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Sparse resultant formulas for differential polynomials

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Algebraic polynomials

Macaulay resultant (formulas), 1916.

Definition of sparse resultant

Sturmfels 1993, Gelfand,

Kapranov and Zelevinsky 1994

Macaulay style formulas

Canny and Emiris 2000

D'Andrea 2002

Differential polynomials

Carrà-Ferro formulas 1997

Linear Complete formulas 2010

Perturbation methods 2011

Rueda and Sendra

Definition of

differential resultant 2011

sparse differential resultant 2012

Gao, Li and Yuan

Macaulay style formulas for sparse differential resultants

Sparse differential resultants can be computed with characteristic set methods (Boulier, Hubert, diffalg 2004). Single exponential algorithm based on order and degree bounds (Li, Yuan and Gao 2011).

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It would be useful to represent the sparse differential resultant as the quotient of two determinants, as done for the algebraic case (D'Andrea 2002).

- improve the existing bounds for degree and order.
- Development of methods to predict the support of the sparse differential resultant $\Rightarrow$ Reduces elimination to an interpolation problem in (numerical) linear algebra.

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In the linear case, consider only the problem of taking the appropriate set of derivatives of the elements in $\mathfrak{P}$.

$$f_1(x) = y' + yx + x' + xx' + yx^2 + y'(x')^2,$$
$$f_2(x) = y + y'x + yx' + y^2xx' + x^2 + (x')^2.$$

$\mathbf{CFRes}(\mathcal{P})$ is the Macaulay algebraic resultant of the polynomial set

$$\mathrm{ps} = \{f_1, f_1', f_2, f_2'\}.$$

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The gcd of the determinant of all the minors of maximal order of a matrix $\mathcal{M}$, whose columns are indexed by all the monomials in x, x' and x'' of degree less than or equal to 5.

The rows of $\mathcal{M}$ are the coefficients of polynomials obtained by multiplying the polynomials in ps by certain monomials in x, x' and x''.

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f_1 and f_2 are nonsparse in x and x' but the extended system ps is sparse. The polynomials in ps do not contain the monomial $(x'')^2$, thus the columns indexed by $(x'')^i$, $i = 2, \ldots, 5$ are all zero and $\mathrm{CFRes}(f_1, f_2) = 0$.

$$\mathcal{P}=\{f_1=z+x+y+y',\ f_2=z+tx'+y'',\ f_3=z+x+y'\}$$

$\mathrm{CFRes}(\mathcal{P})=0$ is the determinant of the next coefficient matrix, whose columns are indexed by y^v, x^v, . . . ,y', x', y,x,1,

$$\begin{bmatrix}
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & z''' \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & z'' \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & z' \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & z \\
1 & 0 & 0 & t & 0 & 2 & 0 & 0 & 0 & 0 & z'' \\
0 & 0 & 1 & 0 & 0 & t & 0 & 1 & 0 & 0 & z' \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & t & 0 & 0 & z \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & z''' \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & z'' \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & z' \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & z
\end{bmatrix}.$$

- Rueda, S.L. and Sendra, J.F., 2010. Linear complete differential resultants and the implicitization of linear DPPEs. Journal of Symbolic Computation, 45, 324-341.
- Rueda, S.L., 2011. A perturbed differential resultant based implicitization algorithm for linear DPPEs. Journal of Symbolic Computation, 46, 977-996.
- Rueda, S.L., 2013. Linear sparse differential resultant formulas. Linear Algebra and Its Applications 438, 4296 - 4321.

- Differential resultant formulas
- ...for nonlinear Laurent differential polynomials
- Order and degree bounds for sparse differential resultants

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Rueda, S.L., 2014. Differential elimintation by differential specialization of Sylvester Style matrices. arXiv: 1310.2081v1.

$\mathcal{D}$ ordinary differential domain, with derivation ∂ (e.g. $\mathbb{Q}(t)$, $\partial = \frac{d}{dt}$)

$U = \{u_1, \ldots, u_{n-1}\}$ set of differential indeterminates over $\mathcal{D}$.

$$k \in \mathbb{N},\ u_{j,k} = \partial^k u_j \text{ and } u_j = u_{j,0}.$$

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$$\mathcal{D}\{U^{\pm}\} := \mathcal{D}[u_{j,k}, u_{j,k}^{-1}]$$

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$f = \sum_{\iota=1}^{m} \theta_\iota \omega_\iota$ in $\mathcal{D}\{U^{\pm}\}$, with ω_ι Laurent differential monomial in $\mathcal{D}\{U^{\pm}\}$.
Differential support in u_j of f

$$\mathfrak{S}_j(f) = \{k \in \mathbb{N} \mid u_{j,k}^{\pm 1}/\omega_\iota \text{ for some } \iota \in \{1, \ldots, m\}\}.$$

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$$\mathrm{ord}(f, u_j) := \max \mathfrak{S}_j(f),\ \ \mathrm{lord}(f, u_j) := \min \mathfrak{S}_j(f) \text{ if } \mathfrak{S}_j(f) \neq \emptyset,$$
$$\mathrm{ord}(f, u_j) = \mathrm{lord}(f, u_j) = -\infty \text{ if } \mathfrak{S}_j(f) = \emptyset.$$

The order of f equals

$$\max\{\mathrm{ord}(f, u_j)\}.$$

System of differential polynomials in $\mathcal{D}\{U^{\pm}\}$.

$$\mathcal{P} := \{f_1, \ldots, f_n\}$$

1. The order of f_i is $o_i \geq 0$, $i = 1, \ldots, n$. So that no f_i belongs to $\mathcal{D}$.
2. $\mathcal{P}$ contains n distinct polynomials.

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Goal: Define differential resultant formulas to compute elements of the differential elimination ideal

$$[\mathcal{P}] \cap \mathcal{D}.$$

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Determinant of the coefficient matrix of $f_1(x)$, $f_2(x)$ and $f_2'(x)$,

$$\rho((y')^2 - yy'' + \alpha y y' - \alpha\gamma y^2 - \beta y^2 y' + \beta\gamma y^3)$$

in $[f_1(x), f_2(x)] \cap \mathcal{D}$.

$\mathrm{PS} \subset \partial\mathcal{P} := \{\partial^k f_i\}$, $\mathcal{U} \subset \{U\}$ and
sets of Laurent differential monomials Ω_f, Ω in $\mathcal{D}[\mathcal{U}^{\pm}]$, $f \in \mathrm{PS}$, verifying:

(ps1) $\mathrm{PS} = \cup_{i=1}^{n} f_i^{[L_i]} = \cup_{i=1}^{n}\{f_i, \partial f_i, \ldots, \partial^{L_i} f_i\}$, $L_i \in \mathbb{N}$,

(ps2) $\mathrm{PS} \subset \mathcal{D}[\mathcal{U}^{\pm}]$ and $|\mathcal{U}| = |\mathrm{PS}| - 1$,

(ps3) $\sum_{f\in \mathrm{PS}} |\Omega_f| = |\Omega|$ and $\cup_{f\in\mathrm{PS}} \Omega_f f \in \oplus_{\omega\in\Omega} \mathcal{D}\omega$.

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Total set of differential polynomials $\mathrm{TS} := \cup_{f\in\mathrm{PS}} \Omega_f f$ whose elements are

$$p = \sum_{\omega\in\Omega} \theta_{p,\omega}\omega, \text{ with } \theta_{p,\omega} \in \mathcal{D}.$$

$\mathcal{M}(\mathrm{TS}, \Omega) = (\theta_{p,\omega})$, is an $|\Omega| \times |\Omega|$ matrix. We call

$$\det(\mathcal{M}(\mathrm{TS}, \Omega)) \tag{1}$$

a differential resultant formula for $\mathcal{P}$.

- Differential resultant formulas
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The diagonals of the matrix $\mathcal{O}(\mathcal{P}_i)$ are indexed by the set Γ_i of all possible bijections

$$\mu : \{1, \ldots, n\} \backslash \{i\} \to \{1, \ldots, n-1\}.$$

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The Jacobi number J_i of the matrix $\mathcal{O}(\mathcal{P}_i)$,

$$J_i := \mathrm{Jac}(\mathcal{O}(\mathcal{P}_i)) = \max \left\{ \sum_{j \in \{1,\ldots,n\} \backslash \{i\}} o_{j,\mu(j)} \mid \mu \in \Gamma_i \right\}.$$

$\mathcal{P} = \{f_1, f_2, f_3, f_4\}$

$$f_1 = 2 + u_1 u_{1,1} + u_{1,2},\ f_2 = u_1 u_{1,2},\ f_3 = u_2 u_{3,1},\ f_4 = u_{1,1} u_2,$$

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$$X_{i,j} := \begin{cases} x_{i,j}, & \mathfrak{S}_j(f_i) \neq \emptyset, \\ 0, & \mathfrak{S}_j(f_i) = \emptyset. \end{cases} \tag{2}$$

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- Every system $\mathcal{P}$ contains a super essential subsystem $\mathcal{P}^*$.
- If $\operatorname{rank}(X(\mathcal{P})) = n - 1$ then $\mathcal{P}^*$ is unique.

Systems $\mathcal{P} = \{f_1, f_2, f_3, f_4\}$ and $\mathcal{P}' = \{f_1, f_2, f_3, f_5\}$

$f_1 = 2 + u_1 u_{1,1} + u_{1,2}$, $f_2 = u_1 u_{1,2}$, $f_3 = u_2 u_{3,1}$, $f_4 = u_{1,1} u_2$, $f_5 = u_{1,2}$,

$$X(\mathcal{P}) = \begin{pmatrix} x_{1,1} & 0 & 0 \\ x_{2,1} & 0 & 0 \\ 0 & x_{3,2} & x_{3,3} \\ x_{4,1} & x_{4,2} & 0 \end{pmatrix} \text{ and } X(\mathcal{P}') = \begin{pmatrix} x_{1,1} & 0 & 0 \\ x_{2,1} & 0 & 0 \\ 0 & x_{3,2} & x_{3,3} \\ x_{4,1} & 0 & 0 \end{pmatrix}.$$

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$\mathcal{P}'$ is not super essential and $\mathbf{rank}(X(P')) < 3$, super essential subsystems are $\{f_1, f_2\}$, $\{f_1, f_3\}$ and $\{f_2, f_3\}$.

For $j = 1, \ldots, n-1$ let us define integers in $\mathbb{N}$

$$\gamma_j := \min\{\operatorname{lord}(f_i, u_j) \mid \mathfrak{S}_j(f_i) \neq \emptyset,\ i = 1, \ldots, n\},$$
$$\gamma := \sum_{j=1}^{n-1} \gamma_j.$$

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Assuming $J_i \geq 0,\ i = 1, \ldots, n$ define

$$\mathrm{ps}(\mathcal{P}) := \cup_{i=1}^{n} f_i^{[J_i - \gamma]},$$

containing $L := \sum_{i=1}^{n} (J_i - \gamma + 1)$ differential polynomials, whose variables belong to

$$\mathcal{V}(\mathcal{P}) := \{u_{j,k} \mid k \in [\gamma_j, M_j] \cap \mathbb{N},\ j = 1, \ldots, n-1\},$$

with $M_j := m_j - \gamma$ and $m_j := \max\{o_{i,j} + J_i - \gamma \mid i = 1, \ldots, n\}$.

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$$J_i \geq 0, i = 1, \ldots, n \Rightarrow \sum_{i=1}^{n} J_i = \sum_{j=1}^{n-1} m_j$$

Thus the number of elements of $\mathcal{V}(\mathcal{P})$ equals

$$\sum_{j=1}^{n-1}(M_j - \gamma_j + 1) = \sum_{j=1}^{n-1}(m_j - \gamma_j - \gamma + 1) = \sum_{i=1}^{n} J_i - n\gamma + n - 1 = L - 1.$$

Li, Yuan, Gao 2013

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Given $j \in \{1, \ldots, n-1\}$ we have

$$\cup_{f \in \mathrm{ps}(\mathcal{P})} \mathfrak{S}_j(f) \subseteq [\gamma_j, M_j] \cap \mathbb{N}. \tag{3}$$

Can we guarantee that the equality holds?

Li, Yuan, Gao 2013

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Can we guarantee that the equality holds?

If there exists j such that (3) is not an equality, $\mathcal{P}$ is sparse in the order.

Theorem If $\mathcal{P}$ is super essential then

$$\cup_{f\in \mathrm{ps}(\mathcal{P})}\mathfrak{S}_j(f) = [\gamma_j, M_j] \cap \mathbb{N}, \quad j = 1, \ldots, n-1.$$

$\mathcal{P}$ is a system of L polynomials in $L-1$ algebraic indeterminates.

- Differential resultant formulas
- ...for nonlinear Laurent differential polynomials
- Order and degree bounds for sparse differential resultants

Ordering on $\mathcal{V}(\mathcal{P})$ through a bijection $\beta : \mathcal{V}(\mathcal{P}) \rightarrow \{1, \ldots, L-1\}$.

$\mathcal{Y} = \{y_1, \ldots, y_{L-1}\}$ algebraic indeterminates over $\mathbb{Q}$.

A bijection $\upsilon : \mathcal{Y} \rightarrow \mathcal{V}(\mathcal{P})$, by $\upsilon(y_l) = \beta^{-1}(l)$ extends to a ring isomorphism

$$\upsilon : \mathcal{D}[\mathcal{Y}^{\pm}] \rightarrow \mathcal{D}[\mathcal{V}(\mathcal{P})^{\pm}].$$

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Monomials in $\mathcal{D}[\mathcal{Y}^{\pm}]$

$$y^{\alpha} = y_1^{\alpha_1} \cdots y_{L-1}^{\alpha_{L-1}}, \quad \alpha = (\alpha_1, \ldots, \alpha_{L-1}) \in \mathbb{Z}^{L-1}.$$

Algebraic support of $f = \sum_{\alpha \in \mathbb{Z}^{L-1}} a_{\alpha} \upsilon(y^{\alpha})$ in $\mathcal{D}[\mathcal{V}(\mathcal{P})^{\pm}]$

$$\mathcal{A}(f) := \left\{\alpha \in \mathbb{Z}^{L-1} \mid a_{\alpha} \neq 0\right\}.$$

Ordering on $\mathrm{ps}(\mathcal{P})$ through a bijection $\boldsymbol{\lambda} : \mathrm{ps}(\mathcal{P}) \rightarrow \{1, \ldots, L\}$.

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We define the algebraic generic system associated to $\mathcal{P}$ as

$$\mathrm{ags}(\mathcal{P}) := \left\{ \sum_{\alpha \in \mathcal{A}(f)} c_{\alpha}^{\lambda(f)} y^{\alpha} \mid f \in \mathrm{ps}(\mathcal{P}) \right\},$$

where $c_{\alpha}^{\lambda(f)}$ are algebraic indeterminates over $\mathbb{Q}$.

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$\rho := \lambda^{-1}$ we have

$$\mathrm{ags}(\mathcal{P}) = \left\{ P_l := \sum_{\alpha \in \mathcal{A}(\rho(l))} c_\alpha^l y^\alpha \mid l = 1, \ldots, L \right\}.$$

$$P_l = c_l T_l + \sum_{h=1}^{h_l} c_{l,h} T_{l,h}, \text{ with } h_l := |A(\rho(l))| - 1.$$

EXAMPLE System $\mathcal{P} = \{f_1, f_2\}$ in $\mathcal{D}\{u\}$, $\mathcal{D} = \mathbb{Q}(t)[a_i, b_j]\{x\}$, $\partial = \frac{\partial}{\partial t}$.

$$f_1 = a_2x + (a_1 + a_4x)u + u' + (a_3 + a_6x)u^2 + a_5u^3,$$
$$f_2 = x' + (b_1 + b_3x)u + (b_2 + b_5x)u^2 + b_4u^3,$$

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$$\partial f_2 = x'' + b_3x'u + (b_3x + b_1)u' + b_5x'u^2 + (2b_5x + 2b_2)uu' + 3b_4u^2u'$$

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System $\mathrm{ags}(\mathcal{P}) = \{P_1, P_2, P_3\}$ of generic polynomials in $\mathcal{Y} = \{y_1, y_2\}$

$$P_1 = c_1 + c_{11} y_1 + c_{12} y_2 + c_{13} y_1^2 + c_{14} y_1^3,$$
$$P_2 = c_2 + c_{21} y_1 + c_{22} y_1^2 + c_{23} y_1^3,$$
$$P_3 = c_3 + c_{31} y_1 + c_{32} y_2 + c_{33} y_1^2 + c_{34} y_1 y_2 + c_{35} y_1^2 y_2$$

$\mathrm{ags}(\mathcal{P})$ is included in $\mathbb{K}[\mathcal{Y}^{\pm}]$, with $\mathbb{K} := \mathbb{Q}(\mathcal{C})$

$$\mathcal{C}_l := \{c^l_\alpha \mid \alpha \in \mathcal{A}(\rho(l))\} \text{ and } \mathcal{C} := \cup_{l=1}^{L} \mathcal{C}_l.$$

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Finite sets of monomials $\Lambda_1, \ldots, \Lambda_L, \Lambda$ in $\mathbb{K}[\mathcal{Y}^{\pm}]$ are determined.

$$\downarrow$$

The matrix $Syl(\mathrm{ags}(\mathcal{P}))$ in the monomial bases of the linear map

$$\langle \Lambda_1 \rangle_{\mathbb{K}} \oplus \cdots \oplus \langle \Lambda_L \rangle_{\mathbb{K}} \rightarrow \langle \Lambda \rangle_{\mathbb{K}} : (g_1, \ldots, g_L) \mapsto \sum g_l P_l,$$

verifies $\det(Syl(\mathrm{ags}(\mathcal{P}))) \neq 0$.

$$\mathbf{det}(\boldsymbol{Syl}(\mathbf{ags}(\boldsymbol{\mathcal{P}}))) \in (\mathbf{ags}(\boldsymbol{\mathcal{P}})) \cap \mathbb{Q}[\boldsymbol{\mathcal{C}}].$$

$S_1(\mathcal{P}) := Syl(\mathrm{ags}(\mathcal{P}))$ assigns a special role to P_1. The same construction can be done choosing P_l, $l = 2, \ldots, L$ as a distinguished polynomial, obtaining a matrix denoted by $S_l(\mathcal{P})$.

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$S_l(\mathcal{P})$ has the minimum number of rows containing coefficients of P_l.

$$D_l = \det(S_l(\mathcal{P})), \quad l = 1, \ldots, L.$$

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$$\boldsymbol{D_l = \det(S_l(\mathcal{P}))}, \quad l = 1, \ldots, L.$$

Let $\mathrm{Res}(\mathcal{P})$ be the algebraic resultant of the system $\mathcal{S} = \mathrm{ags}(\mathcal{P})$,

$$\mathrm{Res}(\mathcal{P}) = \mathrm{gcd}_{\mathbb{Q}[\mathcal{C}]}(D_l).$$

If $\mathrm{Res}(\mathcal{P}) \neq 1$

$$(\boldsymbol{\mathcal{S}}) \cap \mathbb{Q}[\boldsymbol{\mathcal{C}}] = (\mathbf{Res}(\boldsymbol{\mathcal{P}})).$$

If $\mathbf{Res}(\mathcal{P}) \neq 1$,

$$\deg(D_l, \mathcal{C}_l) = \deg(\mathrm{Res}(\mathcal{P}), \mathcal{C}_l) = MV_{-l}(\mathcal{P}) := \mathcal{M}(\mathcal{Q}_h \mid h \in \{1, \ldots, L\} \backslash \{l\}) = \sum_{J \subset \{1,\ldots,L\} \backslash \{l\}} (-1)^{L-|J|} \mathrm{vol}(\sum_{j \in J} \mathcal{Q}_j)$$

$\mathcal{Q}_l$ be the convex hull of $\mathcal{A}(\rho(l))$ in $\mathbb{R}^{L-1}$

$\mathrm{vol}(\mathcal{Q}_l)$ its $L-1$ dimensional volume

$\sum_{j \in J} \mathcal{Q}_j$ the Minkowski sum of $\mathcal{Q}_j$, $j \in J$

Using ”toricres04”, Maple 9 code for sparse (toric) resultant matrices by I.Z. Emiris, obtain a 12×12 matrix $S_1(\mathcal{P})$ whose rows contain the coefficients of the polynomials

$$y_1P_1,\ y_1y_2P_1,\ y_1y_2^2P_1,\ y_1^2P_2,\ y_1y_2P_2,\ y_1^2y_2P_2,$$
$$y_1y_2^2P_2,\ y_1^2y_2^2P_2,\ y_1P_3,\ y_1y_2P_3,\ y_1y_2^2P_3,\ y_1y_2^3P_3$$

in the monomials

$$y_1,\ y_1^2,\ y_1y_2,\ y_1^2y_2,\ y_1y_2^2,\ y_1^2y_2^2,\ y_1y_2^3,\ y_1^2y_2^3,\ y_1y_2^4,\ y_1^2y_2^4,\ y_1y_2^5,\ y_1^2y_2^5.$$

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If the order of the input polynomials is P_2, P_3, P_1, we get a 13×13 matrix $S_2(\mathcal{P})$ and if the order is P_3, P_1, P_2, the matrix $S_3(\mathcal{P})$ obtained is 11×11, namely

$$\begin{bmatrix} c_1 & c_{12} & c_{11} & 0 & c_{13} & 0 & c_{14} & 0 & 0 & 0 & 0 \\ c_3 & c_{32} & c_{31} & c_{34} & c_{33} & c_{35} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_1 & c_{12} & c_{11} & 0 & c_{13} & 0 & c_{14} & 0 & 0 \\ 0 & 0 & c_3 & c_{32} & c_{31} & c_{34} & c_{33} & c_{35} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_1 & c_{12} & c_{11} & 0 & c_{13} & 0 & c_{14} \\ 0 & 0 & 0 & 0 & c_3 & c_{32} & c_{31} & c_{34} & c_{33} & c_{35} & 0 \\ c_2 & 0 & c_{21} & 0 & c_{22} & 0 & c_{23} & 0 & 0 & 0 & 0 \\ 0 & c_2 & 0 & c_{21} & 0 & c_{22} & 0 & c_{23} & 0 & 0 & 0 \\ 0 & 0 & c_2 & 0 & c_{21} & 0 & c_{22} & 0 & c_{23} & 0 & 0 \\ 0 & 0 & 0 & c_2 & 0 & c_{21} & 0 & c_{22} & 0 & c_{23} & 0 \\ 0 & 0 & 0 & 0 & c_2 & 0 & c_{21} & 0 & c_{22} & 0 & c_{23} \end{bmatrix}.$$

The determinants of these matrices are

$$D_1(\mathcal{P}) = -c_3 \mathrm{Res}(\mathcal{P}), D_2(\mathcal{P}) = c_1^2 \mathrm{Res}(\mathcal{P}) \text{ and } D_3(\mathcal{P}) = \mathrm{Res}(\mathcal{P}).$$

SPECIALIZATION to the coefficient set of $\boldsymbol{f} = \sum_{\alpha \in \mathcal{A}(f)} \boldsymbol{a}_{\alpha}^{f} \boldsymbol{v}(\boldsymbol{y}^{\alpha})$ in $\mathrm{ps}(\mathcal{P})$

$$\boldsymbol{A}(\mathcal{P}) := \cup_{f \in \mathrm{ps}(\mathcal{P})} \boldsymbol{A}_f, \text{ wtih } \boldsymbol{A}_f := \{\boldsymbol{a}_{\alpha}^{f} \mid \boldsymbol{\alpha} \in \mathcal{A}(f)\}.$$

SPECIALIZATION to the coefficient set of $f = \sum_{\alpha \in \mathcal{A}(f)} a_\alpha^f \upsilon(y^\alpha)$ in $\mathrm{ps}(\mathcal{P})$

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Given $l \in \{1, \ldots, L\}$, such that $\rho(l) = f$, and $c_\alpha^l \in \mathcal{C}_l$, $a_\alpha^{\rho(l)} \in A_f$.
Ring epimorphism

$$\begin{aligned} \Xi :& \mathbb{Q}[\mathcal{C}][\mathcal{Y}^{\pm}] \to \mathbb{Q}[A(\mathcal{P})][\mathcal{V}^{\pm}], \\ & \Xi(c_\alpha^l) = a_\alpha^{\rho(l)} \\ & \Xi(y_l) = \upsilon(y_l) \end{aligned}$$

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$\Xi(D_l)\in[\mathcal{P}]\cap\mathcal{D}$ is a differential resultant formula for $\mathcal{P}$ with

$$L_i = J_i - \gamma,\ \ \mathcal{U}=\mathcal{V}(\mathcal{P}) \text{ and } \Omega_f = \Xi(\Lambda_{\lambda(f)}), \Omega = \Xi(\Lambda),$$

$f\in\mathrm{ps}(\mathcal{P})$.

ORDER AND DEGREE BOUNDS

Let us consider sets of differential indeterminates over $\mathbb{Q}$,

$$\boldsymbol{A_i := \{\mathfrak{a}^i_\alpha \mid \alpha \in \mathcal{A}(f_i)\}, \quad i = 1, \ldots, n,}$$

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Let us consider sets of differential indeterminates over $\mathbb{Q}$,

$$\boldsymbol{A}_i := \{\mathfrak{a}^i_\alpha \mid \alpha \in \mathcal{A}(f_i)\},\ \ i = 1, \ldots, n,$$

The generic polynomial $\mathbb{F}_i$ in $\mathfrak{D}\{U^\pm\}$ with algebraic support $\mathcal{A}(f_i)$ is

$$\mathbb{F}_i := \sum_{\alpha \in \mathcal{A}(f_i)} \mathfrak{a}^i_\alpha v(y^\alpha).$$

$\mathfrak{P} = \{\mathbb{F}_1, \ldots, \mathbb{F}_n\}$ sparse generic Laurent differential polynomials in

$$\mathfrak{D}\{U^\pm\},\ \ \mathfrak{D} = \mathbb{Q}\{\cup_{i=1}^n \boldsymbol{A}_i\}.$$

If the differential elimination ideal $[\mathfrak{P}] \cap \mathfrak{D}$ has dimension $n-1$ then

$$[\mathfrak{P}] \cap \mathfrak{D} = \mathrm{sat}(\partial\mathrm{Res}(\mathfrak{P})),$$

$\partial\mathrm{Res}(\mathfrak{P})$ is the sparse differential resultant of $\mathfrak{P}$.

Li, W., Yuan, C.M., Gao, X.S., 2012. Sparse Differential Resultant for Laurent Differential polynomials. In arXiv:1111.1084v3.

Assuming $\mathfrak{P}$ super essential, we can compute $D_l, l = 1, \ldots, L$.

$$D_l \in (\mathbf{ags}(\mathfrak{P})) \cap \mathbb{Q}[\mathcal{C}] \text{ prime ideal with generic zero } \epsilon.$$

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$D_l = Q_1 \cdot \ldots \cdot Q_r$ irreducible factors,

$$\mathcal{Q} = \{Q \in \{Q_1, \ldots, Q_r\} \mid Q(\epsilon) = 0\} \neq \emptyset.$$

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(*) If there exists $Q \in \mathcal{Q}$ such that $(\mathcal{S}_Q) \cap \mathbb{Q}[\mathcal{C}_Q] = (Q)$ then

- $\Xi(Q) \neq 0 \Rightarrow \Xi(Q) = \mathfrak{E}\partial\mathrm{Res}(\mathfrak{P})$
- $\Xi(D_l) \neq 0 \Rightarrow \Xi(D_l) = \mathfrak{E}\partial\mathrm{Res}(\mathfrak{P})$, $\mathfrak{E} \in \mathfrak{D}$.

Let $\mathfrak{P} = \{\mathbb{F}_1, \mathbb{F}_2, \mathbb{F}_3\}$, $(J_1 - \gamma, J_2 - \gamma, J_3 - \gamma) = (1, 1, 2)$.

$$\mathbf{ps}(\mathfrak{P}) = \cup_{i=1}^{n} \mathbb{F}_i^{[J_i - \gamma]}$$

$$\begin{aligned}
\{&\partial\mathbb{F}_1 = \partial\mathfrak{a}_1 + \partial\mathfrak{a}_{11}u_1u_2 + \mathfrak{a}_{11}u_{11}u_2 + \mathfrak{a}_{11}u_1u_{21}, \mathbb{F}_1 = \mathfrak{a}_1 + \mathfrak{a}_{11}u_1u_2,\\
&\partial\mathbb{F}_2 = \partial\mathfrak{a}_2 + \partial\mathfrak{a}_{21}u_1u_{22} + \mathfrak{a}_{21}u_{11}u_{22} + \mathfrak{a}_{21}u_1u_{23}, \mathbb{F}_2 = \mathfrak{a}_2 + \mathfrak{a}_{21}u_1u_{22},\\
&\partial^2\mathbb{F}_3 = \partial^2\mathfrak{a}_3 + \partial^2\mathfrak{a}_{31}u_{21} + 2\partial\mathfrak{a}_{31}u_{22} + \mathfrak{a}_{31}u_{23},\\
&\partial\mathbb{F}_3 = \partial\mathfrak{a}_3 + \partial\mathfrak{a}_{31}u_{21} + \mathfrak{a}_{31}u_{22}, \mathbb{F}_3 = \mathfrak{a}_3 + \mathfrak{a}_{31}u_{21}\}.
\end{aligned}$$

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$$\{\partial\mathbb{F}_1 = \partial\mathfrak{a}_1 + \partial\mathfrak{a}_{11}u_1u_2 + \mathfrak{a}_{11}u_{11}u_2 + \mathfrak{a}_{11}u_1u_{21}, \mathbb{F}_1 = \mathfrak{a}_1 + \mathfrak{a}_{11}u_1u_2,$$
$$\partial\mathbb{F}_2 = \partial\mathfrak{a}_2 + \partial\mathfrak{a}_{21}u_1u_{22} + \mathfrak{a}_{21}u_{11}u_{22} + \mathfrak{a}_{21}u_1u_{23}, \mathbb{F}_2 = \mathfrak{a}_2 + \mathfrak{a}_{21}u_1u_{22},$$
$$\partial^2\mathbb{F}_3 = \partial^2\mathfrak{a}_3 + \partial^2\mathfrak{a}_{31}u_{21} + 2\partial\mathfrak{a}_{31}u_{22} + \mathfrak{a}_{31}u_{23},$$
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$$\mathcal{S} = \mathrm{ags}(\mathfrak{P}) =$$

$$\{P_1 = c_{10} + c_{11}y_2y_1 + c_{12}y_2y_3 + c_{13}y_4y_1, P_2 = c_{20} + c_{21}y_2y_1,$$
$$P_3 = c_{30} + c_{31}y_5y_1 + c_{32}y_5y_3 + c_{33}y_6y_1, P_4 = c_{40} + c_{41}y_5y_1,$$
$$P_5 = c_{50} + c_{51}y_4 + c_{52}y_5 + c_{53}y_6, P_6 = c_{60} + c_{61}y_4 + c_{62}y_5,$$
$$P_7 = c_{70} + c_{71}y_4\}.$$

We compute $D_1(\mathfrak{P})$ which has the next irreducible factors

$$
\begin{aligned}
&Q_1 = c_{62}, Q_2 = c_{40}, Q_3 = -c_{70}c_{62}c_{51} + c_{70}c_{61}c_{52} - c_{60}c_{71}c_{52} + c_{62}c_{50}c_{71}\\
&Q_4 = -c_{61}c_{70} + c_{71}c_{60}, Q_5 = c_{70}, Q_6 = c_{20}c_{40}c_{12}c_{41}c_{33}c_{71}^2c_{62}c_{50}\\
&- c_{62}c_{40}c_{70}c_{53}c_{32}c_{13}c_{21} - c_{62}c_{40}c_{70}c_{20}c_{51}c_{12}c_{41}c_{33} - c_{71}c_{40}c_{20}c_{12}c_{41}c_{60}c_{33}c_{52}\\
&+ c_{71}c_{60}c_{40}c_{53}c_{10}c_{32}c_{41}c_{21} + c_{71}c_{40}c_{20}c_{12}c_{41}c_{60}c_{53}c_{31} - c_{71}c_{20}c_{60}c_{30}c_{12}c_{41}^2c_{53}\\
&- c_{71}c_{40}c_{20}c_{32}c_{41}c_{60}c_{53}c_{11} + c_{40}c_{70}c_{20}c_{61}c_{52}c_{12}c_{41}c_{33} - c_{40}c_{70}c_{53}c_{10}c_{61}c_{32}c_{41}c_{21}\\
&- c_{40}c_{70}c_{20}c_{12}c_{41}c_{53}c_{31}c_{61} + c_{70}c_{20}c_{30}c_{61}c_{12}c_{41}^2c_{53} + c_{40}c_{70}c_{20}c_{32}c_{41}c_{53}c_{11}c_{61}.
\end{aligned}
$$

Only $Q_6(\epsilon) = 0$, $\Xi(Q_6) \neq 0$, $\Xi(Q_6) = -\mathfrak{a}_{21}H_0$.

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$$Q_4 = -c_{61}c_{70} + c_{71}c_{60}, Q_5 = c_{70}, Q_6 = c_{20}c_{40}c_{12}c_{41}c_{33}c_{71}^2c_{62}c_{50}$$
$$- c_{62}c_{40}c_{70}c_{53}c_{32}c_{13}c_{21} - c_{62}c_{40}c_{70}c_{20}c_{51}c_{12}c_{41}c_{33} - c_{71}c_{40}c_{20}c_{12}c_{41}c_{60}c_{33}c_{52}$$
$$+ c_{71}c_{60}c_{40}c_{53}c_{10}c_{32}c_{41}c_{21} + c_{71}c_{40}c_{20}c_{12}c_{41}c_{60}c_{53}c_{31} - c_{71}c_{20}c_{60}c_{30}c_{12}c_{41}^2c_{53}$$
$$- c_{71}c_{40}c_{20}c_{32}c_{41}c_{60}c_{53}c_{11} + c_{40}c_{70}c_{20}c_{61}c_{52}c_{12}c_{41}c_{33} - c_{40}c_{70}c_{53}c_{10}c_{61}c_{32}c_{41}c_{21}$$
$$- c_{40}c_{70}c_{20}c_{12}c_{41}c_{53}c_{31}c_{61} + c_{70}c_{20}c_{30}c_{61}c_{12}c_{41}^2c_{53} + c_{40}c_{70}c_{20}c_{32}c_{41}c_{53}c_{11}c_{61}.$$

Only $Q_6(\epsilon) = 0$, $\Xi(Q_6) \neq 0$, $\Xi(Q_6) = -\mathfrak{a}_{21}H_0$.

$$\partial\mathrm{Res}(\mathfrak{P}) = H_0 =$$
$$- \mathfrak{a}_{21}\mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_{11}\mathfrak{a}_{31}^2\partial^2\mathfrak{a}_3 + 2\mathfrak{a}_{21}\mathfrak{a}_{31}\mathfrak{a}_2\mathfrak{a}_1\mathfrak{a}_{11}\partial\mathfrak{a}_3\partial\mathfrak{a}_{31} - \mathfrak{a}_{21}\mathfrak{a}_{31}^2\partial\mathfrak{a}_3\mathfrak{a}_2\partial\mathfrak{a}_1\mathfrak{a}_{11}$$
$$+ \mathfrak{a}_{21}\mathfrak{a}_{31}\mathfrak{a}_2\mathfrak{a}_3\mathfrak{a}_1\partial^2\mathfrak{a}_{31}\mathfrak{a}_{11} + \mathfrak{a}_{21}\mathfrak{a}_{31}^2\mathfrak{a}_2\mathfrak{a}_1\partial\mathfrak{a}_3\mathfrak{a}_1 - 2\mathfrak{a}_{21}\mathfrak{a}_2\mathfrak{a}_3\mathfrak{a}_1\partial\mathfrak{a}_{31}^2\mathfrak{a}_{11}$$
$$+ \mathfrak{a}_{21}\mathfrak{a}_2\mathfrak{a}_3\mathfrak{a}_{31}\partial\mathfrak{a}_1\partial\mathfrak{a}_{31}\mathfrak{a}_{11} + \mathfrak{a}_{21}\mathfrak{a}_{31}^2\mathfrak{a}_1\partial\mathfrak{a}_3\partial\mathfrak{a}_2\mathfrak{a}_{11} - \mathfrak{a}_{21}\mathfrak{a}_3\mathfrak{a}_1\partial\mathfrak{a}_2\partial\mathfrak{a}_{31}\mathfrak{a}_{11}\mathfrak{a}_{31}$$
$$- \mathfrak{a}_{21}\mathfrak{a}_2\mathfrak{a}_3\mathfrak{a}_1\mathfrak{a}_{31}\partial\mathfrak{a}_{11}\partial\mathfrak{a}_{31} + \mathfrak{a}_{31}^2\mathfrak{a}_2^2\mathfrak{a}_3\mathfrak{a}_{11}^2 + \mathfrak{a}_2\mathfrak{a}_3\mathfrak{a}_1\mathfrak{a}_{11}\mathfrak{a}_{31}\partial\mathfrak{a}_{21}\partial\mathfrak{a}_{31} - \mathfrak{a}_{31}^2\mathfrak{a}_2\mathfrak{a}_1\mathfrak{a}_{11}\partial\mathfrak{a}_3\partial\mathfrak{a}_{21}$$

If $\Xi(Q) \neq 0$ then

$$\mathbf{ord}(\partial\mathbf{Res}(\mathfrak{P}), \boldsymbol{A}_i) \leq \mathbf{ord}(\Xi(\boldsymbol{Q}), \boldsymbol{A}_i) \leq \boldsymbol{J}_i - \boldsymbol{\gamma}, \quad \boldsymbol{i} = 1, \ldots, \boldsymbol{n}$$

If $\Xi(Q) \neq 0$ then

$$\mathbf{ord}(\partial\mathbf{Res}(\mathfrak{P}), \boldsymbol{A}_i) \leq \mathbf{ord}(\Xi(Q), \boldsymbol{A}_i) \leq \boldsymbol{J}_i - \gamma, \quad \boldsymbol{i} = 1, \ldots, \boldsymbol{n}$$

and if $(\mathcal{S}_Q) \cap \mathbb{Q}[\mathcal{C}_Q] = (Q) = (\mathrm{Res}(\mathcal{S}_Q))$,

$$\mathbf{deg}(\partial\mathbf{Res}(\mathfrak{P}), \cup_{k=0}^{\tau_i} \partial^k \boldsymbol{A}_i) \leq \mathbf{deg}(\Xi(Q), \cup_{k=0}^{\tau_i} \partial^k \boldsymbol{A}_i) \leq \sum_{k=0}^{\tau_i} \mathbf{deg}(Q, C_{\lambda(\partial^k f_i)}) = \sum_{k=0}^{\tau_i} MV_{-\lambda(\partial^k f_i)}(\mathfrak{P}).$$

$\partial^k A_i = \{\partial\mathfrak{a} \mid \mathfrak{a} \in A_i\}$, $\tau_i = \mathrm{ord}(\Xi(Q), A_i)$.