Applications of Differential Algbera to Algebraic Independence of Arithmetic Functions

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CUNY
Kolchin Seminar

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Then

$$\mathrm{td}_C\, C(y_1, \ldots, y_n, z_1, \ldots, z_n) \geq n + \mathrm{rank}_F(Dy_i)$$

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- $\sum \alpha_k g^k$ converges $\iff g^k \to 0 \iff \|g\| < 1 \iff g \in \mathcal{A}_0$.

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$(\mathcal{A}, +, *)$ is a UFD. [Cashwell-Everett (59)]

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2. For each prime p, the p-basic derivation: $\partial_p \longleftrightarrow \dfrac{\partial}{\partial t_p}$. $(\partial_p f)(n) = v_p(np) f(np)$.
3. $\ker \partial_L = \mathbb{C}$.
4. $\partial_p f = 0 \iff p \notin [\operatorname{supp} f]$. Thus $\ker\{\partial_p : p \text{ prime}\} = \mathbb{C}$.

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$$D = \sum_p De_p * \partial_p.$$

Exp and Log

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For any k *and continuous derivations* $D_1, \ldots, D_n$*,*

$$\det(D_j f_i) = 0 \iff \det(D_j \operatorname{Exp}^k f_i) = 0$$

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Theorem (P)

Suppose $\mathcal{C} = \ker_{\mathcal{F}} \Delta$ *for some set of continuous derivations* Δ *of* $\mathcal{A}$ *and* $f_1, \ldots, f_n \in \mathcal{A}$ *such that either*

1. *no non-trivial power product of* $\mathrm{Exp}(f_1), \ldots, \mathrm{Exp}(f_n)$ *is in* $\mathcal{C}$*; or*
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Then

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Proof.

Immediately follows from Ax's Theorem and the previous proposition. □

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Suppose $\mathcal{C} = \ker_{\mathcal{F}} \Delta$ *for some set of continuous derivations* Δ *of* $\mathcal{A}$*. Then for any* $f \in \mathcal{A}_+ \setminus \ker \Delta$*, and* $c_1, \ldots, c_n \in \ker \Delta$, $\operatorname{Log} f$ *is transcendental over* $\mathcal{C}(f, f^{c_1}, \ldots, f^{c_n})$.

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2. 1 and the c_i are $\mathbb{Q}$-linearly independent. Then no nontrivial power product of f, f^{c_i} is in $\mathcal{C}$.

Otherwise, for some $k_i \in \mathbb{Z}$ not all zero,

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Generalizations

Theorem (Shapiro-Sparer)

Suppose $[\operatorname{supp} f] \not\subseteq \bigcup_{i\in I}[\operatorname{supp} g_i]$, *then* $\operatorname{Exp}^*\{f\}$ *is algebraically independent over* $\mathbb{C}[g_i \colon i \in I]$.

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- $\mathcal{T}$ is not alg. closed, e.g. $\mathbf{1}_2 \notin \mathcal{T}$ but its inverse $1 - e_2 \in \mathcal{T}$.

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Let $f_1, \ldots, f_n \in \mathcal{A}$*. Suppose* $D_1, \ldots, D_n$ *are continuous derivations of* $\mathcal{A}$ *such that* $f_i \left(\bigcap_{i<j} \ker D_j\right) \setminus \ker D_i$*. Then* $\mathrm{Exp}^*\{\boldsymbol{f}\}$ *alg. ind. over* $\ker\{\boldsymbol{D}\}$.

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Example: For $f, g \in \mathcal{A} \setminus \mathbb{C}$, with $[\operatorname{supp} f] \neq [\operatorname{supp} g]$, then f, g are alg. ind. over $\mathbb{C}$. A result of Ruengsinsub, Laohakosol, Udomkavanich (05).

Generalizations

Theorem (Komatsu, Laohakosol, Ruengsinsub (11))

For any $f_1, \ldots, f_n \in \mathcal{A}$, if there exits $p_1, \ldots, p_n$ such that

$$\sum_{k_1 \cdots, k_n = m} \left(\prod_{j=1}^{n} v_{p_j}(k_j p_j) \right) \det \left(f_i(k_j p_j) \right) \neq 0$$

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The left side of the equation above is the value of $\det(\partial_{p_j} f_i)$ at m. $\square$

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Corollary

If $\det\left(f_i(p_j)\right) \neq 0$ then $\operatorname{Exp}^*\{\boldsymbol{f}\}$is alg. ind. over $\ker\{\partial_{p_1}, \ldots, \partial_{p_n}\}$.

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- In particular, f needs not be hyper-transcendental even if $\operatorname{supp} f$ is infinite. E.g. $\mathbf{1}_2$ satisfies:

$$\partial_L X = \log(2)(X^2 - X).$$

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$\mathrm{Exp}^*\{\mathbf{1}_p, \mathbf{1}_{\mathbb{P}} : p \in \mathbb{P}\}$. is alg. ind. over $\mathbb{C}$.

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Generalizations

Theorem (P)

Let $f_1, \ldots, f_n \in \mathcal{A}$ and $D_1, \ldots, D_n$ be continuous derivations. For $1 \le j \le n$, suppose

$$m_j \le \min\{v(D_j f_i) : 1 \le i \le n\}$$

and $\det\left(D_j f_i(m_j)\right) \neq 0$ *then* $\mathrm{Exp}^*\{\boldsymbol{f}\}$ *is alg. ind. over* $\ker\{D_j\}$.

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Proposition (P)

Let $f_{ij} \in \mathcal{A}$ $(1 \leq i, j \leq n)$. Suppose a_i, b_i $(1 \leq i \leq n)$ are positive reals such that $a_i b_j \leq v(f_{ij})$ for all $1 \leq i, j \leq n$. Then

$$\det(f_{ij}) \left(\prod_{k=1}^{n} a_k b_k\right) = \det\left(f_{ij}(a_i b_j)\right).$$

Orders

Lemma (Shapiro-Sparer,P)

Suppose $f_1, \ldots, f_n \in \mathcal{A} \setminus \{0\}$ *with* $\det\left(\partial_{p_j} f_i\right) = 0$ *for some choice of* $p_1, \ldots, p_n$. *Then* $\det\left(v_{p_j}(v(f_i))\right) = 0$.

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$$\det\left(\partial_{p_j} f_i\right)\left(\prod_{k=1}^{n} \frac{m_k}{p_k}\right) = \left(\prod_{i=1}^{n} f_i(m_i)\right) \det\left(v_{p_j}(m_i)\right).$$

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Theorem (R-L-U (05))

Suppose $W \subset \mathcal{A} \setminus \{0\}$ with the property that the orders of its members are pairwise relatively prime then W is alg. ind. over $\mathbb{C}$.

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Example

For $N \subset \mathbb{N}$, $\mathrm{Exp}^*\{e_n : n \in N\}$ is alg. ind. over $\mathbb{C}$ if and only if no nontrivial product of elements of N equals 1.

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Suppose $\mathrm{Exp}^*\{\boldsymbol{f}\}$ is alg. dependent over $\mathbb{C}$ for some $f_1, \ldots, f_n \in W$.

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$$0 = \sum_{i=1}^{n} k_i v_p(vf_i) = v_p\left(\prod_{i=1}^{n} (v(f_i))^{k_i}\right)$$

for all p. □

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For $g \in \mathcal{A}$, consider the (continuous) operator $\mathfrak{m}_g$ from $\mathcal{A}$ to itself:

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- E.g. $\mathfrak{m}_{\boldsymbol{I}}$ where $\boldsymbol{I}$ is the identity map of $\mathbb{N}$. More generally, for $\alpha \in \mathbb{C}$, $\mathfrak{m}_{n^\alpha}(f)(n) := n^\alpha f(n)$ is an automorphism of $\mathcal{A}$.

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Theorem (P)

Let $f, g \in \mathcal{A}$. Suppose $p_1, \ldots, p_n \in [\operatorname{supp} f]$ such that $g(v(\partial_{p_j} f)p_j)$ $(1 \leq j \leq n)$ are distinct and nonzero. Then for any $k \geq 0$,

$$\operatorname{Exp}^*\{\mathfrak{m}_g^i f \colon k \leq i \leq k+n-1\}$$

is alg. ind. over $\ker\{\partial_{p_j} \colon 1 \leq j \leq n\}$. Moreover, if g is nowhere vanishing then the same is true for any $k \in \mathbb{Z}$.

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- However, that is not true for e_n when n has at least two distinct prime factors, since e_n satisfies the differential equation:

$$\partial_L X - \log(n) X = 0.$$

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Suppose $f \in \mathcal{A} \setminus \mathcal{S}$ and g is eventually 1-1. Then $E := \mathrm{Exp}^*\{\mathfrak{m}_g^i f \colon i \geq 0\}$ is alg. ind. over $\ker \Delta_I$ for any infinite $I \subseteq [\mathrm{supp}\, f]$ and hence over $\mathcal{S}$. In addition if g is nowhere vanishing, then i can range through $\mathbb{Z}$.

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- Take $g = \log$ and $f = \mathbf{1}$, we conclude that $\mathbf{1}$ is ∂_L-transcendental (better known as hyper-transcendental) over $\mathcal{S}$, i.e. $\zeta(s)$ is transcendental over $\mathbb{C}$ and hence over $\mathbb{C}(s)$ by eliminating s. (Hilbert, Stadigh).

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Differential-Difference transcendence

Theorem (Shapiro-Sparer (85))

For any sequence (α_i) of complex numbers with distinct real parts, ,

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Ideas.

Let the real part of α_i be increasing. Given an infinite $I \subseteq [\operatorname{supp} f]$, choose an increasing sequence $(p_{uv})_{(u,v) \in L}$ from I such that each term is sufficiently larger than the previous term to achieve

$$\det\left((m_{uv} p_{uv})^{\alpha_i} (\log(m_{uv} p_{uv}))^j\right) \neq 0$$

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- The same is true with "algebraic" replaced by "holomorphic" using a theorem of Axel Reich (84). Steuding (08) (Fibonacci), Komatsu (09) (Lucas).

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- However, it is unclear to us that $D|_{\mathcal{A}}$ maps $\mathcal{A}$ into itself.

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- e.g. $\ker \partial_\Omega = \mathbb{C}$ but $e_p / e_q \in \ker_{\mathcal{F}} \partial_\Omega$.

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