Software for Symbolic Boundary Problems and Applications in Actuarial Mathematics

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4. **Applications in Actuarial Mathematics**
   Joint work with G. Regensburger and our great actuarial maths collaborators [SIAM12, IME10].

Outline

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Note: Any Noetherian partial order extends to a well-order.

Definition

A word order is a partial order $\leq$ on the monoid $\langle X \rangle$ such that $B < B'$ implies $ABC < AB'C$. A reduction order for $R \subseteq \langle X \rangle \times \mathbf{k}\langle X \rangle$ is a Noetherian word order with $W > f$ for all $(W, f) \in R$.

Clearly rules out infinite descending chains $\implies$ normal forms exist.

Converse valid iff $\mathbf{k}$ is free of zero divisors:

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But a Noetherian word order need not extend to word well-order.

Assume $A = \mathbf{k}\langle X|R\rangle$ has normal forms for $R \subseteq \langle X\rangle \times \mathbf{k}\langle X\rangle$.
When are they unique?

Lemma (Newman 1942)

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If $(\mathbf{k}, +, -, \cdot, /)$ as well as X and R are recursive, then so is $A \cong \mathbf{k}\langle X\rangle_\downarrow$ with operations $f(\bar{a}_1, \dots, \bar{a}_n) := f(a_1, \dots, a_n)\downarrow$.

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- P.M. Cohn 1956: Diamond lemma for $\mathrm{sg}(X|R)$
- Shirshov 1962: Shirshov bases, compositions for Lie polynomials
- Hironaka 1964: Standard bases for $\mathbf{k}[[X]]$
- Buchberger 1965: Gröbner bases, S-polynomials for $\mathbf{k}[X]$
- Robinson 1965: Resolution principle for $L_\Sigma(X)$
- Knuth/Bendix 1970: Term rewriting, critical pairs for $\mathcal{T}_\Sigma(X)$
- Bergman 1978: Diamond lemma for $\mathbf{k}\langle X\rangle$
- Mora 1988: Effective computation of noncommutative Gröbner bases

Outline

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Striking similarities as well as differences between $\mathrm{A}_1(\ell)$ and $\mathrm{A}_1(\partial)$.

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In fact, one has (two-sided) ideals $\left\{ \sum_i a_i(\ell)\, x^i \mid a_i \in (\ell^n) \right\}$ for $n > 0$.

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Unlike in $\mathrm{A}_1(\partial)$ there is also a mid basis in $\mathrm{A}_1(\ell)$, namely $(x^m, x^m \ell x^n)$:

Bases in the Integro Weyl Algebra

As in $\mathrm{A}_1(\partial)$, interchangeable left and right bases:

Fact

We have the identities

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for changing between the left/right and the mid basis.

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The ring of constant-coefficient integro-differential operators is

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Enables 3×3 block scheme for multiplication (see below).

Isomorphisms via Localization and Specialization

Proposition (Localization)

The map $\varphi\colon K\langle\partial,\ell\rangle/(\mathtt{E}) \to K[\partial,\partial^{-1}]$ defined by $\partial + (\mathtt{E}) \mapsto \partial$ and $\ell + (\mathtt{E}) \mapsto \partial^{-1}$ is a differential isomorphism, inducing the isomorphisms

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Close relations to Jacobian algebra $\mathbb{A}_n := A_n\langle (\partial_1 x_1)^{-1}, \dots, (\partial_n x_n)^{-1}\rangle$.

Outline

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\hline
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Proposition (R. 2005; Rosenkranz/R. 2008)

One has $\mathcal{F}[\partial, \int] = \mathcal{F}[\partial] \dotplus \mathcal{F}[\int] \dotplus (\mathcal{F}^\bullet)$, and the evaluation ideal $(\mathcal{F}^\bullet)$ is generated by $|\,\mathcal{F}^\bullet)$ as a left $\mathcal{F}$-module.

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Arithemtic in $\mathcal{F}[\partial, \int]$ is basis of all operations on boundary problems.

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For more information please refer to:

`http://www.risc.jku.at/research/theorema/software/`

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- **Reasoning aspect**: Transport properties from input domain(s) to result domains (e.g. preservation of properties.

Functor Example: Word Monoid

$$\mathbf{Definition}\Big[\text{"Word Monoid"},\ \mathbf{any}[L],$$

$$\mathbf{LexWords}[L] = \mathbf{Functor}\Big[W,\ \mathbf{any}\big[v,\ w,\ \xi,\ \eta,\ \bar{\xi},\ \bar{\eta}\big],$$

$$\frac{\mathbb{s} = \langle\rangle}{\underset{W}{\in}[w] \Longleftrightarrow \bigwedge\begin{cases}\mathbf{is\text{-}tuple}[w] \\ \underset{i=1,..,|w|}{\forall}\ \underset{L}{\in}[w_i]\end{cases}}$$

$$\underset{W}{\square} = \langle\rangle$$

$$v \underset{W}{*} w = v \asymp w$$

$$\left(\langle\eta,\ \bar{\eta}\rangle \underset{W}{>} \langle\rangle\right) \Leftrightarrow \mathbf{True}$$

$$\left(\langle\rangle \underset{W}{>} \langle\bar{\eta}\rangle\right) \Leftrightarrow \mathbf{False}$$

$$\left(\langle\eta,\ \bar{\eta}\rangle \underset{W}{>} \big\langle\xi,\ \bar{\xi}\big\rangle\right) \Leftrightarrow \bigvee\begin{cases}\eta \underset{L}{>} \xi \\ (\eta = \xi)\ \bigwedge \langle\bar{\eta}\rangle \underset{W}{>} \langle\bar{\xi}\rangle\end{cases}$$

$$\Big]\Big]$$

Functor Example: Monoid Algebra

$$\mathbf{MonoidAlgebra}[\mathbf{K},\ \mathbf{W}] = \mathbf{where}\Big[\mathbf{V} = \mathbf{FreeModule}[\mathbf{K},\ \mathbf{W}],$$

$$\mathbf{Functor}\Big[\mathbf{P},\ \mathbf{any}[\mathbf{c},\ \mathbf{d},\ \mathbf{f},\ \mathbf{g},\ \xi,\ \eta,\ \bar{\mathbf{m}},\ \bar{\mathbf{n}}],$$

$$\mathbb{s} = \langle\rangle$$

...(* *linear operations from V* *)

(* multiplication *)

$$\langle\rangle \underset{\mathbf{P}}{*} \mathbf{g} = \langle\rangle$$

$$\mathbf{f} \underset{\mathbf{P}}{*} \langle\rangle = \langle\rangle$$

$$\langle\langle \mathbf{c},\ \xi\rangle,\ \bar{\mathbf{m}}\rangle \underset{\mathbf{P}}{*} \langle\langle \mathbf{d},\ \eta\rangle,\ \bar{\mathbf{n}}\rangle = \Big\langle\Big\langle \mathbf{c} \underset{\mathbf{K}}{*} \mathbf{d},\ \xi \underset{\mathbf{W}}{*} \eta\Big\rangle\Big\rangle \underset{\mathbf{P}}{+} \langle\langle \mathbf{c},\ \xi\rangle\rangle \underset{\mathbf{P}}{*} \langle\bar{\mathbf{n}}\rangle \underset{\mathbf{P}}{+} \langle\bar{\mathbf{m}}\rangle \underset{\mathbf{P}}{*} \langle\langle \mathbf{d},\ \eta\rangle,\ \bar{\mathbf{n}}\rangle \Big]\Big]$$

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Nine cases for multiplications, for example:

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\mathbf{E}\partial^i \cdot f\partial^j &= \textstyle\sum_k (\mathbf{E} \cdot f_k)\mathbf{E}\partial^{j+k}, \\
\textstyle\int b \cdot f\mathbf{E}\partial^i &= (\textstyle\int b \cdot f)\mathbf{E}\partial^i, \\
\mathbf{E}\partial^i \cdot f\textstyle\int b &= \textstyle\sum_k (\mathbf{E} \cdot g_k)\mathbf{E}\partial^l,
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& \int g \int f \partial & \\
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&(\textstyle\int \cdot g)\int f\partial - \int (\int \cdot g)\, f\partial - \int g f + \int g \int f' + \int g\, (\mathtt{E}\cdot g)\, \mathtt{E} \\
&\quad = (\textstyle\int \cdot g) f - (\int \cdot g)\int f' - (\int \cdot g)\, (\mathtt{E}\cdot f)\, \mathtt{E} - (\int \cdot g) f + \int \partial \cdot ((\int \cdot g)\cdot f) \\
&\qquad\quad + (\mathtt{E}\cdot ((\textstyle\int \cdot g)\cdot f)\, \mathtt{E} - \int (g\cdot f) + (\int \cdot g)\int f' - \int (\int \cdot g) f' + (\mathtt{E}\cdot f)(\int \cdot g)\, \mathtt{E} \\
&\quad = \textstyle\int \partial \cdot ((\int \cdot g)\cdot f) + (\mathtt{E}\cdot ((\int \cdot g)\cdot f)\, \mathtt{E} - \int (g\cdot f) - \int (\int \cdot g) f' \\
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- Automated proof with integro-differential polynomial coefficients in $\mathcal{F}\{f, g\}$, hence internalizing integro-differential axioms.

Short Demo of IntDiffOp

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Current version for download at:

http://www.risc.jku.at/~akorpora/

Outline

A Simple Insurance Model

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First scenario (zero interest rate):

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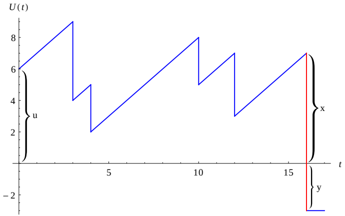

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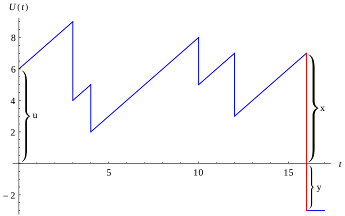

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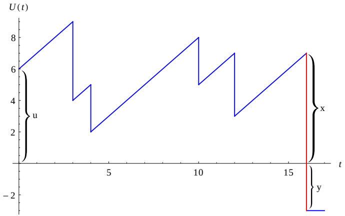

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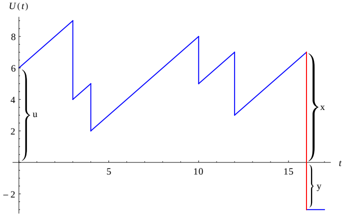

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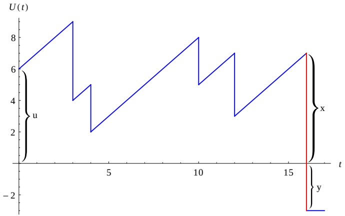

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Then the Gerber-Shiu function is given by

$$m(u) = \mathbb{E}\Big(e^{-\delta T_u}\, w\big(U(T_u-), -U(T_u)\big)\, 1_{T_u<\infty} \mid U(0) = u\Big)$$
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From initial values and net profit condition extract boundary conditions

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$$\boxed{\underbrace{p_X(\partial_u)\, p_\tau^*(c\partial_u - \delta)\, m(u) - a_0 b_0\, m(u)}_{Tm} = \underbrace{a_0\, p_X(\partial_u)\, \omega(u)}_{f}\;.}$$

From initial values and net profit condition extract boundary conditions

$$\boxed{\begin{array}{l} m(0) = M_0, m'(0) = M_1, \ldots, m^{(m-1)}(0) = 0 \\ m(\infty) = 0 \end{array}}$$

Formulation as Boundary Problem

Assuming f_τ and f_X are of phase type:.

- Standard renewal argument (Feller 1971) $\rightarrow$ integral equation
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Here $T = (\partial_u - \rho_1) \cdots (\partial_u - \rho_n)(\partial_u - \sigma_1) \cdots (\partial_u - \sigma_m), \qquad \begin{bmatrix} \mathrm{Re}\,(\rho_i) > 0 \\ \mathrm{Re}\,(\sigma_j) < 0 \end{bmatrix}.$

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Note: We have $m+1$ conditions and not $m+n$ ones!

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Theorem (Albrecher, Constantinescu, Pirsic, Regensburger, Rosenkranz 2010)

If $X \sim E(m)$ and $\tau \sim E(n)$ then

$$m(u) = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}\Big(\Big(\int_0^u e^{\sigma_i(u-\xi)} + \int_u^\infty e^{\rho_j(u-\xi)}\Big) \times f(\xi)\,d\xi - \hat{f}(\rho_j)e^{\sigma_i u}\Big) + m^p(u),$$

where $\hat{f}$ is the Laplace transform and $c_{ij} = c_{ij}(\rho,\sigma) \in \mathbb{R}$.

Special Case and Perturbation

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Theorem ((Albrecher, Constantinescu, Pirsic, Regensburger, Rosenkranz 2010))

Setting $m = 1, n = 2$ one has

$$m(u) = \frac{e^{\sigma u}}{\rho_1 - \rho_2} \left(\frac{\hat{f}(\rho_1)}{\rho_1 - \sigma} - \frac{\hat{f}(\rho_2)}{\rho_2 - \sigma} - \left(\frac{\lambda}{c}\right)^2 (\hat{\omega}(\rho_1) - \hat{\omega}(\rho_2)) \right)$$
$$- \frac{1}{\rho_1 - \rho_2} \int_u^\infty \left(\frac{1}{\rho_1 - \sigma} e^{\rho_1 (u-\xi)} - \frac{1}{\rho_2 - \sigma} e^{\rho_2 (u-\xi)} \right) f(\xi)\, d\xi$$
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$$- \frac{1}{\rho_1 - \rho_2} \int_u^\infty \left(\frac{1}{\rho_1 - \sigma} e^{\rho_1 (u-\xi)} - \frac{1}{\rho_2 - \sigma} e^{\rho_2 (u-\xi)} \right) f(\xi)\, d\xi$$
$$+ \frac{1}{\rho_1 - \sigma} \frac{1}{\rho_2 - \sigma} \int_0^u e^{\sigma(u-\xi)} f(\xi)\, d\xi,$$

Adding Browning motion (double order) for $m = n = 1$ yields

$$m(u) = -\frac{1}{(\rho - \sigma_1)(\rho - \sigma_2)} \int_u^\infty e^{\rho(u-\xi)} f(\xi)\, d\xi - \frac{\hat{f}(\rho)}{\sigma_2 - \sigma_1} \left(\frac{e^{\sigma_1 u}}{\rho - \sigma_1} - \frac{e^{\sigma_2 u}}{\rho - \sigma_2} \right)$$
$$+ \frac{1}{\sigma_2 - \sigma_1} \int_0^u \left(\frac{e^{\sigma_1 (u-\xi)}}{\rho - \sigma_1} - \frac{e^{\sigma_2 (u-\xi)}}{\rho - \sigma_2} \right) f(\xi)\, d\xi$$
$$+ \frac{1}{\sigma_2 - \sigma_1} \Big([\sigma_2 m(0) - m'(0)] e^{\sigma_1 u} + [-\sigma_1 m(0) + m'(0)] e^{\sigma_2 u} \Big),$$

generalizing the representations of [Chen et al.'07] and [Li-Garrido'05].

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- Let T have the fundamental system $s_1, \ldots, s_m, r_1, \ldots, r_n$.
- Then we call $s_i(u)$ **stable** if $s_i(\infty) = 0$.
- Likewise we call $r_j(u)$ **instable** if $r_j(\infty) = \infty$.

Factoring Stable and Instable Green's Operators

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Lemma

The Gerber-Shiu function is $m(u) = c_1 s_1(u) + \cdots + c_m s_m(u) + Gg\,(u)$ with $G = G_s G_r$ and $G_s = A_{s_1} \cdots A_{s_m}$, $G_r = (-1)^n B_{r_1} \cdots B_{r_n}$ where

$$A_{t_i} = \tfrac{\omega(i)}{\omega(i-1)}\, A\, \tfrac{\omega(i-1)}{\omega(i)} \qquad \text{for } 1 \leq i \leq m,$$

$$B_{t_j} = B_{r_{j-m}} = \tfrac{\omega(j)}{\omega(j-1)}\, B\, \tfrac{\omega(j-1)}{\omega(j)} \qquad \text{for } m+1 \leq j \leq m+n,$$

setting $\omega(0) = 1$ for convenience.

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Proposition

The above Green's operator can be decomposed as

$$G = \sum_{i=1}^{m+n} t_i\, C_i\, \tfrac{d_i(m+n)}{\omega(m+n)} - \sum_{j=1}^{n} \tilde{a}_j\, F\, \tfrac{d_{m+j}(m+n)}{\omega(m+n)},$$

where C_i is $\int_0^x$ for $1 \leq i \leq m$ and $\int_\infty^x$ for $1 \leq i - m \leq n$, with $F := \int_0^\infty$ and certain constants $\tilde{a}_j$. Here ω is the Wronskian with minors d_i.

General Representation of Solution

Theorem (Albrecher, Constantinescu, Palmowski, Regensburger, Rosenkranz)

For $\tau \sim E(1/\lambda)$ and $X \sim E(\mu)$ we have

$$m(u) = \gamma\, s(u)\Big(- s(u) \int_0^u \tfrac{r(v)}{w(v)} - r(u) \int_u^\infty \tfrac{s(v)}{w(v)} + \tfrac{r(0)}{s(0)} s(u) \int_0^\infty \tfrac{s(v)}{w(v)} \Big) f(v)\, dv,$$

where $\gamma = \big(\lambda\, \omega(0) + c\, \frac{r(0)s'(0) - r'(0)s(0)}{s(0)} \int_0^\infty \frac{s(v)}{w(v)}\, f(v)\, dv\big) / \big((\lambda + \delta)\, s(0) - c\, s'(0)\big) \in \mathbb{R}$.

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Corollary (Albrecher, Constantinescu, Palmowski, Regensburger, Rosenkranz)

In the special case $\delta = 0$ one gets

$$m(u) = \frac{\lambda\omega(0) - p(0)\frac{s'(0)}{s(0)} \int_0^\infty \frac{s(v)}{s'(v)}\, f(v)\, dv}{\lambda s(0) - p(0) s'(0)}\, s(u)$$
$$+ \Big(s(u) \int_0^u \tfrac{1}{s'(v)} + \int_u^\infty \tfrac{s(v)}{s'(v)} - \tfrac{s(u)}{s(0)} \int_0^\infty \tfrac{s(v)}{s'(v)} \Big) f(v)\, dv$$

where γ is already included.

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- Rational premium $p(u) = c + 1/(1 + u)$: Up to quadratures (exp).
- Quadratic premium $p(u) = c + u^2$: Up to quadratures (exp, arctan).

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Homogeneous solutions t_1, t_2:

- $t_i(u) \sim \exp\big(\int_0^u (\varrho_i(t) + \tilde{\varrho}_i(t))\,dt\big)$ where

$$2\varrho_{1,2} = -\left(\mu + \frac{p'(u)}{p(u)} - \frac{\lambda+\delta}{p(u)}\right) \pm \sqrt{\left(\mu + \frac{p'(u)}{p(u)} - \frac{\lambda+\delta}{p(u)}\right)^2 + 4\frac{\delta\mu}{p(u)}}$$

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- If $p(\infty)$ explodes polynomially then $m(u) \sim (\dots)\, s(u) + K_2 u\, f(u)$

Future Work in Actuarial Mathematics

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THANK YOU

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