Analytic theory of difference equations with rational coefficients

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Correlation functions of diverse statistical models, gap probabilities in the Random Matrix Theory can be expressed in terms of solutions of the Painlevé type *differential* equations. In the recent years discrete analogs of the Painlevé equations have attracted considerable interest due to their connections to discrete probabilistic models. A. Borodin observed that the general setup for these equations is provided by the theory of *isomonodromy* transformations of linear systems of difference equations with rational coefficients.

The analytic theory of matrix linear difference equations

$$\Psi(z+1) = A(z)\Psi(z), \quad A = A_0 + \sum_{m=1}^{n} \frac{A_m}{z - z_m} \tag{1}$$

with rational coefficients is a subject of its own interest. It goes back to the fundamental results of Birkhoff (1911,1913) which have been developed later by many authors (see van der Put, Singer "Galois Theory of Difference Equations").

- They are classified in a rough way by terms: *regular, regular singular, mild and wild*
- The equation is regular singular if $A_0 = 1$.
- It is regular if in addition $\sum_{m=1}^{n} A_m = 0$.
- The mild equations are those for which the matrix A_0 is invertible.

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We will consider the case of mild equations with a diagonalizable leading coefficient A_0. It will be assumed also that the poles z_m are not congruent, $z_l - z_m \notin Z$.

If A_0 is diagonalizable, then using the transformations

$$\Psi' = \rho^z \Psi,\ A' = \rho A(z),\ \ \rho \in \mathbb{C};\ \ \Psi' = g\Psi, \mathcal{A}' = gA(z)g^{-1},\ g \in SL_r$$

we may assume without loss of generality that A_0 is a diagonal matrix of determinant 1,

$$A_0^{ij} = \rho_i \delta^{ij},\ \ \det A_0 = \prod_j \rho_j = 1.$$

In addition, it will be assumed that

$$\mathrm{Tr}\,(\mathrm{res}_\infty Adz) = \mathrm{Tr}\left(\sum_{m=1}^{n} A_m\right) = 0.$$

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$$\mathrm{Tr}\left(\mathrm{res}_\infty A dz\right) = \mathrm{Tr}\left(\sum_{m=1}^{n} A_m\right) = 0.$$

If $\rho_i \neq \rho_j$, then equation (1) has a unique formal solution

$$Y(z) = \left(1 + \sum_{s=1}^{\infty} \chi_s z^{-s}\right) e^{z \ln A_0 + K \ln z},$$

where $K^{ij} = k_i \delta^{ij}$ is a diagonal matrix.

Birkhoffs' results

Birkhoff considered difference equations with *polynomial* coefficients $\widetilde{A}$. The general case of rational $A(z)$ is reduced to the polynomial one by the transformation

$$\widetilde{A} = A(z)\prod_m (z - z_m), \quad \widetilde{\Psi} = \Psi \prod_m \Gamma(z - z_m),$$

where $\Gamma(z)$ is the Gamma-function.

Birkhoff proved that,

- if the ratios of the eigenvalues ρ_i are not real, $\Im(\rho_i/\rho_j) \neq 0$, then equation (1) with polynomial coefficients has two canonical meromorphic solutions $\widetilde{\Psi}_r(z)$ and $\widetilde{\Psi}_l(z)$ which are holomorphic and asymptotically represented by $\widetilde{Y}(z)$ in the half-planes $\Re z >> 0$ and $\Re z << 0$, respectively.
- the connection matrix

$$\widetilde{S}(z) = \widetilde{\Psi}_r^{-1}(z)\widetilde{\Psi}_l(z),$$

is a rational function in $\exp(2\pi iz)$.

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is a rational function in $\exp(2\pi i z)$.

- This function has just as many constants involved as there are parameters in $\widetilde{A}$.
- if two polynomial matrix functions $\widetilde{A}'(z)$ and $\widetilde{A}(z)$ have the same connection matrix $S(z)$ then there exists a rational matrix $R(z)$ such that

$$\widetilde{A}'(z) = R(z+1)\widetilde{A}(z)R^{-1}(z).$$

*Remark.*The condition $\Im(\rho_i/\rho_j) \neq 0$ under which Birkhoff's results are valid is due to Deligne.

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Birkhoff's approach to the construction of the canonical solutions:

$$\Psi_l = A(z-1)A(z-2)\cdots$$

$$\Psi_r = \cdots A^{-1}(z+1)A^{-1}(z)$$

For regular case the products converge. For other cases the canonical solutions are defined as it regularization of the products.

Over the years key ideas of Birhoff's approach have remained intact. A construction of actual solutions of (1) having prescribed asymptotic behavior in various sectors at infinity resembles rather the Stocks' theory of differential equations with irregular singularities, then the conventional theory of differential equations with regular singularities. The monodromy representation of $\pi_1(C\backslash\{z_1,\ldots,z_n\})$ which provides the integrals of motion for the Schlesinger equations, has no obvious analog in discrete situation.

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Alternative approach

Meromorphic solutions of the difference equations can be constructed via the following auxiliary Riemann-Hilbert type problem.

Problem I: *To find in the strip* $\Pi_x : \; x \leq \Re z \leq x+1$, *a continuous matrix function* $\Phi(z)$ *which is meromorphic inside* Π_x, *and such that its boundary values on the two sides of the strip satisfy the equation*

$$\Phi^{+}(\xi+1) = A(\xi)\Phi^{-}(\xi), \;\; \xi = x + iy\,.$$

Fundamental results of the theory of singular integral equations imply that:

- The Problem 1 always has solutions.
- If

$$ind_x A = \frac{1}{2\pi i} \int_L d \ln \det A = 0, \quad z \in L : \Re z = x.$$

then for a generic $A(z)$ this problem has a unique (up to the transformation $\Phi'(z) = \Phi(z)g, \; g \in SL_r$) *sectionally holomorphic*)

$$\exists \;\; 0 \le \alpha < 1, \quad |\Phi(z)| < e^{2\pi\alpha|\Im z|}, \quad |\Im z| \to \infty.$$

non-degenerate solution.

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Mild equations with real exponents

Let $\mathcal{P}_x$ be the space of continuous functions $\Phi(z)$ in the strip Π_x, which are holomorphic inside the strip and have at most polynomial growth at the infinity.

Lemma

Let x be a real number such that $x \neq \Re z_j$. Then:
(a) for $|x| >> 0$ there exists a unique up to normalization non-degenerate solution $\Phi_x \in \mathcal{P}_x$ of the Problem 1;
(b) for a generic $A(z)$ the solution $\Phi_x \in \mathcal{P}_x$ exists and is unique up to normalization for all x such that $\mathrm{ind}_x A = 0$;
(c) at the two infinities of the strip the function Φ_x asymptotically equals

$$\Phi_x(z) = Y(z) g_x^{\pm}, \;\; \Im z \to \pm\infty.$$

The matrix $g_x = g_x^+ (g_x^-)^{-1}$ can be seen as a "transition" matrix along a thick path Π_x.

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Theorem

Let $\rho_i \neq \rho_j, \Im\rho_i = 0$. *Then:*

(A) *there are unique meromorphic solutions* Ψ_l, Ψ_r *of equation (1), which are holomorphic in the domains* $\Re z << 0$ *and* $\Re z >> 0$*, respectively, and which are asymptotically represented by* $Yg^{\pm}_{l(r)},\ g^{-}_{l(r)} = 1$, *as* $\Im z \to \pm\infty$*;*

$$g^{ii}_{r(l)} = 1,\ g^{ij}_r = 0,\ \text{if } \rho_i < \rho_j,\ \ g^{ij}_l = 0,\ \text{if } \rho_i > \rho_j,$$

(B) *the connection matrix* $S = (\Psi_r)^{-1}\Psi_l$ *has the form*

$$S(z) = 1 - \sum_{m=1}^{n} \frac{S_m}{e^{2\pi i(z-z_m)} - 1},\ S_\infty = 1 + \sum_{m=1}^{n} S_m = g_r^{-1} e^{2\pi i K} g_l\,;$$

Local monodromies

Lemma

Let $x < y$ be real numbers such that the corresponding canonical solutions Ψ_x and Ψ_y do exists. Then the function $M_{x,y} = \Psi_y^{-1}\Psi_x$ has the form

$$M_{x,y} = 1 - \sum_{k \in J_{x,y}} \frac{m_{k,(x,y)}}{e^{2\pi i(z-z_k)} - 1},$$

where the sum is taking over a subset of indices $J_{x,y}$ corresponding to the poles such that $x < \Re z_k < y$.

Let us assume that $\Re z_k < \Re z_m,\ k < m$ and the matrix $K = \sum_k A_k$ is diagonal.

Theorem

There exists ϵ such that, if $|A_k| < \epsilon$, then equation (1) has a set of unique canonical normalized solutions $\Psi_k,\ k = 0, \dots, n$, which are holomorphic for $r_k + \epsilon < \Re z < r_{k+1} + 1$.

$$\Psi_k \sim Y g_k^{\pm}, \quad \Im z \to \pm\infty;\ g_k^- = 1$$

The local connection matrices $M_k = \Psi_k^{-1}\Psi_{k-1}$ have the form

$$M_k = 1 - \frac{m_k}{e^{2\pi i(z-z_k)} - 1},$$

where $(1 + m_n)\dots(1 + m_1) = e^{2\pi i K}$.

Regular singular equations. Small norm case.

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Theorem

In the limit $h \to 0$:
(A) *the canonical solution Ψ_k of the difference equation*

$$\Psi(z+h) = \left(1 + hA_0 + h\sum_{m=1}^{n} \frac{A_m}{z - z_m}\right)\Psi(z)$$

uniformly in D_k converges to a solution $\hat{\Psi}_k$ of the differential equation

$$\frac{d\widehat{\Psi}}{dz} = \left(A_0 + \sum_{m=1}^{n} \frac{A_m}{z - z_m}\right)\widehat{\Psi}(z).$$

which is holomorphic in D_k;
(B) *the local monodromy matrix $g_k g_{k+1}^{-1}$ converges to the monodromy of $\hat{\Psi}_k$ along the closed path from $z = -i\infty$ and goes around the pole z_k;*
(C) *the upper- and lower-triangular matrices (g_r, g_l) for the cases of real exponents, converge to the Stokes' matrices.*

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