \eta(x)\eta(y)\int f$, we can see that η must be a (continuous!) representation of G.

$V = \mathbb{C}^n$ is a (left!) rep. of G with $\eta : G \to \mathrm{End}(V)$ **iff** V is a right $R(G)$-comodule. Moreover, a linear map $\varphi : V \to W$ is a morphism of G-modules **iff** it is a morphism of $R(G)$-comodules.

$\int(x \cdot f) = \eta(x)\int f = x \cdot \int(f)$, $\Rightarrow$ $\int \in \mathrm{Hom}^{R(G)}(R(G), V)$ so $\int \in \int_{l,R(G)}$.

Existence and Uniqueness of algebraic Integrals

In analogy to Hopf algebras and compact groups, we may think of the existence and uniqueness properties for integrals:

"Existence of integrals": $\dim(\int_{l,M}) \geq \dim(M)$ (Hopf algebras: $\dim(\int_l) \geq 1 = \dim K$, $\int_l = \int_{l,K} \dots$)
"Uniqueness of integrals": $\dim(\int_{l,M}) \leq \dim(M)$ (Hopf algebras: $\dim(\int_l) \leq 1 = \dim K$, $\int_l = \int_{l,K} \dots$)

co-Frobenius properties and integrals

Proposition

If C is left QcF then:
(i) $\int_{l,T} \neq 0$ *for all rational simple left* C^**-modules* $T \Leftrightarrow C$ *is right QcF*
(ii) $\dim(\int_{l,T}) \geq \dim(T)$ *for all rational simple left* C^**-modules* $T \Leftrightarrow C$ *is right QcF.*

co-Frobenius properties and integrals

Proposition

If C is left QcF then:
(i) $\int_{l,T} \neq 0$ *for all rational simple left C^*-modules T* $\Leftrightarrow$ *C is right QcF*
(ii) $\dim(\int_{l,T}) \geq \dim(T)$ *for all rational simple left C^*-modules T* $\Leftrightarrow$ *C is right QcF.*

Proposition

left co-Frobenius $\Rightarrow$ *uniqueness of left integrals and existence of right integrals for all finite dimensional rational modules.*

Some other Main Results

Theorem

A coalgebra C is co-Frobenius (both on the left and on the right) if and only if $\dim(\int_{l,M}) = \dim(M)$ for all finite dimensional right C-comodules M, equivalently, $\dim(\int_{r,N}) = \dim(N)$ for all finite dimensional left C-comodules N.

Some other Main Results

Theorem

A coalgebra C is co-Frobenius (both on the left and on the right) if and only if $\dim(\int_{l,M}) = \dim(M)$ for all finite dimensional right C-comodules M, equivalently, $\dim(\int_{r,N}) = \dim(N)$ for all finite dimensional left C-comodules N.

Corollary

• *"Another Proof for the existence and uniqueness of integrals of Hopf algebras and the equivalent characterizations".*

• *Further characterizations of co-Frobenius coalgebras!*

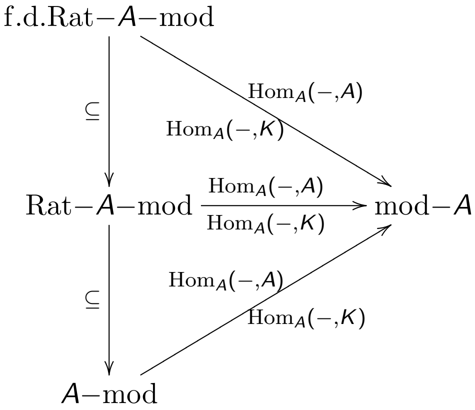
Categorical characterizations

C coalgebra, $A = C^*$.

Generalized(quasi)Frobenius

Generalized(quasi)Frobenius

(quasi)Frobenius



Quasi-co-Frobenius = weak isomorphism
co-Frobenius = isomorphism

Other Results

- Examples showing that the results are the best possible;
- Examples showing that all the possible inclusions between the above classes of coalgebras and other important ones (& combinations of left & right of these) are strict. Also, left QcF $\Rightarrow$ left semiperfect, but also right semiperfect (new)!
- Other connections and applications to compact groups;
- For algebras, the cogenerator and the self-injective do not imply each other. For coalgebras, projective (left) implies generator (right); we prove the converse is not true, and give the precise conditions when it is.

Antipodes

H-dual quasi-Hopf algebra (co-quasi Hopf): H coassociative coalgebra but not necessarily associative as an algebra. Same compatibility.
$\varphi \in (H \otimes H \otimes H)^*$ - *reassociator*, invertible with respect to the convolution algebra structure of $(H \otimes H \otimes H)^*$. For all $h, g, f, e \in H$:

$$\begin{aligned} h_1(g_1 f_1)\varphi(h_2, g_2, f_2) &= \varphi(h_1, g_1, f_1)(h_2 g_2) f_2 \\ 1h = h1 &= h \\ \varphi(h_1, g_1, f_1 e_1)\varphi(h_2 g_2, f_2, e_2) &= \varphi(g_1, f_1, e_1)\varphi(h_1, g_2 f_2, e_2)\varphi(h_2, g_3, f_3) \\ \varphi(h, 1, g) &= \varepsilon(h)\varepsilon(g) \end{aligned}$$

Antipodes

H-dual quasi-Hopf algebra (co-quasi Hopf): H coassociative coalgebra but not necessarily associative as an algebra. Same compatibility.
$\varphi \in (H \otimes H \otimes H)^*$ - *reassociator*, invertible with respect to the convolution algebra structure of $(H \otimes H \otimes H)^*$. For all $h, g, f, e \in H$:

$$\begin{aligned}
h_1(g_1 f_1)\varphi(h_2, g_2, f_2) &= \varphi(h_1, g_1, f_1)(h_2 g_2) f_2 \\
1h = h1 &= h \\
\varphi(h_1, g_1, f_1 e_1)\varphi(h_2 g_2, f_2, e_2) &= \varphi(g_1, f_1, e_1)\varphi(h_1, g_2 f_2, e_2)\varphi(h_2, g_3, f_3) \\
\varphi(h, 1, g) &= \varepsilon(h)\varepsilon(g)
\end{aligned}$$

$\exists$ a coalgebra antimorphism S of H and elements $\alpha, \beta \in H^*$ such that for all $h \in H$:

$$S(h_1)\alpha(h_2)h_3 = \alpha(h)1, \qquad h_1\beta(h_2)S(h_3) = \beta(h)1$$

$$\varphi(h_1\beta(h_2), S(h_3), \alpha(h_4)h_5) = \varphi^{-1}(S(h_1), \alpha(h_2)h_3, \beta(h_4)S(h_5)) = \varepsilon(h).$$

Antipodes

$0 \neq t \in \int_l$; $kt \subseteq Rat({}_{H^*}H^*) = Rat(H^*_{H^*})$ is a two sided ideal $\Rightarrow$ kt also has a left comultiplication $t \mapsto a \otimes t$. i.e. $t \cdot \alpha = \alpha(a)t, \forall \alpha \in H^*$.
a - **the distinguished grouplike of** H.
For $M \in \mathcal{M}^H$, let ${}^aM \in {}^H\mathcal{M}$ be (**well**) defined by

$$M \ni m \mapsto m^a_{-1} \otimes m^a_0 = aS(m_1) \otimes m_0 \in H \otimes M$$

Antipodes

$0 \neq t \in \int_l$; $kt \subseteq Rat({}_{H^*}H^*) = Rat(H^*_{H^*})$ is a two sided ideal $\Rightarrow$ kt also has a left comultiplication $t \mapsto a \otimes t$. i.e. $t \cdot \alpha = \alpha(a)t$, $\forall \alpha \in H^*$.
a - **the distinguished grouplike of** H.
For $M \in \mathcal{M}^H$, let ${}^aM \in {}^H\mathcal{M}$ be (**well**) defined by

$$M \ni m \mapsto m^a_{-1} \otimes m^a_0 = aS(m_1) \otimes m_0 \in H \otimes M$$

Note

The map $p : H \to Rat(H^*)$, $p(x) = x \rightharpoonup t$ is a bijective morphism of left H-comodules (right H^*-modules). In fact, we have an isomorphism of left H-comodules $H \otimes \int_r \to Rat(H^*)$, $(x, t) \mapsto (x \rightharpoonup t)$

Antipodes

$0 \neq t \in \int_l$; $kt \subseteq Rat({}_{H^*}H^*) = Rat(H^*_{H^*})$ is a two sided ideal $\Rightarrow$ kt also has a left comultiplication $t \mapsto a \otimes t$. i.e. $t \cdot \alpha = \alpha(a)t$, $\forall \alpha \in H^*$.
a - **the distinguished grouplike of** H.
For $M \in \mathcal{M}^H$, let ${}^aM \in {}^H\mathcal{M}$ be (**well**) defined by

$$M \ni m \mapsto m^a_{-1} \otimes m^a_0 = aS(m_1) \otimes m_0 \in H \otimes M$$

Note

The map $p : H \to Rat(H^*)$, $p(x) = x \rightharpoondown t$ is a bijective morphism of left H-comodules (right H^*-modules). In fact, we have an isomorphism of left H-comodules $H \otimes \int_r \to Rat(H^*)$, $(x, t) \mapsto (x \rightharpoondown t)$

Proposition

The map $p : {}^aH \to Rat(H^*)$, $p(x) = x \rightharpoonup t$ is a surjective morphism of left H-comodules (right H^*-modules).

Antipodes

Theorem (Radford)

The antipode of a co-Frobenius Hopf algebra is bijective.

Antipodes

Theorem (Radford)

The antipode of a co-Frobenius Hopf algebra is bijective.

Proof.[New] Only need S surjective (the map $H \ni x \mapsto t \leftharpoondown x \in H^*$ is injective $\Rightarrow$ S-injective)
Put $\pi := {}^aH \xrightarrow{p} Rat(H^*_{H^*}) \xrightarrow{\sim} H \otimes \int_r \simeq H$; it splits ($H$ projective in ${}^H\mathcal{M}$) so $\exists \varphi \in {}^H\mathcal{M}$ s.t. $\pi\varphi = \mathrm{Id}_H$.

Antipodes

Theorem (Radford)

The antipode of a co-Frobenius Hopf algebra is bijective.

Proof.[New] Only need S surjective (the map $H \ni x \mapsto t \leftharpoondown x \in H^*$ is injective $\Rightarrow$ S-injective)

Put $\pi := {}^aH \xrightarrow{p} Rat(H^*_{H^*}) \xrightarrow{\sim} H \otimes \int_r \simeq H$; it splits ($H$ projective in ${}^H\mathcal{M}$) so $\exists \varphi \in {}^H\mathcal{M}$ s.t. $\pi\varphi = \mathrm{Id}_H$.

$$\varphi(x)^a_{-1} \otimes \varphi(x)^a_0 = x_1 \otimes \varphi(x_2) \Rightarrow$$

Antipodes

Theorem (Radford)

The antipode of a co-Frobenius Hopf algebra is bijective.

Proof.[New] Only need S surjective (the map $H \ni x \mapsto t \leftharpoondown x \in H^*$ is injective $\Rightarrow$ S-injective)
Put $\pi := {}^aH \xrightarrow{p} Rat(H^*_{H^*}) \xrightarrow{\sim} H \otimes \int_r \simeq H$; it splits ($H$ projective in ${}^H\mathcal{M}$) so $\exists \varphi \in {}^H\mathcal{M}$ s.t. $\pi\varphi = \mathrm{Id}_H$.

$$\varphi(x)^a_{-1} \otimes \varphi(x)^a_0 = x_1 \otimes \varphi(x_2) \Rightarrow$$
$$aS(\varphi(x)_2) \otimes \varphi(x)_1 = x_1 \otimes \varphi(x_2) \Rightarrow$$

Antipodes

Theorem (Radford)

The antipode of a co-Frobenius Hopf algebra is bijective.

Proof.[New] Only need S surjective (the map $H \ni x \mapsto t \leftharpoondown x \in H^*$ is injective $\Rightarrow$ S-injective)

Put $\pi := {}^aH \xrightarrow{p} Rat(H^*_{H^*}) \xrightarrow{\sim} H \otimes \int_r \simeq H$; it splits ($H$ projective in ${}^H\mathcal{M}$) so $\exists \varphi \in {}^H\mathcal{M}$ s.t. $\pi\varphi = \mathrm{Id}_H$.

$$\varphi(x)^a_{-1} \otimes \varphi(x)^a_0 = x_1 \otimes \varphi(x_2) \Rightarrow$$
$$aS(\varphi(x)_2) \otimes \varphi(x)_1 = x_1 \otimes \varphi(x_2) \Rightarrow$$
$$S(a^{-1})S(\varphi(x)_2)\varepsilon\pi(\varphi(x)_1) = x_1\varepsilon\pi\varphi(x_2) = x_1\varepsilon(x_2) = x \Rightarrow$$

Antipodes

Theorem (Radford)

The antipode of a co-Frobenius Hopf algebra is bijective.

Proof.[New] Only need S surjective (the map $H \ni x \mapsto t \leftharpoondown x \in H^*$ is injective $\Rightarrow$ S-injective)

Put $\pi := {}^aH \xrightarrow{p} Rat(H^*_{H^*}) \xrightarrow{\sim} H \otimes \int_r \simeq H$; it splits ($H$ projective in ${}^H\mathcal{M}$) so $\exists \varphi \in {}^H\mathcal{M}$ s.t. $\pi\varphi = \mathrm{Id}_H$.

$$\varphi(x)^a_{-1} \otimes \varphi(x)^a_0 = x_1 \otimes \varphi(x_2) \Rightarrow$$
$$aS(\varphi(x)_2) \otimes \varphi(x)_1 = x_1 \otimes \varphi(x_2) \Rightarrow$$
$$S(a^{-1})S(\varphi(x)_2)\varepsilon\pi(\varphi(x)_1) = x_1\varepsilon\pi\varphi(x_2) = x_1\varepsilon(x_2) = x \Rightarrow$$
$$x = S(\varepsilon\pi(\varphi(x)_1)\varphi(x)_2 a^{-1}).$$

□

This proof adapts to co-quasi Hopf algebras (dual quasi-Hopf algebras), with some technicalities; some assambly (inventivity) required...

Antipodes

A proof of the bijectivity of the antipode without the use of the uniqueness of integrals, which follows then as a consequence This shows a much tighter connection to compact groups then realized before.

For $(M, \rho) \in \mathcal{M}^H$, $\rho : M \longrightarrow M \otimes H, \quad \rho(m) = m_0 \otimes m_1$, we define ${}^S M \in {}^H\mathcal{M}$ with comodule structure given by

$$m \longmapsto m_{(-1)} \otimes m_{(0)} = S(m_1) \otimes m_0$$

Antipodes

A proof of the bijectivity of the antipode without the use of the uniqueness of integrals, which follows then as a consequence This shows a much tighter connection to compact groups then realized before.

For $(M,\rho)\in\mathcal{M}^H$, $\rho: M\longrightarrow M\otimes H$, $\quad\rho(m)=m_0\otimes m_1$, we define ${}^SM\in{}^H\mathcal{M}$ with comodule structure given by

$$m\longmapsto m_{(-1)}\otimes m_{(0)}=S(m_1)\otimes m_0$$

Proposition

${}^S Rat(H^*)$, with left H-module structure given by

$$H\otimes{}^S Rat(H^*)\longrightarrow{}^S Rat(H^*),\quad x\otimes\alpha\longrightarrow x\rightharpoonup\alpha$$

and left H-comodule structure as above is a left H-Hopf module.

Antipodes

By the above and the Fundamental Th of Hopf modules:
${}^{S}Rat(H^*) \simeq H \otimes ({}^{S}Rat(H^*))^{co} = H \otimes \int_l$ and then we get a map

$$\pi : ({}^{S}H)^{(dim \int_l)} \simeq {}^{S}Rat(H^*) \simeq H \otimes ({}^{S}Rat(H^*))^{co} \twoheadrightarrow {}^{H}H$$

Then, looking at the "coalgebras of the coefficients", we get $C_H \subseteq C_{S_H}$ and then imediately $H \subseteq S(H)$.

New Perspective: With this, the classical proof of the uniqueness of the Haar measure can be adopted "mutatis-mutandis" to Hopf algebras.

Articles

1. *The generating condition for coalgebras*, **Bull. Lond. Math. Soc.** 41 (2009), no. 3, 483-494.
2. (with. M. Beattie and S. Raianu) *The antipode of a quasi-Hopf algebra with nonzero integrals is bijective*, **Algebr. Represent. Theory** 12 (2009), no. 2-5, 251-255.
3. *Abstract Integrals in Algebra*, submitted to publication, (31p); old preprint arXiv:0810.3740.
4. *Generalized Frobenius Algebras and the Theory of Hopf Algebras*, submitted to publication, (12p); old preprint arXiv:0803.0775.
5. (with S. Raianu) *The bijectivity of the antipode revisited*, preprint (arxiv).
6. *Co-Frobenius Coalgebras*, **J. Algebra** 303, no. 1 (2006), 146-153.

T

T H

T H A

T H A N

T H A N K

T H A N K   Y

THANK YO

T H A N K   Y O U

T H A N K Y O U !