A Noncommutative Mikusiński Calculus for Linear Boundary Problems

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Overview

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1. Motivation

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Will start with intuitive treatment.

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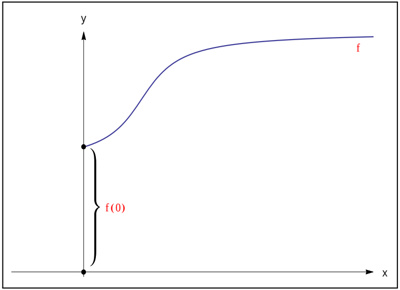

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What is the "normal" meaning of this?!

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Recall Leibniz's integral rule (chain rule $\mathbb{R} \to \mathbb{R}^3 \to \mathbb{R}$):

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Can be interpreted as formal Laplace transform.

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So what is this “formal inverse”?

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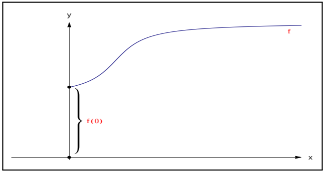

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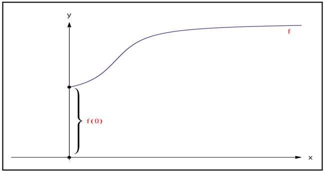

Now write $\mathbb{1} := h \,/\!\!/\, h \in \mathcal{M}$ for the unit and $s := \mathbb{1} \,/\!\!/\, h$ for differentation.

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Characteristic difference: Operational Calculus $\leftrightarrow$ Algebraic Analysis.

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Rather unwieldy, better stay on operator level (action separate)!

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Classical Ring of Fractions

Definition

Let R be a ring, $S \subseteq R$. Then $\varepsilon\colon R \to S^{-1}R$ is a left ring of fractions if

(a) all elements $\varepsilon(s)$ with $s \in S$ are invertible in $S^{-1}R$,

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  Also here $K[x] \subset K(x)$ does not help directly.
- Also $\mathcal{E} = C^\omega(\mathbb{R})$ works.

Suitable Boundary Conditions: Umbral Characters

Definition

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Suitable Boundary Conditions: Umbral Characters

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Lemma (Antiderivative Leibniz Rule)

In any integro-differential algebra $(\mathcal{F}, \partial, \int)$, we have the formula

$$\int f x^n = \sum_{k=0}^{n} (-1)^k n^{\underline{k}} x^{n-k} f^{(-k-1)}$$

for all $f \in \mathcal{F}$. Here we define $f^{(0)} = f$ and $f^{(-k-1)} = \int f^{(-k)}$.

Outline

Relation to Umbral Calculus

Lemma

Let $\beta = \varphi \textstyle\int f$ be a global condition in $\mathcal{F}_\Phi[\partial, \textstyle\int]$. Then $\beta = \varphi \tilde{\beta}$ with

$$\tilde{\beta} = \sum_{k=0}^{\infty} b_k \, \partial^k : \quad K[x] \to K[x]$$

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Shift operator $S_\varphi \colon f(x) \mapsto f(x + \bar{\varphi})$ with $\bar{\varphi} := \varphi(x) \in K$

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Result applies to subalgebras like $C^\omega(\mathbb{R})$ and exponential polynomials.

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  Then $(\varphi \int) e^x = e - 1 \neq 0$ but $(\varphi \int) x^m = \varphi(x^{m+1})/(m+1) = 0$.

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- Use $\det M(x) = V(r)^{s^2}\, \mathrm{sf}(s-1)^r / \mathrm{sf}(n-1)$ and $\bar\varphi_i \neq \bar\varphi_{i'}$.

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- Gram-Schmidt yields Legendre polynomials $e_n \sim \frac{d^n}{dx^n}\,(x^2-1)^n$.

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Lemma

Let φ be a complete character in an integro-differential algebra $(\mathcal{F}, \partial, \smallint)$. Then a nondegenerate global condition $\varphi \smallint f$ never coincides on $K[x]$ with any local condition based on φ.

Regularity Lemma

Lemma

Let $\mathcal{F}$ be an integro-differential algebra and β an umbral Stieltjes condition over $\mathcal{F}$. Then $(\partial^{k+1}, [\mathtt{E}, \dots, \mathtt{E}\partial^{k-1}, \beta])$ is regular for some $k \in \mathbb{N}$.

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- Evaluation matrix $\gamma(u) = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & 0 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & 0 \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \beta(x^k/k!) \end{pmatrix}$.

Outline

The Factorization Theorem for Boundary Problems

Definition

Let $(T, \mathcal{B}) \in \mathcal{E}[\partial] \ltimes K\Phi$ be a an arbitrary boundary problem.

Then $(T_2, \mathcal{B}_2) \in \mathcal{E}[\partial] \ltimes K\Phi$ is called a **subproblem** of $(T, \mathcal{B})$ if T_2 is a right divisor of T and $\mathcal{B}_2 \leq \mathcal{B}$.

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Given $(T, \mathcal{B}) \in \mathcal{E}[\partial]_\Phi$, every factorization $T = T_1 T_2$ of the differential operator lifts to a factorization $(T, \mathcal{B}) = (T_1, \mathcal{B}_1) \cdot (T_2, \mathcal{B}_2)$ of boundary problems such that $(T_1, \mathcal{B}_1), (T_2, \mathcal{B}_2) \in \mathcal{E}[\partial]_\Phi$ and $(T_2, \mathcal{B}_2) \leq (T, \mathcal{B})$.

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Justification by the following Division Lemma.

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Note the left/right asymmetry in the Division Lemma!

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- Then $T_2^\Diamond f_1, \dots, T_2^\Diamond f_m, g_1, \dots, g_n$ is a fundamental system for $T_1 T_2$.

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- Fundamental systems $(f_1, \dots, f_m)$ for T_1 and $(g_1, \dots, g_n)$ for T_2.
- Take K-bases $\beta_1, \dots, \beta_m$ of $\mathcal{B}_1 \leq \mathcal{F}^*$ and $\gamma_1, \dots, \gamma_n$ of $\mathcal{B}_2 \leq \mathcal{F}^*$.
- Then $T_2^\Diamond f_1, \dots, T_2^\Diamond f_m, g_1, \dots, g_n$ is a fundamental system for $T_1 T_2$.
- Conclude $\mathcal{B}_1 T_2 \dotplus \mathcal{B}_2 = [\beta_1 T_2, \dots, \beta_m T_2,\, \gamma_1, \dots, \gamma_n]$.

Converse Closure for Regular Boundary Problems

Closure of $\mathcal{E}[\partial]_\Phi \subset \mathcal{E}[\partial] \ltimes K\Phi$ admits partial converse:

Lemma

Let $(T_1, \mathcal{B}_1), (T_2, \mathcal{B}_2)$ be boundary problems over an integro-differential algebra $\mathcal{F}$ with $\operatorname{ord}(T_1) = \dim \mathcal{B}_1$ and $\operatorname{ord}(T_2) = \dim \mathcal{B}_2$. Then $(T_1, \mathcal{B}_1)$ and $(T_2, \mathcal{B}_2)$ are regular whenever $(T_1, \mathcal{B}_1)(T_2, \mathcal{B}_2)$ is.

<ins>Proof:</ins>

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- Then $T_2^\Diamond f_1, \dots, T_2^\Diamond f_m, g_1, \dots, g_n$ is a fundamental system for $T_1 T_2$.
- Conclude $\mathcal{B}_1 T_2 \dotplus \mathcal{B}_2 = [\beta_1 T_2, \dots, \beta_m T_2,\, \gamma_1, \dots, \gamma_n]$.
- Regularity of $(T_1 T_2, \mathcal{B}_1 T_2 + \mathcal{B}_2)$ yields regular evaluation matrix

$$\begin{pmatrix} (\beta T_2)(T_2^\Diamond f) & (\beta T_2)(g) \\ \gamma(T_2^\Diamond f) & \gamma(g) \end{pmatrix} = \begin{pmatrix} \beta(f) & 0 \\ \gamma(T_2^\Diamond f) & \gamma(g) \end{pmatrix}$$

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- Hence diagonal blocks $\beta(f)$ and $\gamma(g)$ are regular.

Crucial Tool: The Regularization Lemma

Lemma (Regularization)

Let Φ be an umbral character set for an integro-differential algebra $\mathcal{F}$. Then for an **arbitrary** boundary problem $(T, \mathcal{B}) \in \mathcal{E}[\partial] \ltimes K\Phi$ there is a **regular** boundary problem $(S, \mathcal{A}) \in \mathcal{E}[\partial]_\Phi$ that has $(T, \mathcal{B})$ as subproblem.

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- Now $(S, \mathcal{A}) \in \mathcal{E}[\partial]_\Phi$ since $(\tilde{T}, \tilde{\mathcal{B}}), (\tilde{S}, \tilde{\mathcal{A}}) \in \mathcal{E}[\partial]_\Phi$.

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- Since $\mathcal{B}_{k-1} \leq \tilde{\mathcal{A}}$ we have also $(T, \mathcal{B}_k) \leq (S, \mathcal{A})$.

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Left Permutability of Boundary Problems

Lemma

Let Φ be umbral for an integro-differential algebra $\mathcal{F}$ with left extensible coefficient algebra $\mathcal{E}$. Then $\mathcal{E}[\partial]_\Phi$ is a **left permutable** monoid.

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- Then $(T_1, \mathcal{B}_1), (T_2, \mathcal{B}_2) \leq (S, \mathcal{A})$ are regular subproblems.

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- Regular subproblem $(S, \mathcal{A}) \leq (T, \mathcal{B}_1 + \mathcal{B}_2)$ by Regularization Lemma.
- Then $(T_1, \mathcal{B}_1), (T_2, \mathcal{B}_2) \leq (S, \mathcal{A})$ are regular subproblems.
- Division Lemma yields $(\tilde{T}_1, \tilde{\mathcal{B}}_1), (\tilde{T}_2, \tilde{\mathcal{B}}_2) \in \mathcal{E}[\partial]_\Phi$ with

$$(S, \mathcal{A}) = (\tilde{T}_1, \tilde{\mathcal{B}}_1)(T_1, \mathcal{B}_1) = (\tilde{T}_2, \tilde{\mathcal{B}}_2)(T_2, \mathcal{B}_2).$$

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Simplest Example of an Ore Quadruple:

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Lemma

Let Φ be umbral for an integro-differential algebra $\mathcal{F}$ with left extensible coefficient algebra $\mathcal{E}$. Then $\mathcal{E}[\partial]_\Phi$ is a left permutable monoid.

Proof:

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for some right factors $(S, \mathcal{C}_1), (S, \mathcal{C}_2) \in \mathcal{E}[\partial] \ltimes K\Phi$. Then both $(S, \mathcal{C}_1)$ and $(S, \mathcal{C}_2)$ are **singular** whenever $\mathcal{B}_1 \neq \mathcal{B}_2$.

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Conjecture

Let Φ be an umbral character set for an integro-differential algebra $\mathcal{F}$ with left extensible coefficient algebra $\mathcal{E}$. Then we have $\sum_i \lambda_i\,(T_i, \mathcal{B}_i) \in \ker \varepsilon$ iff $\sum_i \lambda_i G_i \in (\Phi)$, where G_i is the Green's operator of $(T_i, \mathcal{B}_i)$.

Outline

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This will justify the terminology of methorious <u>operators</u>.

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Note that $\mathtt{E}_\xi$ is the projector onto $\ker \partial$ along $[\mathtt{E}_\xi]$.

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Recall Mikusiński's construction: $h = \int_0^x \rightsquigarrow s := h^{-1}$.

- Why not start with Green's operators like $h = (\partial, [\mathtt{E}_0])^{-1}$?
  Problem: This is a right but not a left Ore Monoid!
- If we must do $s \rightsquigarrow h$, why not localize $\mathcal{E}[\partial]$ instead of $\mathcal{E}[\partial]_\Phi$?
  Problem: Yields ∂^{-1} as two-sided inverse, losing boundary data!

However, localizing via $K\mathcal{E}[\partial]_\Phi^\star$ brings complications:

- Need action of $K\mathcal{E}[\partial]_\Phi$ on $\mathcal{F}$.
- Hence must extend $\mathcal{F}$ prior to localization, unlike Mikusiński. ☹

Recall Mikusiński's fundamental formula:

$$s_{\xi} f = f' + \mathtt{E}_{\xi}(f)\, \delta_{\xi}$$

Here δ_0 for $h /\!\!/ h$ since tied to $\mathtt{E}_0$. Can generalize to $s_\xi := (\partial, [\mathtt{E}_\xi])$.

Note that $\mathtt{E}_\xi$ is the projector onto $\ker \partial$ along $[\mathtt{E}_\xi]$.
But how to represent carrier objects δ_ξ in $\mathcal{F}_\Phi$?

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But which methorious functions are equal?

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Definition

Let $\mathcal{I} \leq \mathcal{F} \otimes_K K\mathcal{E}[\partial]_\Phi$ generated by $f\,(T, \mathcal{B})$ with $Tf = 0$. Furthermore, let $\mathcal{I}_0$ be the subspace of $\mathcal{I}$ generated by the elements

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Module structure is as intended:

Proposition

Let $(\mathcal{F},\partial,\textstyle\int)$ be an integro-differential algebra with character set Φ. The definitions given above induce a monoid action of $\mathcal{E}[\partial]_\Phi$ on $\mathcal{F}_\Phi$ such that it becomes a $K\mathcal{E}[\partial]_\Phi$-module.

General theory as expected:

Theorem

Let M be a left R-module, and let $S \subseteq R$ be a multiplicative, right permutable and right reversible denominator set $S \subseteq R$. Then there exists a left $S^{-1}R$-module $S^{-1}M$. The kernel of the extension $\mu \colon M \to S^{-1}M$ consists of those $u \in M$ for which there exists an $s \in S$ with $su = 0$.

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Proposition

Let $(\mathcal{F}, \partial, \int)$ be an integro-differential algebra with character set Φ.
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Outline

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\uparrow \varepsilon & & \uparrow \mu \\
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