Hors d'oeuvre to Malgrange ideas: Jet Bundles

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Definition of Jet Bundle

Let $\pi : E \to M$ be a fiber bundle, where E and M are finite dimensional manifolds (real or complex) of dimensions $m+n$ and n respectively. Let $p \in M$. Let $1 \leqslant i \leqslant n$, $1 \leqslant \alpha \leq m$.

- We denote by Γ_p the sections of π around p.
- Let (U, x^i) be a coordinate system of M around p. We say that (x^i, u^α) is an *adapted coordinate* of π if $x^i(a) = x^i(b)$ whenever $\pi(a) = \pi(b)$. One can think about the u^α as coordinate functions of the fibers above U.
- A section $\phi \in \Gamma_p$ is completely defined by the functions

$$\begin{aligned} \phi^\alpha(x^1, \ldots, x^n) &= u^\alpha \circ \phi\,(x^1, \ldots, x^n) \\ &= u^\alpha(x^1, \ldots, x^n). \end{aligned}$$

Definition of Jet Bundle

Let ϕ, $\psi \in \Gamma_p$. Using multi-index notation, we say that ϕ and ψ define the same k-th order jet at p if $\phi(p) = \psi(p)$ and

$$\frac{\partial^{|I|}\phi^\alpha}{\partial x^I}(p) = \frac{\partial^{|I|}\psi^\alpha}{\partial x^I}(p)$$

for all multi-index I with $1 \leq |I| \leq k$ and all $1 \leq \alpha \leq m$. To define the same jet bundle at p is an equivalence relation on Γ_p and so, the equivalence class of ϕ is denoted $j_p^k\phi$
The k-th order Jet Bundle of π is defined as the collection

$$J^k\pi = \{j_p^k\phi |\ p \in M,\ \phi \in \Gamma_p\}$$

First facts about Jet Bundles

- $J^k\pi$ is a manifold
- The map $\pi_k : J^k\pi \longrightarrow M$, $\pi_k(j^k_p\phi) = p$ defines a fiber bundle.
- The map $\pi_{k,0} : J^k\pi \longrightarrow E$, $\pi_{k,0}(j^k_p\phi) = \phi(p)$ defines an affine bundle.
- If $k \geqslant l$, the map $\pi_{k,l} : J^k\pi \longrightarrow J^l\pi$, $\pi_{k,l}(j^k_p\phi) = j^l_p\phi$ defines an affine bundle.
- $\cdots \longrightarrow J^k\pi \longrightarrow \cdots \longrightarrow J^1\pi \longrightarrow E \longrightarrow M$

First facts about Jet Bundles

The adapted coordinate system (x^i, u^α) on E leads to the *induced coordinate* system on $J^k\pi$, $(x^i, u^\alpha, u_I^\alpha)$, where

$$u_I^\alpha(j_p^k\phi) = \frac{\partial^{|I|}\phi^\alpha}{\partial x^I}, \quad 1 \leq |I| \leq k, \; 1 \leq \alpha \leq m.$$

One can think about the induced coordinates of a k-th order jet $j_p^k\phi$ as the coefficients of the k-th order Taylor expansion of the section ϕ around p on adapted coordinates.
Say, $x^i(p) = p^i$, then if

$$\phi^\alpha(x^1, \ldots, x^n) = \phi^\alpha(p) + \sum_i \frac{\partial\phi^\alpha}{\partial x^i}(x^i - p^i) + \ldots + \sum_{|I|=k} \frac{1}{I!}\frac{\partial^{|I|}\phi^\alpha}{\partial x^I}(x-p)^I + \ldots$$

the induced coordinates of $j_p^k\phi$ are $(p^i, \phi^\alpha(p), \frac{\partial^{|I|}\phi^\alpha}{\partial x^I}(p))$.

First facts about Jet Bundles

Jet Bundles are convenient for defining differential equations. Indeed, a k-th order *differential equation* is now defined as a closed embedded submanifold S of $J^k\pi$. A solution is a local section $\phi \in \Gamma_U(\pi)$ ($U \subseteq M$ open), such that $j^k_p\phi \in S$ for every $p \in U$.

First facts about Jet Bundles

Example: Consider the trivial bundle $\pi : \mathbb{C}^2 \times \mathbb{C} \to \mathbb{C}^2$, given by the first projection, with global adapted coordinates (z^1, z^2, w^1). The map $F : J^1\pi \to \mathbb{C}$ defined by:

$$F(z^1, z^2, w^1, w^1_1, w^1_2) = w^1_1 w^1_2 - 2z^2 w^1$$

gives rise to the differential equation:

$$S = \{j^1_p\phi \in J^1\pi |\ F(j^1_p\phi) = 0\}.$$

In traditional notation:

$$\frac{\partial\phi}{\partial z^1}\frac{\partial\phi}{\partial z^2} - 2z^2\phi = 0$$

A particular solution to S is $\phi(z^1, z^2) = z^1(z^2)^2$.

Prolongation of sections

Given an open subset $U \subseteq M$ and a section $\phi \in \Gamma_U(\pi)$ we define the k-th order prolongation of ϕ, $j^k\phi$ by setting

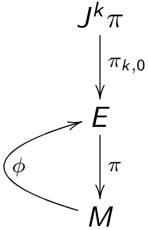
$$j^k\phi(p) = j^k_p\phi$$

In local coordinates

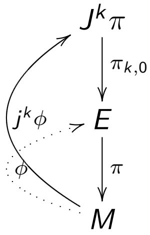
$$j^k\phi(x^1, \ldots, x^n) = (x^i, \phi^\alpha, \frac{\partial^{|I|}\phi^\alpha}{\partial x^I})$$

Proposition: If $\psi \in \Gamma_U(\pi_k)$, then there is a local section $\phi \in \Gamma_U(\pi)$ satisfying $\psi = j^k\phi$ if, and only if, $\psi = j^k(\pi_{k,0} \circ \psi)$.

Prolongation of sections



Prolongation of sections



Prolongation of morphisms

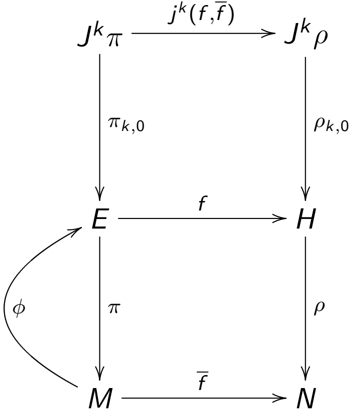
Let $\rho : H \to N$ be another fiber bundle, and $(f, \overline{f})$ be a bundle morphism where $\overline{f} : M \to N$ is a diffeomorphism

$$\begin{array}{ccc} E & \xrightarrow{\;f\;} & H \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \rho} \\ M & \xrightarrow[\sim]{\;\overline{f}\;} & N \end{array}$$

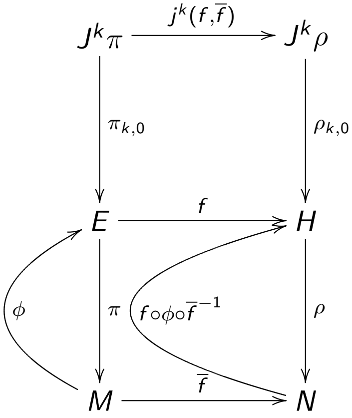
We define the k-th order prolongation $j^k(f, \overline{f}) : J^k\pi \to J^k\rho$ by

$$j^k(f, \overline{f})(j^k_p \phi) = j^k_{\overline{f}(p)}\{f \circ \phi \circ \overline{f}^{-1}\}$$

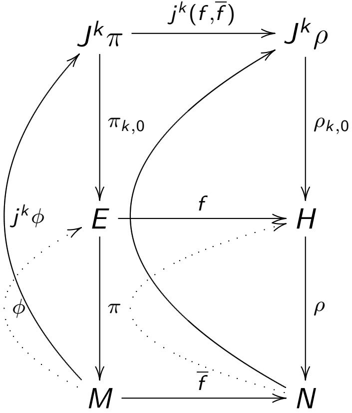
Prolongation of morphisms



Prolongation of morphisms



Prolongation of morphisms



Prolongation of morphisms

Jet Bundles behave "nicely" with respect to product and morphisms. By "nice" I mean:

- $j^k(id_E) = id_{J^k\pi}$.
- if π and ρ are bundles over M then $J^k\{\pi \times \rho\} = J^k\pi \times J^k\rho$.
- if $f : \pi \longrightarrow \rho$ and $g : \rho \longrightarrow \sigma$ are bundle morphisms projecting to diffeomorphisms, then $j^k\{g \circ f\} = j^k g \circ j^k f$.

Prolongation of morphisms

These properties are useful when one has a bundle where the fibers are groups. Indeed one can prolong:

- The identity section.
- The inverse morphism.
- The product morphism.
- The Lie algebra structure of the fibers

Remark: This does NOT mean that $J^k\pi$ turns into a group itself (example?).

The algebraic differential structure

Since $\pi_{k,0} : J^k\pi \to E$ is an affine bundle, if $\mathscr{O}_E(V)$ denotes the sheaf of coordinate functions over some open $V \subseteq \pi^{-1}U$ in E; then in $J^k\pi$:

$$\mathscr{O}_{J^k\pi}(\pi_{k,0}^{-1}V) = \mathscr{O}_E(V)[u_I^\alpha]$$

(I ranges over all multi-indexes $1 \leq |I| \leq k$ and
$1 \leq \alpha \leq m = \dim(E) - \dim(M)$).
i.e. we just append some more variables with no relation in between them.

The algebraic differential structure

- $\cdots \to J^k\pi \to \cdots \to J^1\pi \to E \to M$, so we define the pro-manifold through the projective limit:

$$J\pi = \lim_{\leftarrow} J^k\pi$$

- $\mathscr{O}_M \to \mathscr{O}_E \to \mathscr{O}_{J^1\pi} \to \ldots \to \mathscr{O}_{J^k\pi} \to \ldots$, so we define the sheave through the inductive limit:

$$\mathscr{O}_{J\pi} = \lim_{\to} \mathscr{O}_{J^k\pi}$$

The algebraic differential structure

In order to understand the differential structure on $J\pi$ it is useful to remember that the Jet Bundle is created starting from sections of π.
If $\phi \in \Gamma_U \pi$ is a local section, then we can think of $\phi(U)$ as a "deformation" of U on E, so that if $F : E \to \mathbb{C}$ then, the pull-back, $\phi^*(F) = F \circ \phi$ is a function on the "deformed" copy of U and so:

$$\frac{\partial}{\partial x^i}\phi^*(F)(p) = \frac{\partial F}{\partial x^i}(\phi(p)) + \sum_\alpha \frac{\partial F}{\partial u^\alpha}(\phi(p))\frac{\partial \phi^\alpha}{\partial x^i}(p)$$

Pulling back the functions $\frac{\partial F}{\partial x^i}$ and $\frac{\partial F}{\partial u^\alpha}$ to $J^1\pi$ we get:

$$\frac{\partial}{\partial x^i}\phi^*(F)(p) = (\frac{\partial F}{\partial x^i} + \sum_\alpha u_i^\alpha \frac{\partial F}{\partial u^\alpha})(j_p^1\phi)$$

The algebraic differential structure

So we define the derivation:

$$\begin{array}{rcl} D_i : \mathscr{O}_E & \longrightarrow & \mathscr{O}_{J^1\pi} \\ F & \longmapsto & \dfrac{\partial F}{\partial x^i} + \sum_\alpha u_i^\alpha \dfrac{\partial F}{\partial u^\alpha} \end{array}$$

Similarly, we extend this derivation to:

$$\begin{array}{rcl} D_i : \mathscr{O}_{J^k\pi} & \longrightarrow & \mathscr{O}_{J^{k+1}\pi} \\ F & \longmapsto & \dfrac{\partial F}{\partial x^i} + \sum_\alpha u_i^\alpha \dfrac{\partial F}{\partial u^\alpha} + \sum_{\alpha, 1<|I|\le k} u_{I+\epsilon_i}^\alpha \dfrac{\partial F}{\partial u_I^\alpha} \end{array}$$

So that we get the ring $\mathscr{O}_{J\pi}$ with n-derivations $D_1, \ldots, D_n$.

The algebraic differential structure

Just as we defined k-th order differential equations as embedded closed submanifolds of $J^k\pi$; we define, in more generality, a differential equation as a closed embedded subpro-manifold of $J\pi$. These are given by coherent sheafs of ideals in $\mathscr{O}_{J\pi}$. Actually, we will only consider coherent sheafs of differential ideals (we are looking for objects respecting the differential structure).
So let $\mathscr{I}$ be such a sheaf of ideals, and S the closed pro-manifold in $J\pi$ defined by it. Just as before, a solution to the differential equation is a local section $\phi \in \Gamma_U\pi$ such that

$$j_p^k\phi \in S \cap J^k\pi, \text{ for all } p \in U \text{ and all } k \leq 0$$

The algebraic differential structure

Algebraically: a local solution to the differential system of equations $\mathscr{I}$ is a differential morphism of $\mathscr{O}_M$-algebras:

$$\Phi : \mathscr{O}_{J\pi}/\mathscr{I} \longrightarrow \mathscr{O}_M \restriction_U$$

Indeed, if $\phi^\alpha(x^1, \dots, x^n) = \Phi(u^\alpha)$ then

- $\frac{\partial \phi^\alpha}{\partial x^i} = \Phi(D_i u^\alpha) = \Phi(u_i^\alpha)$
- $\frac{\partial^{|I|} \phi^\alpha}{\partial x^I} = \Phi(D_I u^\alpha) = \Phi(u_I^\alpha)$

So the k-th prolongation of the section $\phi = (x^i, \phi^\alpha)$ is a zero of $\mathscr{I} \cap \mathscr{O}_{J^k\pi}$