# On the quantum torus

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- Let $\mathfrak{Z}(M) = \{m \in M \mid rm = mr \ \forall r \in R\}$, the centre of $M$. When $R$ is commutative, $\mathfrak{Z}(M)$ is an $R$-bimodule. In general $M_0 = R\mathfrak{Z}(M)R$, the $R$-bimodule generated by $\mathfrak{Z}(M)$.

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- Inductively construct for each $i \geq 1$, an $R$-bimodule $M_i$, called the $i$-th *differential part of* $M$, as follows:

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- $M_0 \subset M_1 \subset \cdots \subset M_i \subset \cdots$ The *differential part* of $M$ is the $R$-bimodule $M_{diff} = \cup_{i \geq 0} M_i$.

- The vector space $\mathrm{Hom}_{\Bbbk}(R,R)$ (the vector space of $\Bbbk$-linear homomorphisms) is an $R$-bimodule with

$$(r\cdot\varphi)(s)=r\varphi(s),\qquad (\varphi\cdot r)(s)=\varphi(rs)\qquad \forall r\in R,\varphi\in \mathrm{Hom}_{\Bbbk}(R,R).$$

The bimodule of $\Bbbk$-linear *differential operators* on $R$ is $D_{\Bbbk}(R)=(\mathrm{Hom}_{\Bbbk}(R,R))_{diff}$, while each $D^i_{\Bbbk}(R)=(\mathrm{Hom}_{\Bbbk}(R,R))_i$ is the bimodule of order $i$ *differential operators* on $R$.

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- $D^0_{\Bbbk}(R) \subset D^1_{\Bbbk}(R) \subset \cdots D^i_{\Bbbk}(R) \subset \cdots$.
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- When $R$ is commutative, we have the definition given by Grothendieck ([EGA]):

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$$D^i_{\Bbbk}(R)=\{\varphi\mid[\cdots[[\varphi,a_0],a_1],\cdots a_i]=0,\forall a_0,\cdots,a_i\in R\}.$$

- For $r \in R$, let $\lambda_r, \rho_r \in \mathrm{Hom}_{\Bbbk}(R, R)$ be the left and right multiplication operators. That is, $\lambda_r(s) = rs$ and $\rho_r(s) = sr$. Then $D^0_{\Bbbk}(R)$ is generated by the set $\{\lambda_r, \rho_r | r \in R\}$.

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    More generally, when $R$ is an Azumaya algebra, we have $D_{\Bbbk}(R)$ is generated as an algebra by $D_{\Bbbk}(\mathcal{Z}(R))$ and $D^0_{\Bbbk}(R)$.

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  - When $R$ is a semisimple Hopf algebra, then $D_{\Bbbk}(R) = D^0_{\Bbbk}(R)$. This covers the cases of group algebras and universal enveloping algebras of the semisimple kind.
  - (*Joint with T.C.McCune*) $R$ a finitely generated free associative algebra.

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- For each $a \in R$, and $i \leq n$ let $\partial_i^a$ be the derivation on $R$ defined by $\partial_i^a(x_j) = \delta_{i,j}a$. That is, $[\partial_i^a, x_j] = \delta_{i,j}\lambda_a \in D^0_{\Bbbk}(R)$.

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  - For an $r \in \mathbb{N}, r \geq 2$, let $I = (i_1, i_2, \cdots, i_r)$ be a sequence of natural numbers $i_j \leq n$ and $J = (a_1, \cdots, a_r)$ be a sequence of elements from $R$. Further, let $\widehat{I} = (i_2, \cdots, i_r)$ and $\widehat{J} = (a_2, \cdots, a_r)$. Denote by $\partial_I^J \in D^r_{\Bbbk}(R)$ the operator which satisfies the commutator rules

$$[\partial_I^J, x_{i_1}] = a_1\partial_{\widehat{I}}^{\widehat{J}}, \quad [\partial_I^J, x_l] = 0 \textit{ for } l \neq i_1, \textit{ and } \quad \partial_I^J(1) = 0.$$

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- **Theorem:** For $r \geq 1$ $D^r_{\Bbbk}(R)$ is generated as an $R$-bimodule by the set

$$\{\partial^J_I \mid I = (i_1, \cdots, i_s), J = (a_1, \cdots, a_s), i_j \in \mathbb{N}, i_j \leq n, a_j \in R, 1 \leq s \leq r\}$$

$$\cup\{\rho_a \mid a \in R\}.$$

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  - The operators $\partial_I^J$ satisfy the Hasse-Schmidt-like property:

$$\partial_I^J(ab) = \sum_{I_1 * I_2 = I} \partial_{I_1}^{J_1}(a)\partial_{I_2}^{J_2}(b)$$

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  - In the case of commutative algebras, the subalgebras of constants of fixed derivations form a rich topic of study. Such a study for the noncommutative algebras have not been undertaken. A start could be on the free algebras. Other than works by P.M.Cohn and his students, not much has been done in this area.

# Quantum differential operators on graded algebras

- When the enveloping algebra of a Lie algebra acts on an associative ring via its Hopf structure, the action gives usual differential operators. Likewise, when the enveloping algebra of a Lie superalgebra (respectively, Lie coloured algebra) acts on an associative ring, the action gives super (respectively, coloured) differential operators. Lunts and Rosenberg wanted a notion of differential operators which would allow quantum groups to act via such differential operators, keeping in mind that quantum groups have group-like elements which act as automorphisms.

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- For $m \in M, r \in R$, let $[m, r]_a = mr - \sigma_a(r)m$. Using these notations,

$$\mathcal{Z}_q(M) = \Bbbk\text{-}span\{homogeneous\ m \mid \exists a \in \Gamma \text{ such that } [m, r]_a = 0 \forall r \in R\}.$$

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$$[\partial^{\beta^a}, x]_a = 1, \;\; \partial^{\beta^a} x \partial^{\beta^b} = \partial^{\beta^b} x \partial^{\beta^a}, \; \textit{and} \; \partial^{\beta^{-1}} - q\partial^\beta = (1-q)\partial^{\beta^{-1}} x \partial^\beta.$$

- The algebra $D_q(R)$ is a simple domain. Presently, in a joint work with D.A.Jordan and T.C.McCune, we are trying to understand this algebra. D.A.Jordan has proved that this algebra is left-Noetherian and that the relations mentioned above are the determining relations for this algebra.

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- Generalization to the polynomial algebra: Let $R = \mathbb{k}[x_1, \cdots, x_n]$ which is $\mathbb{Z}^n$-graded by $deg(x_i) = (0, 0, \cdots, 1, 0, \cdots 0) = e_i$ the standard basis. Let $q_1, \cdots, q_n$ be transcendental over $\mathbb{Q}$ and $\mathbb{Q}(q_1, \cdots, q_n) \subset \mathbb{k}$. Let $\beta : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{k}^*$ be $\beta(e_i, e_j) = \begin{cases} q_i & if\ i = j \\ 1 & if\ i \neq j \end{cases}$ Then $D_q(R)$ is generated as an algebra by the set $\{\lambda_{x_i}, \partial_i^{\beta^{-1}}, \partial_i, \partial_i^{\beta} \mid 1 \leq i \leq n\}$. Again, $D_q(R)$ is a simple domain.

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- The following relations can be seen

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**Proposition:** For any $\varphi \in D_q(R_{s,n})$ there exists an $t \in T$ such that $t\varphi \in D_q(R_n)$.

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# Quantum differential operators on the Quantum Torus

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- The group $\mathbb{Z}^n$ acts on $\Delta$ by $\gamma \cdot \rho_{x_i} = \beta(\gamma, e_i)\rho_{x_i}$ and $\gamma \cdot \delta_i = \beta(\gamma, e_i)^{-1}\delta_i$ for $\gamma \in \mathbb{Z}^n$.
- The algebra $\Delta \# \mathbb{Z}^n$ is a left and right Noetherian simple domain of GK-dimension $3n$.
- Consider the homomorphism of algebras

$$f : \Delta \# \mathbb{Z}^n \to D_q(R_n) \quad \textit{with} \quad f(\varphi\gamma) \mapsto \varphi\sigma_\gamma \quad \textit{for } \varphi \in \Delta, \gamma \in \mathbb{Z}^n.$$

- **Theorem:** The map $f$ is an isomorphism of filtered algebras.
- **Theorem:** The algebra $D_q(R_{s,n})$ is a left and right Noetherian, simple domain of GK-dimension $3n$ with basis

$$\{\rho_I \delta_J \sigma_a \mid a, I, J \in \mathbb{Z}^n, i_t \geq 0 \text{ for } t > s, j_t \geq 0 \forall t\}.$$

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- Derivations and skew derivations on $R_n$ and its tori have been studied by various people (J.Alev, V.A.Artamonov, F.Dumas, S.Montgomery, J.P.Osborn, D.Passman, S.P.Smith, and others ... my apologies for my ignorance). A study of these quantum differential operators will contribute to this literature. Thank you.