A Differential Algebra Approach to Linear Boundary Problems

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   Joint work with G. Regensburger [AMPA09]
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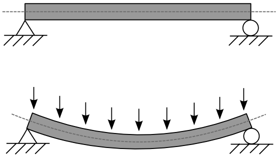
3. **Partial Integro-Differential Operators:**
   Beginnings with G. Regensburger and L. Tec in [CASC09].
   New developement with N. Phisanbut [CASC13]. Ongoing work.

Classical Beam Deflection



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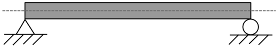
Elastic modulus E, Moment of area I

Normalized horizontal coordinate $x \in [0, 1]$
Deflection $u(x)$, Load $q(x)$



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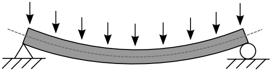

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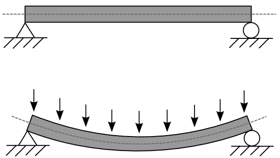

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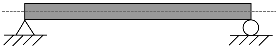

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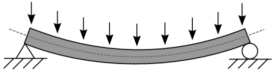

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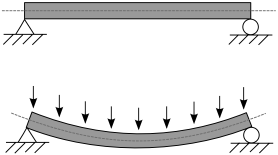

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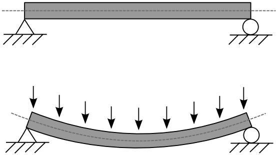

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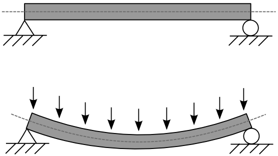

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Classically $u \in C^4[0,1]$.

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Solution for simply supported Euler-Bernoulli beam:

$$g(x,\xi) = \begin{cases} \frac{1}{3}\,x\xi - \frac{1}{6}\,\xi^3 - \frac{1}{2}\,x^2\xi + \frac{1}{6}\,x\xi^3 + \frac{1}{6}\,x^3\xi & \text{für } 0 \le \xi \le x \le 1, \\ \frac{1}{3}\,x\xi - \frac{1}{2}\,x\xi^2 - \frac{1}{6}\,x^3 + \frac{1}{6}\,x\xi^3 + \frac{1}{6}\,x^3\xi & \text{für } 0 \le x \le \xi \le 1 \end{cases}$$

Question: How does **differential algebra** help in finding this solution?

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. . . and the rest is Linear Algebra.

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1. Abstract Boundary Problems
2. Ordinary Integro-Differential Operators
3. Partial Integro-Differential Operators
4. Conclusion

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We write $(T, \mathcal{B})^{-1}$ for G.

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For $(T_1, \mathcal{B}_1)$ and $(T_2, \mathcal{B}_2)$ with $\mathcal{F} \xrightarrow{T_2} \mathcal{G} \xrightarrow{T_1} \mathcal{H}$ define

$$(T_1, \mathcal{B}_1) \cdot (T_2, \mathcal{B}_2) = (T_1 T_2, T_2^*(\mathcal{B}_1) + \mathcal{B}_2),$$

which is again a boundary problem.

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Proposition

The contravariant functor $(T, \mathcal{F}) \mapsto (\ker T, (T, \mathcal{B})^{-1})$ together with its inverse $(\mathcal{S}, G) \mapsto ((\mathcal{S}, G)^{-1}, \operatorname{im}^{\perp} G)$ establishes an isomorphism of categories $\mathbf{BnProb}^* \cong (\mathbf{DuProb}^*)^{\mathrm{op}}$.

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Theorem

Let $(T, \mathcal{B}) \in \mathbf{BnProb}^*$ and $T = T_1 T_2$ a factorization into epimorphisms. Then $(T, \mathcal{B}) = (T_1, \mathcal{B}_1) \cdot (T_2, \mathcal{B}_2)$ is a factorization in $\mathbf{BnProb}^*$ iff

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Incarnations of Boundary Problems

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Lemma

Let $\mathcal{B} \leq \mathcal{F}^*$ be a boundary space with boundary basis $(\beta_i \mid \in I)$. If for any $B_1, B_2 \in \mathcal{B}'$ one has $\overline{B}_1 = \overline{B}_2$ then also $B_1 = B_2$. In particular, the trace f^* of $f \in \mathcal{F}$ depends only on $\overline{f^*} = \beta_i(f)_{i \in I} \in K^I$.

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Usually more realistic to compute P from H:

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Let $(T, \mathcal{B}) \in \mathbf{BnProb}^*$ be given. Then $G = (1 - P)\, T^\Diamond$ and $H = P\mathcal{B}^\Diamond$, hence $F = (1 - P)\, T^\Diamond \oplus P\mathcal{B}^\Diamond$, where P projects onto $\ker T$ along $\mathcal{B}^\perp$.

Usually more realistic to compute P from H:

Proposition

Let $(T, \mathcal{B}) \in \mathbf{BnProb}^*$ be given. Then $\operatorname{trc}|_{\ker T}$ is bijective with state operator H as inverse, and the kernel projector is $P = H \circ \operatorname{trc}$.

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Given a forcing function $f(x) \in C^\infty[a,b]$ and
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Given a forcing function $f(t, x, y) \in C^\omega(\mathbb{R}^3)$ and
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Outline

1. Abstract Boundary Problems
2. Ordinary Integro-Differential Operators
3. Partial Integro-Differential Operators
4. Conclusion

Definition

Let $(\mathcal{F}, \partial)$ be a (noncommutative) differential algebra over a field K. A K-linear operation $\int : \mathcal{F} \to \mathcal{F}$ is called an **integral operator** for ∂ if $\partial \circ \int = 1_{\mathcal{F}}$ and the **differential Rota-Baxter axiom**

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Let $(\mathcal{F}, \partial, \int)$ be an ordinary integro-differential algebra. Then the **ring of integro-differential operators** $\mathcal{F}[\partial, \int]$ is the K-algebra generated by $\{\partial, \int\} \cup \mathcal{F} \cup \mathcal{F}^\bullet$ modulo the Gröbner basis below.

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Stieltjes Conditions versus Two-Point Conditions

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Concrete Boundary Problems for LODEs

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Two-point problems: Normal form of $G \cong$ Green's function $g(x, \xi)$

$$Gf = \int_a^b g(x, \xi)\, f(\xi)\, d\xi$$

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Recall previous two-point problem (taking $a = 0, b = 1$ for simplicity):

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Classical Notation:

$$\begin{aligned} & u''' - (e^x + 2)\, u'' - u' + (e^x + 2)\, u(x) = f \\ & u(0) = u(1) = u'(1) = 0 \end{aligned}$$

Green's Operator:

$$\begin{aligned} G = {} & \left(e^{e^x - x} - e^{e^x}\right) B \left(e^{-e^x} + 2e^{-e} e(x)\right) + \sinh(x)\, B \left(1 + 2e(x)\right) \\ & + \left(2e^{e^x - e}(e^{-x} - 1) - (e - 1)^2\, e^{-x} + 2 \sinh(x)\right) A\, e(x) \end{aligned}$$

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Green's Function: $g(x, \xi) =$

$$= \begin{cases} \left(2e^{e^x - e}(e^{-x} - 1) - (e - 1)^2\, e^{-x} + 2 \sinh(x)\right) e^{2\xi}\, e(\xi) \\ \left(e^{e^x - x} - e^{e^x}\right)\left(e^{-e^\xi} + 2e^{-e} e(\xi)\right) + \sinh(x)\, e^{2\xi} \left(1 + 2e(\xi)\right) \end{cases}$$

$$e(t) := -\tfrac{1}{2}\left(\tfrac{e^t - 1}{e - 1}\right)^2$$

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Outline

1. Abstract Boundary Problems

2. Ordinary Integro-Differential Operators

3. Partial Integro-Differential Operators

4. Conclusion

Basic Example: Smooth Functions

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Similarly use filtered monoid

$$\mathcal{M}(\mathbb{R}) = \bigcup_{n=1}^{\infty} \mathcal{M}_n(\mathbb{R})$$

where $\mathcal{M}_n(\mathbb{R})$ are near-identity matrices with injections $M \hookrightarrow \left(\begin{smallmatrix} I_n & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Action of Integrals and Substitutions

Write $\int^{x_i} \colon \mathcal{F} \to \mathcal{F}$ for Rota-Baxter operator

$$f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots) \mapsto \int_0^{x_i} f(x_1, \ldots, x_{i-1}, \xi, x_{i+1}, \ldots)\, d\xi.$$

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$$g(x_1, x_2, \dots) := f\Big(\textstyle\sum_i M_{1i} x_i, \sum_i M_{2i} x_i, \dots \Big),$$

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Hence note $(MN)^* = N^* M^*$. But what about $\int^{x_i} M^*$?

Notation for Special Matrices

Evaluation at $x_{\boldsymbol{i}}$:

$$E_i = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \leftarrow \boldsymbol{i} \end{pmatrix}$$

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Eliminant for $w \in K^{n-i}$:

$$L_i(w) = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & w_{i+1} & 1 & & \\ & & & \vdots & & \ddots & \\ & & & w_n & & & 1 \end{pmatrix}$$

Substitutive Algebras

Definition

An ascending K-algebra $(\mathcal{F}_n)$ is called **substitutive** if it has a straight contravariant monoid action of $\mathcal{M}(K)$ such that $M^*(\mathcal{F}_n) \subseteq \mathcal{F}_n$ for all $M \in \mathcal{M}_n(K)$ and $E_n^*(\mathcal{F}_n) \subseteq \mathcal{F}_{n-1}$. We write $\mathcal{F} = \varinjlim \mathcal{F}_n$.

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$\longrightarrow$ Complete complemented lattice:

$(\mathcal{F}_\alpha)$ with $\mathcal{F}_\alpha \sqcup \mathcal{F}_\beta = \mathcal{F}_{\alpha \cup \beta}$, $\mathcal{F}_\alpha \sqcap \mathcal{F}_\beta = \mathcal{F}_{\alpha \cap \beta}$

$\mathcal{F}_\emptyset = K$, $\mathcal{F}_{\mathbb{N}} = \mathcal{F}$, $\mathcal{F}'_\alpha = \mathcal{F}_{\mathbb{N} \setminus \alpha}$

Ordinary Rota-Baxter Algebras

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Lemma

Let $(\mathcal{F}, P)$ be an ordinary Rota-Baxter algebra over K. Then $x \mapsto P(1)$ defines an embedding $(K[x], \int_0^x) \hookrightarrow (\mathcal{F}, P)$ of Rota-Baxter algebras.

Definition

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$\rightarrow$ Some subalgebras: $C^\omega(\mathbb{R}^\infty)$, holonomics, $K[x_1, x_2, \ldots]$

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$$\int^x g\, L_x(e_{j-1} + v)^* \int^x = L_j^{-1}(v')^* (I_n \oplus e_j)^* \Big(L_x(e_{j-1})^* \int^x - \int^x L_x(e_{j-1})^* \Big) \int^{x_j} \bar{g}\, L_j(v')^*$$

Crucial example: $C^\infty(\mathbb{R}^\infty)$

→ Some subalgebras: $C^\omega(\mathbb{R}^\infty)$, holonomics, $K[x_1, x_2, \dots]$

→ Exponential polynomials $K[x_1, x_2, \dots, e^{\lambda x_1}, e^{\lambda x_2}, \dots \mid \lambda \in K]$

Verification of the Horizontal and Vertical Rule

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$$\textstyle\int^{x_1} T^* f(x_1, x_2, x_3, \dots) = \int_0^{x_1} f(\xi + x_j, x_2, x_3, \dots)\, d\xi = \int_{x_j}^{x_1 + x_j} f(\bar{\xi}, x_2, x_3, \dots)\, d\bar{\xi}$$

$$= \int_0^{x_1 + x_j} f(\xi, x_2, x_3, \dots)\, d\xi - \int_0^{x_j} f(\xi, x_2, x_3, \dots)\, d\xi$$

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$$L_j(v')^* \textstyle\int^{x_1} g(x_1)\, L_x(e_{j-1} + v)^* \textstyle\int^{x_1} f(x_1, \dots, x_n)$$

$$= L_j(v')^* \int_0^{x_1} g(\eta) \int_0^{\eta} f(\xi, x_{2\dots j-1}, x_j + \eta, x_{j+1\dots n} + v_{j+1\dots n}\eta)\, d\xi\, d\eta$$

$$= L_j(v')^* \int_0^{x_1} \int_{\xi + x_j}^{x_1 + x_j} g(\bar{\eta} - x_j)\, f(\xi, x_{2\dots j-1}, \bar{\eta}, x_{j+1\dots n} + v_{j+1\dots n}(\bar{\eta} - x_j))\, d\bar{\eta}\, d\xi$$

$$= \int_0^{x_1} \int_{\xi + x_j}^{x_1 + x_j} \bar{g}(\eta, x_j)\, f(\xi, x_{2\dots j-1}, \eta, x_{j+1\dots n} + v_{j+1\dots n}\eta)\, d\eta\, d\xi$$

$$= \int_0^{x_1} \int_0^{x_1 + x_j} \dots\, d\eta\, d\xi - \int_0^{x_1} \int_0^{\xi + x_j} \dots\, d\eta\, d\xi$$

Simple Properties

1. For any $\alpha = (\alpha_1, \dots, \alpha_k)$, there is an embedding

$$\iota_\alpha \colon K[X_{\alpha_1}, \dots, X_{\alpha_k}] \hookrightarrow \mathcal{F}_\alpha$$
$$X_{\alpha_j} \mapsto x_{\alpha_j} := \int^{x_{\alpha_j}} 1,$$

and we have $\pi^* p(x_{\alpha_1}, \dots, x_{\alpha_k}) = p(x_{\pi(\alpha_1)}, \dots, x_{\pi(\alpha_k)})$ for all permutations π of $(\alpha_1, \dots, \alpha_k)$.

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Important for applications: $K[x, e^{Kx}] \subset C^\infty(\mathbb{R}^\infty)$

Coalgebra Structure for Coefficients

Can expand every $g \in \mathcal{G}$ as

$$g = \sum_{k=1}^{r} g_{k,1} \cdots g_{k,n}$$

with $g_{k,i} \in \mathcal{G}_{(i)}$ for $i \in \{1, \ldots, n\}$.

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- Similarly, $(i\ j)^* g_{1,(i)'}, (i\ j)^* g_{1,(i)'}, \ldots \in \mathcal{G}_{(j)'}$ written as $g_{(i:j)'}$.

Proposition

Let $(\mathcal{F}_n, \int^{x_n})_{n\in\mathbb{N}}$ be a hierarchical Rota-Baxter algebra over a field K, and let $\mathcal{G}_1$ be an admissible coefficient domain for $\mathcal{F}$. Then for $M \in \mathcal{M}_n$ and $g \in \mathcal{G}_1$ with $g(x_j) := (1\, j)^* g$ and $j \in \{1, \dots, n\}$ we have

$$\int^{x_j} g(x_j) M^* = \begin{cases} M_{ij}^{-1}\, \tilde{g}_{(1:j)'}(1 - E_j^*) \tilde{M}^* \int^{x_i} \hat{M}_{ij}^* \big(\tilde{g}_{(1:i)}\big) L_i(l)^* & \text{if } i \neq \infty, \\ \big(\int^{x_j} g(x_j)\big) M^* & \text{othw.} \end{cases}$$

By definition $i = \min\{k \mid M_{kj} \neq 0\}$, with $\tilde{M} \in \mathcal{M}_n$ and $l \in K_{n-i}$ by one sweep of Gaussian elminiation if the minimum exists, and by convention $i = \infty$ otherwise. Moreover, $\tilde{g} = M_{i\bullet}^* g$ and $\hat{M}_{ij} = d_{i,1/M_{ij}}$.

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$$\int^{x_j} h(x_j) L_j(w)^* \int^{x_i} g(x_i) L_i(v)^* = (1-E_j^*) \int^{x_i} g(x_i) L_i(v')^* \int^{x_j} h(x_j) L_j(w$$

with $v' = L_{j-i}^{-1}(w)\, v \in K^{n-i}$ as earlier.

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$$w_k \int^{x_i} h(x_i)\, L_i(w)^* \int^{x_i} g(x_i)\, L_i(v)^* = L_k^{-1}(w')^* \sigma^*(\bar{h}_{(n+1)}) \times \\ \times \Big(L_i(\bar{w})^* \int^{x_i} \tilde{h}_{(1:k)'} L_i(v')^* - \int^{x_i} \tilde{h}_{(1:k)'} L_i(v' + \bar{w})^*\Big) \int^{x_k} \tilde{h}_{(1:k)} L_k(w')^*$$

where $\bar{h} := (e_k/w_k - e_{n+1}/w_k)^* h = h_{(k)} h_{(n+1)} \in \mathcal{F}_{(k,n+1)}$ with slack transposition $\sigma := (k\ n+1)$, and $\tilde{h} := L_i(v')^*_{k\bullet} (1\ k)^* \bar{h}_{(k)} \in \mathcal{G}_{(i,k)}$.

The remaining notation is as earlier.

The Operator Ring

Definition

Let $(\mathcal{G}_1, \int)$ be an ordinary integro-differential algebra over a field K with induced hiearchy $(\mathcal{G}_n, \int^{x_n})_{n \in \mathbb{N}}$. Then the ring of partial integral operators over $\mathcal{G}$ is defined as the quotient of the K-algebra

$$\mathcal{G}[\textstyle\int] = \mathcal{G} \amalg_K K[\mathcal{M}]^* \amalg_K K[A] \Big/ \cong$$

with the congruence $\cong$ given below.

$M^* g \cong (M \cdot g)\, M^*$	$M^* A_i \cong 0$ if $M_{i\bullet} = 0$
$A_j g(x_i) \cong g(x_i) A_j$	$A_i g(x_j) \cong g(x_j) A_i$
$A_j g(x_j) M^* \cong M_{ij}^{-1}\, \tilde{g}_{(1:j)'} (1 - E_j^*) \tilde{M}^* A_i \hat{M}_{ij}^* \big(\tilde{g}_{(1:i)}\big) L_i(l)^*$	if $i := \min\{k \mid M_{kj} \neq 0\} \neq \infty$
$A_j g(x_j) M^* \cong \big(\int^{x_j} g(x_j)\big) M^*$	if $i := \min\{k \mid M_{kj} \neq 0\} = \infty$
$A_j h(x_j) L_j(w)^* A_i g(x_i) L_i(v)^* \cong (1 - E_j^*)\, A_i g(x_i) L_i(v')^* A_j h(x_j) L_j(w)^*$	
$A_i h(x_i)\, L_i(w)^* A_i g(x_i)\, L_i(v)^* \cong w_k^{-1} L_k^{-1}(w')^* \sigma^*(\bar{h}_{(n+1)}) \times$ $\times \Big(L_i(\bar{w})^*\, A_i \tilde{h}_{(1:k)'} L_i(v')^* - A_i \tilde{h}_{(1:k)'} L_i(v' + \bar{w})^*\Big) A_k \tilde{h}_{(1:k)} L_k(w')^*$	
$A_j g(x_j) A_j \cong \big(\int^{x_j} g(x_j)\big) A_j - A_j \big(\int^{x_j} g(x_j)\big)$	

Natural Action and Termination

Proposition

Let $(\mathcal{F}_n, \int^{x_n})_{n\in\mathbb{N}}$ be a hierarchical Rota-Baxter algebra over a field K, and let $\mathcal{G}_1$ be an admissible coefficient domain for $\mathcal{F}$. Then the natural action $\mathcal{G}[\int] \times \mathcal{F} \to \mathcal{F}$ induced by $g \cdot f = gf$, $M^* \cdot f = M^*(f)$ and $A_i \cdot f = \int^{x_i} f$ is well-defined.

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Now introduce a suitable term order on underlying word monoid.

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Now introduce a suitable term order on underlying word monoid.

Theorem

Let $(\mathcal{G}_1, \int)$ be an ordinary integro-differential algebra over a field K. Orienting the rules of the Table from left to right, one obtains a Noetherian reduction system.

Conjectured Canonical Forms

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- **Volume integrator** is a word of the form $b\, M^* J_1 \cdots J_r$ for line integrators $J_1, \ldots, J_r$ with indices $i_1 < \cdots < i_r$ and $M^* \in \mathcal{M}(K)^*$ with $M_{i_1 \bullet} \neq 0$ if $r > 0$.

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Then we have a system of canonical forms.

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Canonical forms similar but with certain D^α on the right.

LPDE Example Revisited

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Cauchy problem:

$$\boxed{\begin{aligned}&u_{tt} - 4\,u_{tx} + 4\,u_{xx} - 9\,u_{yy} = f,\\ &u(0,x,y) = f_1(x,y), \quad u_t(0,x,y) = f_2(x,y)\end{aligned}}$$

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Signal and state operators:

$$G f\,(t,x,y) = \int_0^t \int_0^\sigma f(\tau, x + 2t - 2\tau, y - 3t - 3\tau + 6\sigma)\,d\tau\,d\sigma.$$

$$H(f_1,f_2) = f_1(x{+}2t, y{-}3t) + \int_0^t (f_2 - 2\,D_x f_1 + 3\,D_y f_1)(x{+}2t, y{-}3t{+}6\tau)\,d\tau$$

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Factor problems:

$$\begin{aligned} &u_t - 2\,u_x \pm 3\,u_y = f, \\ &u(0,x,y) = f^{\pm}(x,y). \end{aligned}$$

$$H^{\pm} f^{\pm}\,(t,x,y) = f^{\pm}(x + 2t, y \mp 3t)$$

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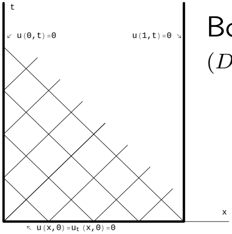

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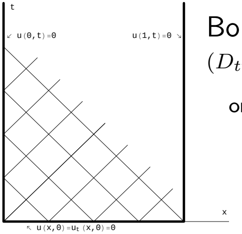

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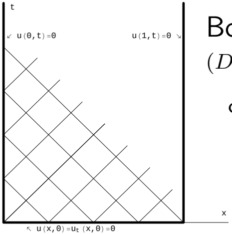

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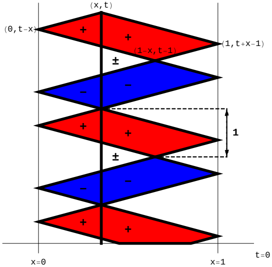

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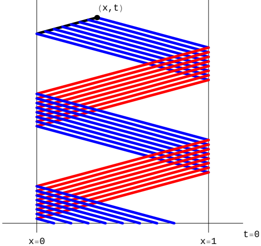

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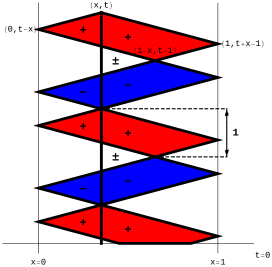

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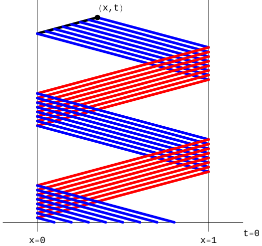

$$G_1 f(x,t) = \int_{(x-t)_+}^{x} f(\xi, \xi - x + t)\, d\xi$$

$$G_2 f(x,t) = \int_x^{x+t} (-1)^{\lfloor \eta \rfloor} f\big(\tfrac{1}{2} + (-1)^{\lfloor \eta \rfloor}(\mathrm{frac}(\eta) - \tfrac{1}{2}), x + t - \eta\big)\, d\eta$$

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THANK YOU

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