Category Theory Meets the First Fundamental Theorem of Calculus
Kolchin Seminar in Differential Algebra

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One of the most important results in the calculus is the First Fundamental Theorem of Calculus, which states that if \( f : [a, b] \rightarrow \mathbb{R} \) is a continuous function and if \( F \) is defined on \( [a, b] \) by \( F(x) = \int_a^x f(t)dt \), then

1. \( F \) is continuous on \([a, b]\),

2. \( F \) is differentiable on \((a, b)\) and

3. \( \frac{d}{dx} \left( \int_a^x f(t)dt \right) = f(x) \) on \((a, b)\).

We will investigate (3) from a categorical viewpoint.
Outline

- Notations & Review of Some Category Theory
- Differential Algebras
- Rota-Baxter Algebras
- Mixed Distributive Laws
- Differential Rota-Baxter Algebras
Notations

- Fix $k$ a commutative ring with identity and fix $\lambda \in k$.
- All algebras are commutative $k$-algebras with identity.
- All homomorphisms preserve the identity.
- All linear maps and tensor products are over $k$.
- $\mathbb{N} = \{0, 1, 2, \ldots\}$ will denote the natural numbers.
- $\mathbb{N}_+ = \{1, 2, 3, \ldots\}$.
**Definition:** For categories \( \mathbf{A} \) and \( \mathbf{B} \) and functors \( F : \mathbf{A} \to \mathbf{B} \) and \( G : \mathbf{A} \to \mathbf{B} \), a **natural transformation** \( \eta : F \to G \) is a family of morphisms in \( \mathbf{B} \), \( \{ \eta_X : FX \to GX \} \), one for each \( X \in \mathbf{A} \), such that for any morphism \( f : X \to Y \) in \( \mathbf{A} \), we have the following in \( \mathbf{B} \):

\[
\eta_Y \circ Ff = Gf \circ \eta_X,
\]

i.e., the diagram

\[
\begin{array}{ccc}
FX & \xrightarrow{\eta_X} & GX \\
Ff \downarrow & & \downarrow Gf \\
FY & \xrightarrow{\eta_Y} &GY
\end{array}
\]

commutes.
Natural Transformations

Example

- Let \( \textbf{FVS} \) denote the category of finite-dimensional vector spaces over some field \( K \). Let \( V \in \textbf{FVS} \) and \( V^* \) be the dual space of \( V \). Then \( V^* \cong V \), but the isomorphism is not “natural” in the sense that it requires the choice of a basis for \( V \).

- However, there is a natural transformation \( \eta_V : V \rightarrow V^{**} \) defined by \( (\eta_V(v))(\varphi) = \varphi(v) \) for any \( v \in V \) and \( \varphi \in V^* \).
Adjoint Functors

**Definition:** Given categories \( A \) and \( B \), an adjunction between \( A \) and \( B \) consists of functors \( F : A \to B \) and \( U : B \to A \) (opposite directions) such that for each \( X \in A \) and \( Y \in B \),

\[
\mathbf{B}(FX, Y) \cong \mathbf{A}(X, UY),
\]

where the isomorphism is natural in \( X \in A \) and \( Y \in B \). In this case, we say that \( F \) is **left adjoint** to \( U \), or that \( U \) is **right adjoint** to \( F \).
Adjoint Functors

**Example**

- Let $\mathbf{A} = \mathbf{SET}$, the category of sets and functions, and let $\mathbf{B} = \mathbf{GRP}$, the category of groups and group homomorphisms.

- Let $F : \mathbf{SET} \to \mathbf{GRP}$ be the free group functor, and let $U : \mathbf{GRP} \to \mathbf{SET}$ be the underlying set functor, i.e. the "forgetful" functor.

- Then $F$ is left adjoint to $U$, since we have the natural isomorphism

\[
\mathbf{GRP}(FX, G) \cong \mathbf{SET}(X, UG)
\]

for any set $X$ and any group $G$. 
Unit-Counit Description of an Adjunction

Given adjoint functors $F : A \to B$ and $U : B \to A$ with $F$ left adjoint to $U$, we can equivalently describe the adjunction by using natural transformations.

In this case, there are natural transformations $\eta : \text{id}_A \to UF$ and $\varepsilon : FU \to \text{id}_B$ such that

$$\varepsilon F \circ F \eta = \text{id}_F$$

and

$$U \varepsilon \circ \eta U = \text{id}_U.$$

$\eta$ is called the **unit** of the adjunction, and $\varepsilon$ is called the **counit** of the adjunction.

We write the adjunction as $\langle F, U, \eta, \varepsilon \rangle : A \leftrightarrow B$. 

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Unit-Counit Description of an Adjunction

**Example**

- As before, let \( F : \text{SET} \rightarrow \text{GRP} \) be the free group functor, and \( U : \text{GRP} \rightarrow \text{SET} \) the forgetful functor.

- The unit is the natural transformation \( \eta : \text{id}_{\text{SET}} \rightarrow UF \) given for each set \( X \) by \( \eta_X : X \rightarrow UFX \), i.e., \( \eta_X \) is the injection of the generators into the (underlying set of the) free group on \( X \).

- The counit is \( \varepsilon : FU \rightarrow \text{id}_{\text{GRP}} \) given for each group \( G \) by \( \varepsilon_G : FUG \rightarrow G \), where \( \varepsilon_G \) maps an element of the free group on \( UG \) to the corresponding element in \( G \) by using the operation in \( G \) to "condense" the string to an element in \( G \).
**Definition:** A monad $\mathbf{T}$ on a category $\mathbf{A}$ is a triple $\mathbf{T} = (T, \eta, \mu)$ where

- $T : \mathbf{A} \to \mathbf{A}$ is a functor (i.e., an endofunctor on $\mathbf{A}$),
- $\eta$ is a natural transformation $\eta : \text{id}_\mathbf{A} \to T$, and
- $\mu$ is a natural transformation $\mu : TT \to T$,

such that

- $\mu \circ T\eta = \text{id}_T = \mu \circ \eta T$, and
- $\mu \circ T\mu = \mu \circ \mu T$

that is, the diagrams
Monads

\[ T \xrightarrow{T\eta} TT \xleftarrow{\eta T} T \]

\[ \text{id}_T \xrightarrow{} \mu \quad \mu \xrightarrow{} \text{id}_T \]

and

\[ TTTT \xrightarrow{T\mu} TTT \]

\[ \mu T \xrightarrow{} \mu \]

\[ \mu \xrightarrow{} T \]

commute.
Example

- Take $\mathbf{A} = \mathbf{SET}$.
- Let $T : \mathbf{SET} \to \mathbf{SET}$ to be the functor that assigns to each set $X$ the underlying set of the free group on $X$.
- Let $\eta_X : X \to TX$ be the "insertion of the generators" as before.
- Let $\mu_X : TTX \to TX$ be the underlying set map of the "partial collapse" of a string of strings to a string using the operation in the free group.
Algebras from a Monad

Given a monad $T = (T, \eta, \mu)$ on a category $A$, we can form the category of $T$-algebras, denoted by $A^T$, as follows.

- The objects of $A^T$ are pairs $(A, f)$, where $A$ is an object of $A$ and $f : TA \to A$ is a morphism in $A$ such that
  - $f \circ \eta_A = \text{id}_A$, and
  - $f \circ \mu_A = f \circ Tf$.
- A morphism $g : (A, f) \to (A', f')$ in $A^T$ is a morphism $g : A \to A'$ in $A$ such that $g \circ f = f' \circ Tg$. 
Algebras from a Monad

The diagrams for objects:

\[ A \xrightarrow{\eta_A} TA \quad \text{and} \quad TTA \xrightarrow{\mu_A} TA \]

and for morphisms:

\[ TA \xrightarrow{Tg} TA' \]

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Monad from an Adjunction

Given adjoint functors $F: A \to B$ and $U: B \to A$ with $F$ left adjoint to $U$, the adjunction $\langle F, U, \eta, \varepsilon \rangle: A \to B$ gives rise to a monad $T = (T, \eta, \mu)$ on the category $A$ by taking:

- $T = UF: A \to A$,
- $\eta = \eta: \text{id}_A \to UF = T$, and
- $\mu = U\varepsilon F: UFUF = TT \to UF = T$. 

Adjunction from a Monad

Given a monad \( T = (T, \eta, \mu) \) on the category \( A \), there are adjoint functors \( F^T : A \to A^T \) and \( U^T : A^T \to A \) defined as follows:

- For any \( X \in A \), define \( F^T X = (TX, \mu_X) \), and define \( F^T \) on morphisms in \( A \) similarly.
- For any \( (A, f) \in A^T \), define \( U^T(A, f) = A \), and similarly for morphisms in \( A^T \).

This adjunction gives rise to the same monad \( T = (T, \eta, \mu) \) on the category \( A \).
Given adjoint functors $F : A \to B$ and $U : B \to A$ with $F$ left adjoint to $U$, let $T = (T, \eta, \mu)$ be the monad on the category $A$ generated by the adjunction, and let $\langle F^T, U^T, \eta^T, \varepsilon^T \rangle : A \to A^T$ be the adjunction given from $T$.

Then there is a “comparison” functor $K : B \to A^T$ given by $KX = (UX, U\varepsilon_X)$, which satisfies $KF = F^T$ and $U^TK = U$.

In some nice cases, $K$ is an isomorphism, in which case we say that $B$ is monadic over $A$. 
A **comonad** $G$ on a category $\mathcal{B}$ is a (co)triple $G = (G, \varepsilon, \delta)$ where $G : \mathcal{B} \to \mathcal{B}$ is a functor (i.e., an endofunctor on $\mathcal{B}$) and $\varepsilon$ and $\delta$ are natural transformations, $\varepsilon : G \to \text{id}_\mathcal{B}$ and $\delta : G \to GG$ such that the following diagrams commute:

![Diagram](attachment:image.png)

and

![Diagram](attachment:image2.png)
We have seen that an adjunction \( \langle F, U, \eta, \varepsilon \rangle : A \rightarrow B \) gives rise to a monad \( T = (T, \eta, \mu) \) on the category \( A \) by taking \( T = UF \) and \( \mu = U\varepsilon F \).

The adjunction \( \langle F, U, \eta, \varepsilon \rangle : A \rightarrow B \) also gives rise to a comonad \( G = (G, \varepsilon, \delta) \) on the category \( B \) by taking \( G = FU \) and \( \delta = F\eta U \).
A comonad \( \mathbf{G} = (G, \varepsilon, \delta) \) on \( \mathbf{B} \) gives a category of \( \mathbf{G} \)-coalgebras, denoted by \( \mathbf{B}_{\mathbf{G}} \), as follows.

The objects of \( \mathbf{B}_{\mathbf{G}} \) are pairs \( (B, g) \), where \( B \) is an object of \( \mathbf{B} \) and \( g : B \to GB \) is a morphism in \( \mathbf{B} \) such that \( \varepsilon_B \circ g = \text{id}_B \) and \( Gg \circ g = \delta_B \circ g \).

A morphism \( f : (B, g) \to (B', g') \) in \( \mathbf{B}_{\mathbf{G}} \) is a morphism \( f : B \to B' \) in \( \mathbf{B} \) such that \( g' \circ f = Gf \circ g \).

Et cetera, et cetera, et cetera.
Derivations with Weight

Let $k$ be a ring, $\lambda \in k$, and let $R$ be an algebra.

- A **derivation of weight $\lambda$ on $R$ over $k$** or more briefly, a **$\lambda$-derivation on $R$ over $k$** is a module endomorphism $d$ of $R$ satisfying both

  $$d(xy) = d(x)y + xd(y) + \lambda d(x)d(y), \text{ for all } x, y \in R$$

  and

  $$d(1_R) = 0.$$ 

- A **$\lambda$-differential algebra** is a pair $(R, d)$ where $R$ is an algebra and $d$ is a $\lambda$-derivation on $R$ over $k$.

- Let $(R, d)$ and $(S, e)$ be two $\lambda$-differential algebras. A **homomorphism of $\lambda$-differential algebras** $f : (R, d) \to (S, e)$ is a homomorphism $f : R \to S$ of algebras such that $f(d(x)) = e(f(x))$ for all $x \in R$.

Note that a 0-derivation is a derivation in the usual sense.
An Example of a $\lambda$-Derivation

**Example**

Let $\mathbb{R}$ denote the field of real numbers, and let $\lambda \in \mathbb{R}$, $\lambda \neq 0$. Let $A$ denote the $\mathbb{R}$-algebra of $\mathbb{R}$-valued continuous functions on $\mathbb{R}$, and consider the usual "difference quotient" operator $d_\lambda$ on $A$ defined by

$$(d_\lambda(f))(x) = (f(x + \lambda) - f(x))/\lambda.$$ 

Then a simple calculation shows that $d_\lambda$ is a $\lambda$-derivation on $A$. 
**Proposition (Leibniz’ Rule):** Let \((R, d)\) be a \(\lambda\)-differential algebra, let \(x, y \in R\), and let \(n \in \mathbb{N}\). Then

\[
d^{(n)}(xy) = \sum_{k=0}^{n} \sum_{j=0}^{n-k} \binom{n}{k} \binom{n-k}{j} \lambda^k d^{(n-j)}(x) d^{(k+j)}(y).
\]

When \(\lambda = 0\), this reduces to the familiar

\[
d^{(n)}(xy) = \sum_{k=0}^{n} \binom{n}{k} d^{(k)}(x) d^{(n-k)}(y).
\]
A Simplification

For the remainder of the talk, to simplify notation, we will assume that

$$\lambda = 0.$$
The Hurwitz Product

For any algebra $A$, let $A^N$ denote the $k$-module of all functions $f : N \to A$. On $A^N$, we define the Hurwitz product $fg$ of any $f, g \in A^N$ by

$$(fg)(n) = \sum_{k=0}^{n} \binom{n}{k} f(k)g(n - k).$$

Compare with Leibniz’ Rule:

$$d^{(n)}(xy) = \sum_{k=0}^{n} \binom{n}{k} d^{(k)}(x)d^{(n-k)}(y).$$
The Algebra of Hurwitz Series

- $A^N$ (with the Hurwitz product) is called the \textbf{algebra of Hurwitz series} over $A$.

- The map $\partial_A : A^N \to A^N$, defined by $\partial_A(f)(n) = f(n + 1)$ for any $f \in A^N$, is a derivation on $A^N$.

- Hence $(A^N, \partial_A)$ is a differential algebra for any algebra $A$.

- For any algebra homomorphism $h : A \to B$, the map $h^N : A^N \to B^N$ defined by $(h^N(f))(n) = h(f(n))$ for any $f \in A^N$ and $n \in \mathbb{N}$ is a differential algebra homomorphism from $(A^N, \partial_A)$ to $(B^N, \partial_B)$.
Let $\textbf{DIF}$ denote the category of differential algebras, and let $\textbf{ALG}$ denote the category of algebras.

We have a functor $G : \textbf{ALG} \rightarrow \textbf{DIF}$ given on objects $A \in \textbf{ALG}$ by $G(A) = (A^N, \partial_A)$ and on morphisms $h : A \rightarrow B$ in $\textbf{ALG}$ by $G(h) = h^N$ as defined above.

Let $V : \textbf{DIF} \rightarrow \textbf{ALG}$ denote the forgetful functor defined on objects $(R, d) \in \textbf{DIF}$ by $V(R, d) = R$ and on morphisms $g : (R, d) \rightarrow (S, e)$ in $\textbf{DIF}$ by $V(g) = g$. 
Beginning of the Big Picture

\[ \text{DIF} \leftarrow \text{ALG} \] (1)

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There are two natural transformations $\eta : \text{id}_{\text{DIF}} \to GV$ and $\varepsilon : VG \to \text{id}_{\text{ALG}}$.

- For any $(R, d) \in \text{DIF}$, define

  $$\eta_{(R,d)} : (R, d) \to (GV)(R, d) = (R^\mathbb{N}, \partial_R)$$

  by $(\eta_{(R,d)}(x))(n) = d^n(x), x \in R, n \in \mathbb{N}$.

- For any $A \in \text{ALG}$, define

  $$\varepsilon_A : (VG)(A) = A^\mathbb{N} \to A$$

  by $\varepsilon_A(f) = f(0), f \in A^\mathbb{N}$. 

Proposition:

- The functor $G : \text{ALG} \rightarrow \text{DIF}$ defined above is the right adjoint of the forgetful functor $V : \text{DIF} \rightarrow \text{ALG}$.

- It follows that $(A^N, \partial_A)$ is a cofree differential algebra on the algebra $A$. 
The Comonad from the Adjunction

The adjunction \( \langle V, G, \eta, \varepsilon \rangle : \text{DIF} \rightarrow \text{ALG} \) gives rise to a comonad \( C = (C, \varepsilon, \delta) \) on the category \( \text{ALG} \), where

- \( C \) is the functor \( C : = V G : \text{ALG} \rightarrow \text{ALG} \) given by \( C(A) = A^\mathbb{N} \) for any \( A \in \text{ALG} \), and

- \( \delta : C \rightarrow CC \) is the natural transformation defined by \( \delta : = V \eta G \).

It follows that for any \( A \in \text{ALG} \),

\[ \delta_A : A^\mathbb{N} \rightarrow (A^\mathbb{N})^\mathbb{N}, \quad (\delta_A(f)(m))(n) = f(m+n), \quad f \in A^\mathbb{N}, m, n \in \mathbb{N}. \]

Note that as a \( k \)-module, \( (A^\mathbb{N})^\mathbb{N} \cong A^{\mathbb{N} \times \mathbb{N}} \), the set of sequences of sequences, or equivalently, doubly-indexed sequences with values in \( A \).
The Comonad Comes from the Monoid \((\mathbb{N}, +, 0)\)

Observe that comonad \(C = (C, \varepsilon, \delta)\) on \(\text{ALG}\) come from the monoid of natural numbers \((\mathbb{N}, +, 0)\) in the following sense:

- The functor \(C\) is given by \(C(A) = A^\mathbb{N}\).
- \(\varepsilon_A : A^\mathbb{N} \to A \cong A^{\{\ast\}}\) is induced by \(0 : \{\ast\} \to \mathbb{N}\), i.e., \(\varepsilon_A \cong A^0\).
- \(\delta_A : A^\mathbb{N} \to (A^\mathbb{N})^\mathbb{N} \cong A^{\mathbb{N} \times \mathbb{N}}\) is induced by \(+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}\), i.e., \(\delta_A \cong A^+\).
The comonad $\mathbf{C}$ induces a category of $\mathbf{C}$-coalgebras, denoted by $\mathbf{ALG}_\mathbf{C}$.

The objects in $\mathbf{ALG}_\mathbf{C}$ are pairs $\langle A, f \rangle$ where $A \in \mathbf{ALG}$ and $f : A \to A^N$ is an algebra homomorphism satisfying the two properties

$$\varepsilon_A \circ f = \text{id}_A, \quad \delta_A \circ f = f^N \circ f$$

A morphism $\varphi : \langle A, f \rangle \to \langle B, g \rangle$ in $\mathbf{ALG}_\mathbf{C}$ is an algebra homomorphism $\varphi : A \to B$ such that $g \circ \varphi = \varphi^N \circ f$. 

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**Proposition:** The cocomparison functor $H : \text{DIF} \to \text{ALG}_C$ is an isomorphism, i.e., DIF is comonadic over ALG.

**Corollary:** For any algebra $A$, there is a one-to-one correspondence among

- derivations $d$ on $A$ over $k$;
- $C$-costructures $f$ on $A$, i.e., algebra homomorphisms $f : A \to A^\mathbb{N}$ satisfying $\varepsilon_A \circ f = \text{id}_A$ and $\delta_A \circ f = f^\mathbb{N} \circ f$;
- sequences of $k$-module homomorphisms $(f_n) : A \to A$ for $n \in \mathbb{N}$ that satisfy $f_0 = \text{id}_A$, $f_m \circ f_n = f_{m+n}$ and
  
  $$f_n(ab) = \sum_{k=0}^{n} \binom{n}{k} f_k(a)f_{n-k}(b)$$
  for all $a, b \in A$. 
The Picture Grows
Definitions

Let $R$ be an algebra.

- A **Rota-Baxter operator** on $R$ is a $k$-linear endomorphism $P$ of $R$ satisfying

  $$P(x)P(y) = P(xP(y)) + P(yP(x)),$$

  for all $x, y \in R$.

- A **Rota-Baxter algebra** is a pair $(R, P)$ where $R$ is an algebra and $P$ is a Rota-Baxter operator on $R$.

- Let $(R, P)$ and $(S, Q)$ be two Rota-Baxter algebras. A **homomorphism of Rota-Baxter algebras**

  $f : (R, P) \rightarrow (S, Q)$ is a homomorphism $f : R \rightarrow S$ of algebras with the property that $f(P(x)) = Q(f(x))$ for all $x \in R$. 
An Example

Example

Let $R = \text{Cont}(\mathbb{R})$ denote the $\mathbb{R}$-algebra of continuous functions on $\mathbb{R}$. Let $P_0$ be the operator on $R$ given by

$$P_0(f)(x) = \int_0^x f(t)\,dt.$$ 

Then $P_0$ is a Rota-Baxter operator on $R$. 
Let $\textbf{RBA}$ denote the category of commutative Rota-Baxter $k$-algebras, and let $\textbf{ALG}$ denote the category of commutative algebras.

Let $U : \textbf{RBA} \to \textbf{ALG}$ denote the forgetful functor given on objects $(R, P) \in \textbf{RBA}$ by $U(R, P) = R$ and on morphisms $f : (R, P) \to (S, Q)$ in $\textbf{RBA}$ by $U(f) = f : R \to S$.

We proved earlier that $U$ has a left adjoint, and below we give an explicit description of the left adjoint, the free commutative Rota-Baxter algebra functor.

There are earlier constructions (e.g., by Cartier and by Rota) of free commutative Rota-Baxter algebras on sets, that is, as a left adjoint of the forgetful functor from $\textbf{RBA}$ to $\textbf{SET}$. 
The Free Rota-Baxter Algebra

We begin with some general observations about the free commutative Rota-Baxter algebra on a commutative algebra $A$ with identity $1_A$.

- The product for this free Rota-Baxter algebra on $A$ is constructed in terms of a generalization of the shuffle product, called the **mixable shuffle product**, which we will describe below and which in its recursive form is a natural generalization of the quasi-shuffle product.

- This free commutative Rota-Baxter algebra on $A$ is denoted by $\mathbb{III}(A)$.

- As a module, we have

$$\mathbb{III}(A) = \bigoplus_{i \geq 1} A^\otimes i = A \oplus (A \otimes A) \oplus (A \otimes A \otimes A) \oplus \cdots$$

where the tensors are defined over $k$. 

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The Mixable Shuffle Product

The multiplication on \( \Pi(A) \) is the product \( \diamond \) defined as follows.

Let \( a = a_0 \otimes \cdots \otimes a_m \in A^{\otimes (m+1)} \) and \( b = b_0 \otimes \cdots \otimes b_n \in A^{\otimes (n+1)} \).

If \( mn = 0 \), define

\[
\begin{align*}
a \diamond b &= \begin{cases} 
(a_0 b_0) \otimes b_1 \otimes \cdots \otimes b_n, & m = 0, n > 0, \\
(a_0 b_0) \otimes a_1 \otimes \cdots \otimes a_m, & m > 0, n = 0, \\
a_0 b_0, & m = n = 0.
\end{cases}
\end{align*}
\]
If $m > 0$ and $n > 0$, then $a \diamond b$ is defined inductively on $m + n$ by

$$a \diamond b = (a_0 b_0) \otimes \left( (a_1 \otimes \cdots \otimes a_m) \diamond (1_A \otimes b_1 \otimes \cdots b_n) \right) + (1_A \otimes a_1 \otimes \cdots \otimes a_m) \diamond (b_1 \otimes \cdots b_n).$$

Extending by additivity, $\diamond$ gives a $k$-bilinear map

$$\diamond : \mathcal{R}(A) \times \mathcal{R}(A) \to \mathcal{R}(A).$$
An Example of the Product

Example

\[(a_0 \otimes a_1)(b_0 \otimes b_1 \otimes b_2) = (a_0 b_0) \otimes \left( a_1 (1_A \otimes b_1 \otimes b_2) + (1_A \otimes a_1)(b_1 \otimes b_2) \right).\]

Now the first term in the right tensor factor is just \(a_1 \otimes b_1 \otimes b_2\). For the second term, we have

\[(1_A \otimes a_1)(b_1 \otimes b_2) = b_1 \otimes (a_1 (1_A \otimes b_2) + (1_A \otimes a_1)b_2) = b_1 \otimes (a_1 \otimes b_2 + b_2 \otimes a_1).\]

Thus we obtain

\[(a_0 \otimes a_1)(b_0 \otimes b_1 \otimes b_2) = (a_0 b_0) \otimes (a_1 \otimes b_1 \otimes b_2 + b_1 \otimes a_1 \otimes b_2 + b_1 \otimes b_2 \otimes a_1).\]
The Rota-Baxter Operator on $\mathfrak{III}(A)$

Define a linear endomorphism $P_A$ on $\mathfrak{III}(A)$ by assigning

$$P_A(x_0 \otimes x_1 \otimes \ldots \otimes x_n) = 1_A \otimes x_0 \otimes x_1 \otimes \ldots \otimes x_n,$$

for all $x_0 \otimes x_1 \otimes \ldots \otimes x_n \in A^{\otimes(n+1)}$ and extending by additivity.
The Rota-Baxter Functors

- Let $F : \text{ALG} \to \text{RBA}$ denote the functor given on objects $A \in \text{ALG}$ by $F(A) = (\Pi(A), P_A)$ and on morphisms $f : A \to B$ in $\text{ALG}$ by

$$F(f) \left( \sum_{i=1}^{k} a_{i0} \otimes a_{i1} \otimes \cdots \otimes a_{in_i} \right) = \sum_{i=1}^{k} f(a_{i0}) \otimes f(a_{i1}) \otimes \cdots \otimes f(a_{in_i})$$

which we also denote by $\Pi(f)$.

- As above, $U : \text{RBA} \to \text{ALG}$ denotes the forgetful functor.
Next define two natural transformations $\eta : \text{id}_{\text{ALG}} \to UF$ and $\varepsilon : FU \to \text{id}_{\text{RBA}}$.

- For any $A \in \text{ALG}$, define $\eta_A : A \to (UF)(A) = \mathcal{III}(A)$ to be just the natural embedding $j_A : A \to \mathcal{III}(A) = \bigoplus A^\otimes i$.

- For any $(R, P) \in \text{RBA}$, define

$$
\varepsilon_{(R,P)} : (FU)(R, P) = (\mathcal{III}(R), P_R) \to (R, P)
$$

by

$$
\varepsilon_{(R,P)} \left( \sum_{i=1}^{k} a_i0 \otimes a_i1 \otimes \cdots \otimes a_{in_i} \right) = \sum_{i=1}^{k} a_i0 P(a_i1 P(\cdots P(a_{in_i}) \cdots)),
$$

for any $\sum_{i=1}^{k} a_i0 \otimes a_i1 \otimes \cdots \otimes a_{in_i} \in \mathcal{III}(R)$. 

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Theorem: The functor $F : \text{ALG} \to \text{RBA}$ defined above is the left adjoint of the forgetful functor $U : \text{RBA} \to \text{ALG}$. More precisely, there is an adjunction $\langle F, U, \eta, \varepsilon \rangle : \text{ALG} \to \text{RBA}$. 

Adjoint Functors for RBA
The adjunction $\langle F, U, \eta, \varepsilon \rangle : \mathbf{ALG} \to \mathbf{RBA}$ gives rise to a monad $T = \langle T, \eta, \mu \rangle$ on $\mathbf{ALG}$.

- $T$ is the functor defined for any $A \in \mathbf{ALG}$ by $T(A) = \mathbf{III}(A)$.
- $\mu$ is the natural transformation $\mu_A : \mathbf{III}(\mathbf{III}(A)) \to \mathbf{III}(A)$ extended additively from

$$
\mu_A((a_{00} \otimes \cdots \otimes a_{0n_0}) \otimes \cdots \otimes (a_{k0} \otimes \cdots \otimes a_{kn_k}))
= (a_{00} \otimes \cdots \otimes a_{0n_0}) P_A(\cdots P_A(a_{k0} \otimes \cdots \otimes a_{kn_k}) \cdots ),
$$

where

$$(a_{00} \otimes \cdots \otimes a_{0n_0}) \otimes \cdots \otimes (a_{k0} \otimes \cdots \otimes a_{kn_k}) \in \mathbf{III}(\mathbf{III}(A))$$

with $a_{i0} \otimes \cdots \otimes a_{in_i} \in A^{\otimes (n_i+1)}$ for $n_0, \ldots, n_k \geq 0$ and $0 \leq i \leq k$. 

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Category Theory Meets the First Fundamental Theorem of Calculus
Algebras for the Rota-Baxter Monad $T$ on $\text{ALG}$

- The monad $T$ induces a category of $T$-algebras, denoted by $\text{ALG}^T$.
- The objects in $\text{ALG}^T$ are pairs $\langle A, h \rangle$ where $A \in \text{ALG}$ and $h : \mathbb{W}(A) \to A$ is an algebra homomorphism satisfying the two properties

$$h \circ \eta_A = \text{id}_A, \quad h \circ T(h) = h \circ \mu_A.$$

- A morphism $\phi : \langle R, f \rangle \to \langle S, g \rangle$ in $\text{ALG}^T$ is an algebra homomorphism $\phi : R \to S$ such that $g \circ T(\phi) = \phi \circ f$. 
RBA is Monadic over ALG

**Theorem:** The comparison functor $K : \text{RBA} \to \text{ALG}^T$ is an isomorphism, i.e., \textbf{RBA} is monadic over \textbf{ALG}.

**Corollary:** For any algebra $A$, there is a one-to-one correspondence between

1. Rota-Baxter operators $P$ on $A$;
2. $T$-structures on $A$, i.e., algebra homomorphisms $h : \mathcal{III}(A) \to A$ satisfying both $h \circ \eta_A = \text{id}_A$ and $h \circ T(h) = h \circ \mu_A$;
3. Sequences of linear maps $h_n : \mathcal{III}(A) \to A$, $n \in \mathbb{N}_+$, satisfying certain conditions.
The Big Picture Grows a Little More

RBA → U → DIF

H ← V ← ALG

T U ↘ ↘ ↙ ↙ ↖ ↖

ALG \rightarrow ALG^T \rightarrow ALG_C \rightarrow ALG

(3)
**Proposition:** Let \((A, P)\) be a Rota-Baxter algebra.

- Define a \(k\)-linear mapping \(\tilde{P} : A^N \rightarrow A^N\) by
  \[
  \tilde{P}(f)(0) = P(f(0)), \quad \tilde{P}(f)(n) = f(n - 1), \quad f \in A^N, n \in \mathbb{N}_+.
  \]
- Then \(\tilde{P}\) is a Rota-Baxter operator on \(A^N\),
  \[
  \varepsilon_A \circ \tilde{P} = P \circ \varepsilon_A
  \]
  and
  \[
  \partial_A \circ \tilde{P} = \text{id}_{A^N}.
  \]
- Compare with the First Fundamental Theorem of Calculus.
**Definition of Mixed Distributive Law**

**Definition:** Given a category $A$, a monad $T = (T, \eta, \mu)$ on $A$ and a comonad $C = (C, \varepsilon, \delta)$ on $A$, then a **mixed distributive law of $T$ over $C$** is a natural transformation $\beta : TC \to CT$ such that

- $\beta \circ \eta C = C \eta$;
- $\varepsilon T \circ \beta = T \varepsilon$;
- $\delta T \circ \beta = C \beta \circ \beta C \circ T \delta$ and
- $\beta \circ \mu C = C \mu \circ \beta T \circ T \beta$. 

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Category Theory Meets the First Fundamental Theorem of Calcul
The Lifting Theorem

**Theorem:** Given a category $\mathcal{A}$, a monad $\mathbf{T} = (T, \eta, \mu)$ on $\mathcal{A}$, a comonad $\mathbf{C} = (C, \varepsilon, \delta)$ on $\mathcal{A}$, and a mixed distributive law of $\mathbf{T}$ over $\mathbf{C}$. Then:

- there is a comonad $\tilde{\mathbf{C}}$ on the category $\mathcal{A}^\mathbf{T}$ of $\mathbf{T}$-algebras which lifts $\mathbf{C}$,
- there is a monad $\tilde{T}$ on the category $\mathcal{A}^\mathbf{C}$ of $\mathbf{C}$-coalgebras which lifts $\mathbf{T}$, and
- there is an isomorphism of categories $(\mathcal{A}^\mathbf{C})^{\tilde{T}} \cong (\mathcal{A}^\mathbf{T})^{\tilde{\mathbf{C}}}$ over $\mathcal{A}$.
The Mixed Distributive Law

- We want to apply this theorem to the case where \( A = \text{ALG}, \)
  \( T \) is the Rota-Baxter monad and \( C \) is the differential comonad. So we need a mixed distributive law of \( T \) over \( C \).

- This means that for each \( A \in \text{ALG} \), we need a natural homomorphism \( \beta_A : \text{III}(A^N) \rightarrow (\text{III}(A))^N \).

- By an earlier result, the Rota-Baxter operator \( P_A \) on \( \text{III}(A) \)
  extends to a Rota-Baxter operator \( \widetilde{P}_A \) on \( (\text{III}(A))^N \).
The Key Lemma

**Lemma:** For any algebra $A$, there is a unique Rota-Baxter algebra homomorphism

$$\beta_A : (\Pi(A^N), P_{AN}) \rightarrow ((\Pi(A))^N, \widetilde{P_A})$$

such that the equation

$$(\eta_A)^N = \beta_A \circ \eta_{AN}$$

holds.
The Main Theorem

- The natural transformation $\beta : TC \to CT$ given by $\beta_A : \Pi(A^N) \to (\Pi(A))^N$ is a mixed distributive law of $T$ over $C$.

- $\beta : TC \to CT$ gives rise to a comonad $\tilde{C}$ on the category $\text{ALG}^T$ of $T$-algebras which lifts $C$ in the sense that the underlying functor $U^T : \text{ALG}^T \to \text{ALG}$ commutes with $\tilde{C}$ and $C$, that is,

$$U^T \tilde{C} = CU^T, \quad U^T \tilde{\varepsilon} = \varepsilon U^T \quad \text{and} \quad U^T \tilde{\delta} = \delta U^T.$$

- Similarly, $\beta$ gives rise to a monad $\tilde{T}$ on the category $\text{ALG}_C$ of $C$-coalgebras which lifts $T$.

- There is an isomorphism $\Phi : (\text{ALG}_C)^\tilde{T} \to (\text{ALG}^T)^\tilde{C}$ of categories.
The Big Picture Grows Larger

(4)
**Definition:** We say that \((R, d, P)\) is a **differential Rota-Baxter algebra** if

- \((R, d)\) is a differential algebra,
- \((R, P)\) is a Rota-Baxter algebra, and
- \(d \circ P = \text{id}_R\).

If \((R, d, P)\) and \((R', d', P')\) are differential Rota-Baxter algebras, then a morphism of differential Rota-Baxter algebras \(f : (R, d, P) \to (R', d', P')\) is an algebra homomorphism \(f : R \to R'\) such that \(d'(f(x)) = f(d(x))\) and \(P'(f(x)) = f(P(x))\) for all \(x \in R\). The category of differential Rota-Baxter algebras will be denoted by \(\text{DRB}\).
There are forgetful functors:

- $U' : \text{DRB} \to \text{DIF}$ and
- $V' : \text{DRB} \to \text{RBA}$

such that $UV' = VU'$. We will see that:

- $U'$ has a left adjoint,
- $V'$ has a right adjoint, and
- $\text{DRB} \cong (\text{ALG}_C)^\sim \cong (\text{ALG}^T)_{\tilde{C}}$, where $\tilde{T}$ and $\tilde{C}$ come from the Main Theorem.
The Right Adjoint to $V'$

- Suppose that $(A, P) \in \text{RBA}$. 

- Let $\partial_A : A^N \to A^N$ and $\tilde{P} : A^N \to A^N$ be as above. 

- Since $\partial_A \circ \tilde{P} = \text{id}_{A^N}$, the triple $(A^N, \partial_A, \tilde{P})$ is a differential Rota-Baxter algebra. 

- Thus we have a functor $G' : \text{RBA} \to \text{DRB}$ given on objects $(A, P) \in \text{RBA}$ by $G'(A, P) = (A^N, \partial_A, \tilde{P})$ and on morphisms $\varphi : (A, P) \to (A', P')$ in \text{RBA} by $(G'(\varphi)(f))(n) = \varphi(f(n))$ for $f \in A^N$ and $n \in \mathbb{N}$. 

- $G'$ is the right adjoint to $V'$. 

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Another Comonad

The adjunction \( \langle V', G', \eta', \varepsilon' \rangle : \text{DRB} \rightarrow \text{RBA} \) gives a comonad \( C' = \langle C', \varepsilon', \delta' \rangle \) on the category \( \text{RBA} \), where

- \( C' := V' G' : \text{RBA} \rightarrow \text{RBA} \) is given by \( C'(A, P) = (A^N, \tilde{P}) \),

- \( \delta' \) is a natural transformation from \( C' \) to \( C'C' \) defined by \( \delta' := V' \eta' G' \).

In other words, for any \( (A, P) \in \text{RBA} \),

\[
\delta'_{(A,P)} : (A^N, \tilde{P}) \rightarrow ((A^N)^N, \tilde{\tilde{P}}), \quad \delta'_{(A,P)}(f) = \delta_A(f), \quad f \in A^N.
\]
And Another Category of Coalgebras

- The comonad $C'$ on $RBA$ gives a category of $C'$-coalgebras, denoted by $RBA_{C'}$.

- The comonad $C'$ also induces an adjunction.

- There is a uniquely defined cocomparison functor $H' : DRB \rightarrow RBA_{C'}$, and

- $H'$ is an isomorphism, so that $DRB \cong RBA_{C'}$. 
More Functors for the Big Picture

\[ \begin{array}{ccc}
\text{RBA} & \xrightarrow{\phi} & \text{DIF} \\
\text{(ALG}_T^C)^\sim & & \text{(ALG}_C)^\sim T \\
\text{ALG}^T & \xrightarrow{U^T} & \text{ALG}_C
\end{array} \]
Some Folklore from Category Theory

**Lemma:** Suppose that $A$ and $B$ are categories, $K : A \to B$ is an isomorphism of categories, $C = \langle C, \varepsilon, \delta \rangle$ is a comonad on $A$ and $C' = \langle C', \varepsilon', \delta' \rangle$ is a comonad on $B$. If $K$ commutes with $C$ and $C'$, i.e., $KC = C'K$, $K\varepsilon = \varepsilon'K$ and $K\delta = \delta'K$, then there exists a unique isomorphism $\widetilde{K} : A_C \to B_{C'}$ that lifts $K$, i.e., $U_{C'}\widetilde{K} = KU_C$.

**Corollary:** There is an isomorphism of categories $\widetilde{K} : RBA_{C'} \to (\text{ALG}^T)_{\widetilde{C}}$ such that $V_{\widetilde{C}}\widetilde{K} = KV_{C'}$. 
Let \((A, d)\) be a differential algebra.

- There is a derivation \(\tilde{d}\) on \(\Pi(A) \rightarrow \Pi(A)\) extending \(d\).

- \((\Pi(A), \tilde{d}, P_A)\) is a free differential Rota-Baxter algebra on the differential algebra \((A, d)\).

- There is a functor \(F' : \text{DIF} \rightarrow \text{DRB}\) that is left adjoint to the forgetful \(U' : \text{DRB} \rightarrow \text{DIF}\).

- There is a monad \(T'\) on \(\text{DIF}\) such that \(\text{DIF}^{T'} \cong \text{DRB}\).
To Complete the Picture

\[ \begin{array}{c}
\text{RBA} \\
\text{T\text{ALG}} \\
\text{ALG} \\
\text{DIF} \\
\text{DRB} \\
\end{array} \]

\[ \begin{array}{c}
\text{RBA}_C' \\
\text{ALG}_C^T \sim \tilde{C} \\
\text{ALG}_C \\
\text{DIF}^T \sim \tilde{T} \\
\end{array} \]

\[ \begin{array}{c}
\text{K} \\
\text{K'} \\
\text{H} \\
\text{H'} \\
\text{U} \\
\text{U'} \\
\end{array} \]

\[ \begin{array}{c}
\text{V} \\
\text{V'} \\
\text{V_C} \\
\text{V_C'} \\
\text{U} \\
\text{U'} \\
\text{U^T} \\
\text{U^T'} \\
\end{array} \]

\[ \begin{array}{c}
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