Hors d'oeuvre to Malgrange ideas: Jet Bundles

Camilo Sanabria

The CUNY Graduate Center

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Recall

Given a fiber bundle $\pi : E \to M$, we define the k-th order jet bundle $J^k(\pi)$ as the collection of equivalent classes of germs of sections of π, where two sections are identified if their derivatives, up to the k-th order, coincide. The class of $\phi \in \Gamma_p\pi$ is denoted by $j^k_p\phi$.
A local coordinate chart is given by the functions $(x^i, u^\alpha, u^\alpha_I)$, where:

- x^i are the coordinate functions of M.
- u^α are coordinate functions of the fibers of π.
- $I = (i_1, \ldots, i_n)$ is a multi-index, $|I| \leq k$, and $u^\alpha_I(j^k_p\phi) = \frac{\partial^{|I|}\phi^\alpha}{\partial x^I}$

Recall

- $\cdots \longrightarrow J^k\pi \longrightarrow \cdots \longrightarrow J^1\pi \longrightarrow E \longrightarrow M$, and we define:

$$J\pi = \lim_{\leftarrow} J^k\pi$$

- $\mathscr{O}_M \longrightarrow \mathscr{O}_E \longrightarrow \mathscr{O}_{J^1\pi} \longrightarrow \ldots \longrightarrow \mathscr{O}_{J^k\pi} \longrightarrow \ldots$, and we define:

$$\mathscr{O}_{J\pi} = \lim_{\rightarrow} \mathscr{O}_{J^k\pi}$$

- Locally, $\mathscr{O}_{J\pi}$ turns into a sheave of differential rings by setting

$$D_i F = \frac{\partial F}{\partial x^i} + \sum_{\alpha} u_i^{\alpha} \frac{\partial F}{\partial u^{\alpha}} + \sum_{\alpha, 1 < |I| \leq k} u^{\alpha}_{I+\epsilon_i} \frac{\partial^{|I|} F}{\partial u^{\alpha}_I}$$

Recall

With all this paraphernalia, we want to think about a system of differential equations as a coherent sheaf of ideals on $\mathscr{O}_{J\pi}$ that are locally differential ideals. The local solutions are given by differential morphisms of $\mathscr{O}_M$-algebras:

$$\Phi : \mathscr{O}_{J\pi}/\mathscr{I} \longrightarrow \mathscr{O}_M \restriction_U$$

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Remark: In this setting, differential ideals are huge (in general they are algebraically generated by infinitely many elements). In practice, from this point of view, solving a system amounts to satisfy a huge a amount of conditions (in general, an infinite amount).

We turn our attention to a different perspective. We model everything over $\mathbb{C}$ and we see the geometric objects involved as varieties rather than manifolds. We recall that a k-th order system of differential equations is defined as a closed subvariety S_k of $J^k\pi$. A solution is a local section $\phi \in \Gamma_U(\pi)$ ($U \subseteq M$ open), such that $j^k_p\phi \in S_k$ for every $p \in U$. A point in S_k is called a k-th order jet solution to S_k.

S_k is locally defined as the zeroes of some functions $F_1, \ldots, F_N$. We would like to know how well behaved is the system given by the F_j in various senses:

Differentiating the F_j, the functions F_j and the functions D_iF_j define a closed subvariety pr_1S_k of $J^{k+1}\pi$ projecting to $J^k\pi$.

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- Does every solution of S_k lifts to a solution of pr_1S_k?
- Can we build a power series solution from any jet solution?
- Does every jet solution at the non-singular parts lifts?

Example 1

Consider $\pi : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$, with adapted coordinates $(x^1, \ldots, x^n, u)$ and the first order system of differential equations given by the zeros of:

$$f_1(x^i, u, u_i) = u_1, \quad f_2(x^i, u, u_i) = x^1 u_2 + x^2 u_3 + \ldots + x^{n-1} u_n$$

In traditional notation, we search for a $y(x^1, \ldots, x^n)$ such that:

$$\frac{\partial y}{\partial x^1} = 0, \quad x^1 \frac{\partial y}{\partial x^2} + x^2 \frac{\partial y}{\partial x^3} + \ldots + x^{n-1} \frac{\partial y}{\partial x^n} = 0$$

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- $D_i f_1 = u_{1,i}$ and $D_1 f_2 = u_2 + x^1 u_{1,2} + x^2 u_{1,3} + \ldots, x^{n-1} u_{1,n}$ so the first prolongation is defined by u_1, u_2, f_2 and some other.

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- $D_i u_2 = u_{2,i}$ and $D_2\{f_2 - x^1 u_2\} = u_3 + x^2 u_{2,3} + \ldots + x^{n-1} u_{2,n}$ so the second prolongation is defined by u_1, u_2, u_3, f_2 and some other; recursively,

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- $D_n u_{n-1} = u_{n-1,n}$ and $D_{n-1}\{f_2 - x^1 u_2 - \ldots - x^{n-2} u_{n-1}\} = u_n + x^{n-1} u_{n-1,n}$ so the n-th prolongation is defined by the u_is and some other.

Example 1

Through differentiation we see that the solutions to the system

$$S_1: \; u_1 = 0, \; x^1 u_2 + x^2 u_3 + \ldots + x^{n-1} u_n = 0$$

are solutions to the system

$$S_1' : u_1 = 0, \ldots, u_n = 0$$

Also, the solutions to S_1' are solutions to S_1, so both systems are equivalent.
Now, the point of coordinates
$(x^1, x^2, \ldots, x^n, u, u_1, u_2 \ldots, u_n) = (0, 0, \ldots, 0, 0, 0, 1, \ldots, 0) \in S_1$ does not lift to $pr_1 S_1$, for $u_2 \neq 0$. This does not happen in S_1'. So by prolonging S_1 n-times we get a well behaved system S_1'.

Example 2

Consider $\pi : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, with adapted coordinates (x, u) and the first order differential equation:

$$S_1 : xu' + u - f = 0, \quad f = \sum_i \frac{a_i}{i!} x^i$$

The theory of linear ordinary differential equations says that for $x \neq 0$ any k-th order jet solution can be lifted to a $k+1$st jet solution.
Above zero things are not so nice:
$D^{q-1}(xu' + u - f) = xu^{(q)} + qu^{(q-1)} - f^{(q-1)}$, so
$(x, u, u', \ldots, u^{(q-1)}, u^{(q)}) = (0, a_0, \frac{a_1}{2}, \ldots, \frac{a_{q-1}}{q}, b)$ with b arbitrary, is a q-th order jet solution of S that cannot be lifted to a $q+1$st jet solution unless $b = \frac{a_q}{q+1}$. No amount of differentiation is going to fix this issue.

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Remark 2: We may need to take out some singularities to get a well behaved system.

Lifting jet solutions

Assume we are given F locally define on $J^k\pi$ and a zero of F with coordinates $(x^i, u^\alpha, u_I^\alpha)$. We want to lift this zero to the first prolongation. For that matter we differentiate F to get:

$$D_i F = \frac{\partial F}{\partial x^i} + \sum_{\alpha, 1 \le |I| \le k} u^\alpha_{I+\epsilon_i} \frac{\partial^{|I|} F}{\partial u_I^\alpha}$$

And to get a zero we solve for $\{u^\alpha_{I+\epsilon_i}\}_{|I|=k, i\in\{1,\dots,n\}}$ the system:

$$\sum_{|I|=k} \frac{\partial^{|I|} F}{\partial u_I^\alpha}(x^i, u^\alpha, u_I^\alpha) u^\alpha_{I+\epsilon_i} = -(\frac{\partial F}{\partial x^i} + \sum_{\alpha, 1 \le |I| < k} u^\alpha_{I+\epsilon_i} \frac{\partial^{|I|} F}{\partial u_I^\alpha})(x^i, u^\alpha, u_I^\alpha)$$

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Which, above $(x^i, u^\alpha, u_I^\alpha)$, is in fact a linear system of the form

$$\sum_{|I|=k} \frac{\partial^{|I|} F}{\partial u_I^\alpha} u^\alpha_{I+\epsilon_i} = a^\alpha_{I,i}$$

Lifting jet solutions

We want to prolong once more, so we differentiate again. Now, the $\frac{\partial F}{\partial x^i}$ and the $\frac{\partial F}{\partial u_I^\alpha}$ are actually function in $J^k\pi$, so they are constant in the $u^\alpha_{I+\epsilon_i}$, so we get:

$$
\begin{aligned}
D_{j,i}F &= \frac{\partial^2 F}{\partial x^j \partial x^i} + \sum_{\alpha, 1\leq |I| \leq k} u^\alpha_{I+\epsilon_i} \frac{\partial^{|I|+1} F}{\partial x^j \partial u_I^\alpha} \\
&+ \sum_{\alpha, 1\leq |J| \leq k} u^\alpha_{J+\epsilon_i} \frac{\partial^{|J|+1} F}{\partial u_J^\alpha \partial x^i} + \sum_{\alpha, 1\leq |I| \leq k, 1\leq |J| \leq k} u^\alpha_{I+\epsilon_i} \frac{\partial^{|I|+|J|} F}{\partial u_J^\alpha \partial u_I^\alpha} \\
&+ u^\alpha_{I+\epsilon_i+\epsilon_j} \sum_{|I|=k} \frac{\partial^{|I|} F}{\partial u_I^\alpha}
\end{aligned}
$$

Once more, if we want to lift the jet, we have a system:

$$
\sum_{|I|=k} \frac{\partial^{|I|} F}{\partial u_I^\alpha} u^\alpha_{I+\epsilon_i+\epsilon_j} = a^\alpha_{I,i,j}
$$

Lifting jet solutions

Conclusion: The behavior of the prolongations is dominated by the nature of the $\frac{\partial^{|I|} F}{\partial u_I^\alpha}$.

Involutivity

Consider a system of k-th order differential equations $\tilde{S}_k$. So it is, algebraicly, finitely generated (it is the zeroes of finitely many functions defined locally in $J^k\pi$). The system generates, differentialy, a reduced system of differential equations on $J\pi$, that is a coherent sheave of locally differential ideals $\mathscr{I}$. We set:

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Malgrange's generic involutivity theorem says in rough words:
For some q, S_q, outside a subvariety of codimension one, is locally nice
By nice we understand that any q-th order jet solution of S_q can be lifted infinitely many times and that the limit is a convergent power series.

I finish stating Malgrange's conditions implying that S_q is "nice". They are sufficient, but it is not known whether or not they are necessary (there is no counterexample showing this). The conditions are the definition of q-involutivity.

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The system $\mathscr{I}_q$ is called q-involutive if:

- S_q is smooth.
- M_q and M_{q+1} are locally free.
- for every $a \in S_q$, $M(a)$ is q-involutive.
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The involutivity of a graded module is a condition on the homogeneous homology of the Koszul complex of the module. Here the module M is a module translating the "nature" of the $\frac{\partial^{|I|} F}{\partial u_I^\alpha}$ we respect to the differential algebra of $\mathscr{O}_{J\pi}$.