Singular Boundary Problems and Generalized Green's Operators

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Kolchin Seminar in Differential Algebra
8 July 2014

We acknowledge support from EPSRC First Grant EP/I037474/1.

Overview

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This talk is fully based on A. Korporal's PhD thesis [Korporal2012].

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2. Generalized Boundary Problems

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3. Multiplying Generalized Boundary Problems

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Outline

Generalized Inverses for Linear Maps

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They always exist

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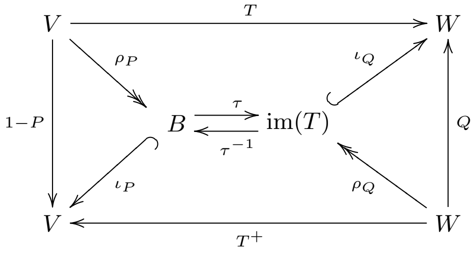

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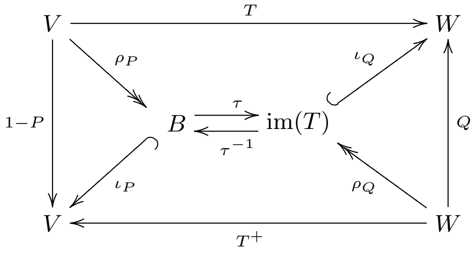

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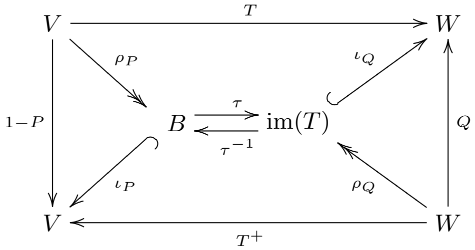

Bijection $(P, Q) \leftrightarrow T^+(P, Q)$. Special case: Moore-Penrose inverse $T^\dagger$.

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- Outer inverse again via $T^{\dashv}(P, Q) := T|_B^{-1} Q \leftrightarrow (P, Q)$.
  Also written as $T^{\dashv}(B, E)$.

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$$\boxed{A \colon E \to V \quad \text{arbitrary}} \qquad \text{as an extension of } \mathrm{im}(T) \xrightarrow{\sim} B.$$

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- Inner inverse given by $T_A^{\vdash}(P,Q) := T|_B^{-1} \oplus A$. Also $T_A^{\vdash}(B,E)$.
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- Expands T^+ by using A on E, hence $\ker(G) \leq E$ and $B \leq \mathrm{im}(G)$.
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- Regain T^+ for trivial extension $A = 0$.

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- We need **fewer constraints—fewer exceptions**.
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Note that G is an outer inverse of T iff T is an inner inverse of G.

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- Expands T^+ by using A on E, hence $\ker(G) \le E$ and $B \le \mathrm{im}(G)$.
- We need fewer constraints—fewer exceptions.
- Regain T^+ for trivial extension $A = 0$.

Note that G is an outer inverse of T iff T is an inner inverse of G.

In the sequel we shall only use <u>outer inverses</u>.

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As we shall see, generalized Green's operators for LODEs are Fredholm.

Proposition

Let $F \leq V$ be generated by $u_1, \dots, u_m$ and $B \leq V^*$ by $\beta_1, \dots, \beta_n$. Choosing a basis $k_1, \dots, k_r \in K^m$ for $\ker(\beta(u))$ and $\kappa_1, \dots, \kappa_s \in F^n$ for $\ker(\beta(u)^\top)$, the intersections $F \cap B^\perp$ and $F^\perp \cap B$ are generated by

$$k_1 \cdot (u_1, \dots, u_m), \quad \dots, \quad k_r \cdot (u_1, \dots, u_m)$$

and

$$\kappa_1 \cdot (\beta_1, \dots, \beta_n), \quad \dots, \quad \kappa_s \cdot (\beta_1, \dots, \beta_n),$$

respectively.

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$$\beta(u) = \begin{pmatrix} \beta_1(u_1) & \cdots & \beta_1(u_m) \\ \vdots & \ddots & \vdots \\ \beta_n(u_1) & \cdots & \beta_n(u_m) \end{pmatrix}$$

for $u = (u_1, \ldots, u_m) \in V^m$ and $\beta = (\beta_1, \ldots, \beta_n) \in (V^*)^n$.

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Lemma

Let $F \leq V$ have basis $u_1, \ldots, u_m$ and $B \leq V^*$ have basis $\beta_1, \ldots, \beta_n$. Then we have $F \cap B^\perp = O$ iff $\beta(u)$ has full column rank.

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Proposition

Let $U \leq V$ and $B \leq V^*$ be such that $U \cap B^{\perp} = O$. For arbitrary bases $u_1, \ldots, u_m$ of U and $\beta_1, \ldots, \beta_n$ of B let $\beta(u)^{\#}$ be any left inverse of $\beta(u)$ and define $(\tilde{\beta}_1, \ldots, \tilde{\beta}_m) = \beta(u)^{\#}(\beta_1, \ldots, \beta_n)$. Then

$$Pu = \sum_{i=1}^{m} \tilde{\beta}_i(u)\, u_i$$

is a **projector** onto U along $[\tilde{\beta}_1, \ldots, \tilde{\beta}_m]^{\perp} \geq B^{\perp}$.

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In particular, if $V = U \dotplus B^\perp$ then P projects along $[\tilde{\beta}_1, \ldots, \tilde{\beta}_m]^\perp = B^\perp$.

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for subspaces $U \le V$, $Z \le W$, $C \le W^*$ and orthogonally closed $B \le V^*$.

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Outline

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Predicts number of conditions, for example $\dim \mathcal{C} = 3 - 2$ above.

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In the sequel, restrict ourselves to semi-regular problems.

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Recall multiplication of plain boundary problems:

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Product of Generalized Boundary Problems

Definition

Let $(T_1, \mathcal{B}_1, E_1)$ and $(T_2, \mathcal{B}_2, E_2)$ be generalized problems. Then

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Now for the **reverse order law**:

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Let $(T_1, \mathcal{B}_1, E_1)$ and $(T_2, \mathcal{B}_2, E_2)$ be regular with $G_1 = (T_1, \mathcal{B}_1, E_1)^{-1}$ and $G_2 = (T_2, \mathcal{B}_2, E_2)^{-1}$. **If** $G_2 G_1$ is an outer inverse of $T_1 T_2$ **then**

$$\Big((T_1, \mathcal{B}_1, E_1)(T_2, \mathcal{B}_2, E_2)\Big)^{-1} = (T_2, \mathcal{B}_2, E_2)^{-1}\,(T_1, \mathcal{B}_1, E_1)^{-1},$$

with direct sums $\mathcal{B}_2 \dotplus T_2^*(\mathcal{B}_1 \cap E_2^\perp)$ and $E_1 \dotplus T_1(\mathcal{B}_1^\perp \cap E_2)$.

Ensuring the Reverse Order Law

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Theorem (Good Exceptions)

Let $(T_1, \mathcal{B}_1)$ and $(T_2, \mathcal{B}_2)$ be semi-regular with $T_1 \in L(V, W)$ and $T_2 \in L(U, V)$. Then there exists an exceptional space $E_2 \leq V$ for $(T_2, \mathcal{B}_2)$ such that the reverse order law holds with all possible exceptional spaces $E_1 \leq W$ for $(T_1, \mathcal{B}_1)$.

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Constructive Proof.

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Constructive Proof.

Proposition (Sufficient Conditions)

Let $(T_1, \mathcal{B}_1, E_1)$ and $(T_2, \mathcal{B}_2, E_2)$ be regular with $G_1 = (T_1, \mathcal{B}_1, E_1)^{-1}$ and $G_2 = (T_2, \mathcal{B}_2, E_2)^{-1}$. Then $G_2 G_1$ is the Green's operator of the product if one of the following five conditions hold:

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Corollary

For regular boundary problems $(T_1, \mathcal{B}_1, \mathcal{C}_1)$ and $(T_2, \mathcal{B}_2)$ the reverse order law always holds, and the product is given by $(T_1 T_2, \mathcal{B}_2 + T_2^*(\mathcal{B}_1), E_1)$.

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$(T, \mathcal{B}, E) = (\partial^2, [\mathtt{E}_0, \mathtt{E}_0\partial, \mathtt{E}_1\partial], [e(x)])$

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In general, one needs necessary & sufficient conditions similar to earlier ones.

Necessary and Sufficient Conditions

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Let $(T_1, \mathcal{B}_1, E_1)$ and $(T_2, \mathcal{B}_2, E_2)$ be regular boundary problems with Green's operators $G_1 = (T_1, \mathcal{B}_1, E_1)^{-1}$ and $G_2 = (T_2, \mathcal{B}_2, E_2)^{-1}$. Then the following are equivalent:

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Result algorithmic for Fredholm operators ($\rightarrow$ outer inverses Fredholm).

Outline

Theorem (Regensburger/R. 2009)

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and $\mathcal{B}_2 \leq \mathcal{B}$ is orthogonally closed such that $\mathcal{B} = (\mathcal{B} \cap K_2^\perp) \dotplus \mathcal{B}_2$.
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Now try to lift $T = T_1 T_2$ to generalized boundary problems:

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As to be expected, new complications arise:

- Possible loss of information on $\mathcal{B}_1$ and E_2 in intersections.
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▷ We consider important special cases.

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In both cases, reverse order law applies.

Factorizations with Plain Right Factor

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Lemma

Let $(T, \mathcal{B})$ be semi-regular with factorization $T = T_1 T_2$ into epimorphisms. Then there exists a **regular core** $(T_2, \mathcal{B}_2) \leq (T_2, \mathcal{B})$.

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Theorem

Let $(T, \mathcal{B}, E)$ be regular with factorization $T = T_1T_2$ into epimorphisms. Then there exists a unique regular boundary problem $(T_1, \mathcal{B}_1, E)$ such that for each $\mathcal{B}_2 \leq \mathcal{B}$ with $(T_2, \mathcal{B}_2)$ regular we have

$$(T, \mathcal{B}, E) = (T_1, \mathcal{B}_1, E)(T_2, \mathcal{B}_2) = (T_1T_2, \mathcal{B}_2 \dotplus T_2^*(\mathcal{B}_1), E)$$

and $(T, \mathcal{B}, E)^{-1} = (T_2, \mathcal{B}_2)^{-1}(T_1, \mathcal{B}_1, E)^{-1}$.

Factorizations with Plain Right Factor

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and $(T, \mathcal{B}, E)^{-1} = (T_2, \mathcal{B}_2)^{-1}(T_1, \mathcal{B}_1, E)^{-1}$.

Moreover, the left boundary conditions are $\mathcal{B}_1 = H_2^*(\mathcal{B} \cap \ker(T)^\perp)$ for an arbitrary right inverse H_2 of T_2.

Outline

1. Outer and Inner Inverses
2. Generalized Boundary Problems
3. Multiplying Generalized Boundary Problems
4. Factorization of Generalized Boundary Problems
5. Algorithm Library

Algorithm 6.1 (Intersection 1).

Input Generating sets $f_1,\dots,f_s \in \mathscr{F}$ and $\beta_1,\dots,\beta_r \in |\Phi)$ of $\mathscr{F}_1 \leq \mathscr{F}$ and $\mathscr{B}_1 \leq |\Phi)$.

Output A generating set of $\mathscr{F}_1 \cap \mathscr{B}_1^\perp$.

1. Compute the evaluation matrix $M = \beta(f) \in F^{r \times s}$.
2. Compute a basis $v^1,\dots,v^k \in F^s$ of $\operatorname{Ker} M$.
3. For $1 \leq i \leq k$ set $g_i = \sum_{j=1}^s v_j^i f_j$.
4. Return $g_1,\dots,g_k$.

Algorithm 6.2

Algorithm 6.2 (Intersection 2).

Input Generating sets $f_1,\dots,f_s \in \mathscr{F}$ and $\beta_1,\dots,\beta_r \in |\Phi)$ of $\mathscr{F}_1 \le \mathscr{F}$ and $\mathscr{B}_1 \le |\Phi)$.

Output A generating set of $\mathscr{F}_1^\perp \cap \mathscr{B}_1$.

1. Compute the evaluation matrix $M = \beta(f) \in F^{r\times s}$.
2. Compute a basis $v^1,\dots,v^k \in \mathscr{F}^r$ of $\operatorname{Ker} M^t$.
3. For $1 \le i \le k$ set $\alpha_i = \sum_{j=1}^r v_j^i \alpha_j$.
4. Return $\alpha_1,\dots,\alpha_k$.

Algorithm 6.3

Algorithm 6.3 (Direct Sum Test)**.**

Input Bases $f_1,\dots,f_r \in \mathscr{F}$ and $\beta_1,\dots,\beta_r \in |\Phi)$ of $\mathscr{F}_1 \leq \mathscr{F}$ and $\mathscr{B}_1 \leq |\Phi)$.

Output **true** if $\mathscr{F}_1 \dotplus \mathscr{B}_1^{\perp} = \mathscr{F}$ and **false** otherwise.

1. Compute the evaluation matrix $M = \beta(f) \in F^{r\times r}$.
2. Compute the determinant $d = \det M$ of the evaluation matrix.
3. If $d \neq 0$, return **true**, else return **false**.

Algorithm 6.4 (Projector).

Input Bases $f_1,\dots,f_s \in \mathscr{F}$ and $\beta_1,\dots,\beta_t \in |\Phi)$ of $\mathscr{F}_1 \le \mathscr{F}$ and $\mathscr{B}_1 \le |\Phi)$ such that $\mathscr{F}_1 \cap \mathscr{B}_1^\perp = \{0\}$.

Output A projector P with $\operatorname{Im} P = \operatorname{span}(f_1,\dots,f_s)$ and $\operatorname{Ker} P \le \operatorname{span}(\beta_1,\dots,\beta_t)^\perp$.

1. Compute the evaluation matrix $M = \beta(f) \in F^{t\times s}$.
2. Compute a left inverse $M^- = (m_{i,j})_{1\le i\le s, 1\le j\le t}$ of the evaluation matrix.
3. For $1 \le i \le s$ set $\alpha_i = \sum_{k=1}^t m_{i,k}\beta_k$.
4. Set $P = \sum_{j=1}^s f_j \alpha_j$.
5. Return P.

Algorithm 6.6 (Fundamental Right Inverse)**.**

Input A regular fundamental system $s_1, \ldots, s_n$ of a monic differential operator T.

Output The fundamental right inverse $T^\blacklozenge$ of T.

1. Compute the Wronskian matrix W and $d = \det W$.
2. For $1 \le i \le n$ compute $d_i = \det W_i$ for W_i as in Proposition 6.5.
3. Compute $T^\blacklozenge = \sum_{i=1}^{n} s_i \int d^{-1} d_i \in \mathcal{F}[\partial, \int]$.
4. Return $T^\blacklozenge$.

Algorithm 6.7 (Compatibility Conditions).

Input A boundary problem $(T, \mathscr{B})$ where $\beta_1, \ldots, \beta_n$ is a basis of $\mathscr{B}$. A regular fundamental system $s_1, \ldots, s_m$ of T.

Output A basis of the space of compatibility conditions $\mathscr{C} = T(\mathscr{B}^\perp)^\perp$.

1. Compute the fundamental right inverse $T^\blacklozenge$ of T with Algorithm 6.6.
2. Compute a basis $\alpha_1, \ldots, \alpha_r$ of $(\operatorname{span}(s_1, \ldots, s_m))^\perp \cap \mathscr{B}$ with Algorithm 6.2.
3. For $1 \le i \le r$ multiply $\gamma_i = \alpha_i T^\blacklozenge \in \mathscr{F}_\Phi\langle \partial, \textstyle\int \rangle$.
4. Return $\gamma_1, \ldots, \gamma_r$.

Algorithm 6.8

Algorithm 6.8 (Compatibility Conditions 2).

Input A semi-regular boundary problem $(T, \mathscr{B})$ where $\beta_1, \ldots, \beta_n$ is a basis of $\mathscr{B}$. A regular fundamental system $s_1, \ldots, s_m$ of T.

Output A basis of the space of compatibility conditions $T(\mathscr{B}^\perp)^\perp$.

1. Compute the fundamental right inverse $T^\blacklozenge$ of T (Algorithm 6.6).
2. Compute a projector P with $\operatorname{Im} P = S$ and $\mathscr{B}^\perp \leq \operatorname{Ker} P$ with Algorithm 6.4.
3. Multiply $G = (\mathbf{1} - P)T^\blacklozenge \in \mathscr{F}_\Phi\langle \partial, \textstyle\int \rangle$.
4. For $1 \leq i \leq n$ multiply $\gamma_i = \beta_i G \in \mathscr{F}_\Phi\langle \partial, \textstyle\int \rangle$.
5. Return a basis of $\operatorname{span}(\gamma_1, \ldots, \gamma_r)$.

Algorithm 6.9 (Regularity Test)**.**

Input A boundary problem $(T,\mathscr{B},E)$ where $\beta_1,\dots,\beta_n$ is a basis of $\mathscr{B}$ and $e_1,\dots,e_r$ is a basis of E. A regular fundamental system $s_1,\dots,s_m$ of T.

Output **true** if the problem is regular and **false** otherwise.

1. If $E=\{0\}$ and $m\neq n$: Return **false**.
2. If $E=\{0\}$ and $m=n$: Test if $\mathscr{B}^\perp \dotplus \operatorname{span}(s_1,\dots,s_n)=\mathscr{F}$ with Algorithm 6.3.
3. If $E\neq\{0\}$ and $m\geq n$: Return **false**.
4. If $E\neq\{0\}$ and $m<n$: Compute the evaluation matrix $M=\beta(s)\in F^{n\times m}$.

Algorithm 6.9b

(a) If $\operatorname{rank} M < m$: return false.

(b) If $\operatorname{rank} M = m$:

   i. Compute the compatibility conditions $\gamma_1, \ldots, \gamma_r$ of $(T, \mathscr{B})$ with Algorithm 6.7.

   ii. Test if $E \dotplus \operatorname{span}(\gamma_1, \ldots, \gamma_r)^\perp = \mathscr{F}$ with Algorithm 6.3.

Algorithm 6.10 (Generalized Green's Operator).

Input A regular boundary problem $(T, \mathscr{B}, E)$ where $\beta_1, \ldots, \beta_n$ is a basis of $\mathscr{B}$ and $e_1, \ldots, e_r$ is a basis of E. A regular fundamental system $s_1, \ldots, s_m$ of T.

Output The generalized Green's operator G with $\operatorname{Im} G = \mathscr{B}^\perp$ and $\operatorname{Ker} G = E$.

1. Compute the fundamental right inverse $T^\blacklozenge$ of T (Algorithm 6.6).
2. Compute a projector P with $\operatorname{Im} P = S$ and $\mathscr{B}^\perp \leq \operatorname{Ker} P$ (Algorithm 6.4).
3. Multiply $\tilde{G} = (\mathbf{1} - P)T^\blacklozenge \in \mathscr{F}_\Phi\langle\partial, \textstyle\int\rangle$.
4. If $E = \{0\}$ return $\tilde{G}$, else
5. Compute the compatibility conditions $\mathscr{C}$ of $(T, \mathscr{B})$ with Algorithm 6.8.
6. Compute the projector Q with $\operatorname{Im} Q = E$ and $\mathscr{C}^\perp \leq \operatorname{Ker} Q$ (Algorithm 6.4).
7. Multiply $G = \tilde{G}(\mathbf{1} - Q) \in \mathscr{F}_\Phi\langle\partial, \textstyle\int\rangle$.
8. Return G.

Algorithm 6.11 (Inhomogeneous Boundary Conditions)**.**

Input A regular boundary problem $(T, \mathscr{B})$ where $\beta_1, \ldots, \beta_n$ is a basis of $\mathscr{B}$, boundary values $c_1, \ldots, c_n$, and a forcing function f. A regular fundamental system $s_1, \ldots s_m$ of T.

Output The solution of the boundary problem (6.3).

1. Compute the evaluation matrix $M = \beta(s) \in F^{n \times n}$.
2. Compute the Green's operator G of $(T, \mathscr{B})$ with Algorithm 6.10.
3. Compute the solution $\lambda \in F^n$ of the linear system $M\lambda = (c_1, \ldots, c_n)^t$.
4. Set $k = \sum_{i=1}^{n} \lambda_i s_i$.
5. Compute the application of the Green's operator $u = G(f) \in \mathscr{F}$.
6. Return $u + k$.

Algorithm 6.12

Algorithm 6.12 **($T^{-1}(E)$).**

Input A monic differential operator T and a subspace $E \leq \mathscr{F}$, where $e_1, \ldots, e_r$ is a basis of E. A regular fundamental system $s_1, \ldots s_m$ of T.

Output A basis of $T^{-1}(E)$.

1. Compute the fundamental right inverse $T^{\blacklozenge}$ with Algorithm 6.6.
2. For $1 \leq i \leq r$ compute $k_i = T^{\blacklozenge}(e_i)$.
3. Return $s_1, \ldots s_m, k_1, \ldots, k_r$.

Algorithm 6.13

Algorithm 6.13 (Check Reverse Order Law).

Input Two regular boundary problems $(T_1, \mathscr{B}_1, E_1)$ and $(T_2, \mathscr{B}_2, E_2)$, where $\beta_1, \dots, \beta_n$ and $\tilde{\beta}_1, \dots, \tilde{\beta}_\nu$ are bases for $\mathscr{B}_1$ and $\mathscr{B}_2$ and $e_1, \dots, e_t$ and $\tilde{e}_1, \dots, \tilde{e}_\tau$ are bases for E_1 and E_2. Fundamental systems $s_1, \dots, s_m$ of T_1 and $\tilde{s}_1, \dots, \tilde{s}_\ell$ of T_2.

Output **true** if $(T_2, \mathscr{B}_2, E_2)^{-1}(T_1, \mathscr{B}_1, E_1)^{-1} = ((T_1, \mathscr{B}_1, E_1) \circ (T_2, \mathscr{B}_2, E_2))^{-1}$ and **false** otherwise.

1. Compute $T_1^{-1}(E_1)$ with Algorithm 6.6.
2. Compute a basis of $I = E_2 \cap T_1^{-1}(E_1)$.
3. Compute a basis of $B = \mathscr{B}_1 \cap I^\perp$ with Algorithm 6.2.
4. Compute a basis of $K = \mathscr{B}_1 \cap E_2^\perp$ with Algorithm 6.2.
5. Compute the compatibility conditions $\gamma_1, \dots, \gamma_r$ of $(T_2, \mathscr{B}_2)$ with Algorithm 6.7.
6. Compute $C = \operatorname{span}(\gamma_1, \dots, \gamma_r) + K$.
7. If $B \leq C$ return **true**, else return **false**.

Algorithm 6.14

Algorithm 6.14 (Composition).

Input Two boundary problems $(T_1, \mathscr{B}_1, E_1)$ and $(T_2, \mathscr{B}_2, E_2)$, where $\beta_1, \ldots, \beta_n$ and $\tilde{\beta}_1, \ldots, \tilde{\beta}_\nu$ are bases of $\mathscr{B}_1$ and $\mathscr{B}_2$ and $e_1, \ldots, e_t$ and $\tilde{e}_1, \ldots, \tilde{e}_\tau$ are bases of E_1 and E_2.

Output The composite boundary problem $(T_1, \mathscr{B}_1, E_1) \circ (T_2, \mathscr{B}_2, E_2)$.

1. Multiply $T = T_1 T_2 \in \mathscr{F}_\Phi\langle \partial, \textstyle\int \rangle$.
2. Compute a basis $b_1, \ldots, b_k$ of $\mathscr{B}_1 \cap E_2^\perp$ with Algorithm 6.2.
3. Compute a basis $v_1, \ldots, v_\ell$ of $I = \mathscr{B}_1^\perp \cap E_2$ with Algorithm 6.1.
4. For $1 \le i \le k$ multiply $c_i = b_i T_2 \in \mathscr{F}_\Phi\langle \partial, \textstyle\int \rangle$.
5. For $1 \le j \le \ell$ compute the application $t_j = T_1(v_j) \in \mathscr{F}$.
6. Compute a basis $\alpha_1, \ldots, \alpha_q$ of $\tilde{\beta}_1, \ldots, \tilde{\beta}_t, c_1, \ldots, c_k$.
7. Compute a basis $f_1, \ldots, f_r$ of $e_1, \ldots, e_t, t_1, \ldots, t_\ell$.
8. Return $(T, (\alpha_1, \ldots, \alpha_q), (f_1, \ldots, f_r))$.

Algorithm 6.15 (Right Regular Factorization).

Input A regular boundary problem $(T,\mathscr{B},E)$, where $\beta_1,\dots,\beta_n$ is a basis of $\mathscr{B}$ and $e_1,\dots,e_r$ is a basis of E. A factorization $T = T_1T_2$ and a regular fundamental system $s_1,\dots,s_\mu$ of T_2.

Output Two regular boundary problems $(T_1,\mathscr{B}_1,E)$ and $(T_2,\mathscr{B}_2)$ with $(T,\mathscr{B},E) = (T_1,\mathscr{B}_1,E)\circ(T_2,\mathscr{B}_2)$.

1. Compute the evaluation matrix $M = \beta(s) \in F^{n\times\mu}$.
2. Compute $C = (c_{i,j})_{1\le i,j\le n} \in F^{n\times n}$ such that CM is in reduced row echelon form.
3. For $1 \le i \le n$ set $\tilde{\beta}_i = \sum_{k=1}^{n} c_{i,k}\beta_k$.
4. Compute a right inverse H_2 of T_2 with Algorithm 6.6.
5. For $\mu+1 \le j \le n$ multiply $\alpha_{j-\mu} = \tilde{\beta}_j H_2 \in \mathscr{F}_\Phi\langle\partial,\textstyle\int\rangle$.
6. Return $(T_1,(\alpha_1,\dots,\alpha_{n-\mu}),(e_1,\dots,e_r))$ and $(T_2,(\tilde{\beta}_1,\dots,\tilde{\beta}_\mu))$.

That's all folks. . .

THANK YOU

References

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Table of Contents