Notes on Malgrange's General Involutivity Theorem [1]

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The setting is a fiber bundle $\pi : E \to M$ of manifolds modeled over $\mathbb{C}$ ($\dim M = n$, $\dim E = n + m$). Let $Z \subseteq J^k\pi$ be a locally closed embedded sub-manifold (a k-th order differential equation); and, fix $p \in Z$ together with a adapted chart $(U, (x^i, u^\alpha, u^\alpha_I))$ around p of $J^k\pi$. We denote $x^i(p) = p^i$, $u^\alpha(p) = q^\alpha$, $u^\alpha(p)_I = q^\alpha_I$.

Let $\mathscr{I}$ be the sheaf of ideals defined over U of sections of $\mathscr{O}_{J^k\pi}\restriction_U$ vanishing on $U \cap Z$. Note that if $F \in \mathscr{I}_p$ does not involve the u^α_I, $|I| = k$, then $D_iF \in I_p$.

1 Setting the necessary and sufficient conditions for a lifting of p to exist.

For $F \in \mathscr{I}_p$, set

$$D_iF(p) = \underbrace{\frac{\partial F}{\partial x_i}(p) + \sum_{\alpha,|I|<k} \frac{\partial F}{\partial u^\alpha_I}(p) q^\alpha_{I+\epsilon_i}}_{D'_iF(p)} + \underbrace{\sum_{\alpha,|I|=k} \frac{\partial F}{\partial u^\alpha_I}(p) u^\alpha_{I+\epsilon_i}}_{D''_iF(p)(u^\alpha_{I+\epsilon_i})},$$

where $D'_iF(p)$ is a complex number and $D''_iF(p)(u^\alpha_{I+\epsilon_i})$ is a linear form on the complex vector space with coordinates $\{u^\alpha_{I+\epsilon_i}\}_{i,\alpha,|I|=k}$.

The point $p \in Z$ (the k-th order jet solution to the differential equation) can be lifted to the first prolongation if and only if the system of equations

$$D''_iF(p)(u^\alpha_{I+\epsilon_i}) = -D'_iF(p), \quad F \in \mathscr{I}_p,\ i \in \{1, \dots, n\}$$

has a solution. Alternatively, if we take a set of generators of $\mathscr{I}_p$, $F_1, \dots, F_r$, then to solve the previous system of equation is equivalent the following system

$$D''_iF_j(p)(u^\alpha_{I+\epsilon_i}) = -D'_iF_j(p), \quad j \in \{1, \dots, r\},\ i \in \{1, \dots, n\}.$$

When one allows i to swipe the whole interval $\{1, \dots, n\}$ the association:

$$(u^\alpha_{I+\epsilon_i})_{i,\alpha,|I|=k} \longmapsto (D''_iF(p)(u^\alpha_{I+\epsilon_i}))_i$$

is a map from the complex vector space with coordinates $\{u^{\alpha}_{I+\epsilon_i}\}_{i,\alpha,|I|=k}$, into $\mathbb{C}^n$. So the system of equation has a solution if and only if each vectors $(D'_i F_j(p))_i$ is a zero of the linear forms annihilating the image of the map $(u^{\alpha}_{I+\epsilon_i}) \mapsto (D''_i F(p)(u^{\alpha}_{I+\epsilon_i}))$.

Namely, the system has a solution if and only if:

$$\sum_{i,j} \lambda^{i,j} D''_i F_j(p) = 0,\ \lambda^{i,j} \in \mathbb{C} \Longrightarrow \sum_{i,j} \lambda^{i,j} D'_i F_j(p) = 0$$

2 Writing the conditions in terms of Koszul complexes.

Denote by $A = \mathbb{C}[\xi_1, \dots, \xi_n]$ the ring of polynomial in n-variables over $\mathbb{C}$. We look at A as a ring graded by degree, and we put $T = A_0 = \mathbb{C}\xi_1 + \dots + \mathbb{C}\xi_n$.

We make from the free module of rank m over $\mathscr{O}_{J^k\pi,p}$ an A-module, B, by tensoring it. Explicitly:

$$\begin{aligned} B &= \left(\oplus_\alpha \mathscr{O}_{J^k\pi,p} \delta u^\alpha\right) \otimes_{\mathbb{C}} A \\ &\simeq \oplus_\alpha \mathscr{O}_{J^k\pi,p}[\xi_1, \dots, \xi_n] \delta u^\alpha \end{aligned}$$

where $\delta u^1, \dots, \delta u^m$ is a free basis. In particular, conveying the notation

$$\xi_I \delta u^\alpha =: \delta u^\alpha_I,$$

where I is a multi-index, B has a natural graded A-module structure

$$B = \bigoplus_{l \geq 0} \left(\oplus_{\alpha,|I|=l} \mathscr{O}_{J^k\pi,p} \delta u^\alpha_I\right)$$

Given $F \in \mathscr{O}_{J^k\pi,p}$ we define the (k-th order) symbol of F to be the element

$$\delta F = \sum_{\alpha,|I|=k} \frac{\partial F}{\partial u^\alpha_I} \delta u^\alpha_I$$

("the differential of F modulo the dx^i and the du^α_I, $|I| < k$").

Going back to the $D''_i F(p)$, we express the identity $\sum_{i,j} \lambda^{i,j} D''_i F_j(p) = 0$ in terms of the Koszul complex:

$$
\begin{aligned}
0 &= \sum_{i,j} \lambda^{i,j} D''_i F_j(p) \\
&\Updownarrow \\
0 &= \sum_{i,j} \lambda^{i,j} \left(\sum_{\alpha,|I|=k} \frac{\partial F_j}{\partial u^\alpha_I}(p) \delta u^\alpha_{I+\epsilon_i} \right) \\
&= \sum_{i,j} \lambda^{i,j} \left(\xi_i \sum_{\alpha,|I|=k} \frac{\partial F_j}{\partial u^\alpha_I}(p) \delta u^\alpha_I \right) \\
&= \sum_i \xi_i \sum_j \lambda^{i,j} \delta F_j(p) \\
&= d(\sum_i \xi_i \otimes \sum_j \lambda^{i,j} \delta F_j)(p)
\end{aligned}
$$

where d is the boundary operator in the Koszul complex $K_\bullet(\underline{\xi}, B)$:

$$
\begin{aligned}
d : \left(\bigwedge^p T \right) \otimes B &\longrightarrow \left(\bigwedge^{p-1} T \right) \otimes B \\
\xi_{i_1} \wedge \ldots \wedge \xi_{i_p} \otimes g &\longmapsto \sum_{j=1}^{p} (-1)^{j+1} \xi_{i_1} \ldots \wedge \widehat{\xi_{i_j}} \wedge \ldots \wedge \xi_{i_p} \otimes \xi_{i_j} g
\end{aligned}
$$

So that

$$
\sum_{i,j} \lambda^{i,j} D''_i F_j(p) = 0 \iff \sum_i \xi_i \otimes \left(\sum_j \lambda^{i,j} \delta F_j \right)(p) \in \mathrm{Z}[K_1(\underline{\xi}, B_k)(p)]
$$

On the other hand the identity $\sum_{i,j} \lambda^{i,j} D''_i F_j(p) = 0$ is equivalent to the fact that in the expression $\sum_{i,j} \lambda^{i,j} D_i F_j(p)$ the $u^\alpha_{I+\epsilon_i}$ vanishes, in particular

$$
\sum_{i,j} \lambda^{i,j} D'_i F_j(p) = \sum_{i,j} \lambda^{i,j} D_i F_j(p).
$$

Now we consider the situation over Z around p. So instead of B we consider

$$
B/\mathscr{I}_p = \bigoplus_{l \geq 0} \left(\oplus_{\alpha,|I|=l} \mathscr{O}_{Z,p} \delta u^\alpha_I \right)
$$

and we set N to be the sub-module of $B/\mathscr{I}_p$ generated by the (k-th order) symbols (mod $\mathscr{I}_p$) of the $F \in \mathscr{I}_p$. And we define the torsion of Z at p to be the map:

$$
\begin{aligned}
\tau_p : \mathrm{Z}[K_1(\underline{\xi}, N_k)] &\longrightarrow \mathbb{C} \\
\sum_i \xi_i \otimes \delta g_i &\longmapsto \sum_i D_i g_i(p)
\end{aligned}
$$

So the discussion above means p can be lifted if and only if $\tau_p \equiv 0$.

3 Two remarks

3.1

Let N' be the sub-module of generated $B/\mathscr{I}_p$ generated by the $k-1$st order symbols $\pmod{\mathscr{I}_p}$ of the $F \in \mathscr{I}_p$ not involving the u_I^α, $|I| = k$. Then:

Fact: *τ_p vanishes in the image of $d : \wedge^2 T \otimes N'_{k-1} \to T \otimes N_k$.*

Indeed

$$\begin{aligned} \tau_p(d(\xi_i \wedge \xi_j \otimes \delta g)) &= \tau_p(\xi_j \otimes \delta D_i g - \xi_i \otimes \delta D_j g) \\ &= (D_j D_i g - D_i D_j g)(p) = 0 \end{aligned}$$

3.2

Fact: *$H_p(K_\bullet(\underline{\xi}, N)(p)) = \oplus_l H_p(K_\bullet(\underline{\xi}, N_l)(p))$ is a finite dimensional complex vector space.*

So that $H_1(K_\bullet(\underline{\xi}, N_l)(p)) = 0$ for $l >> 0$; or equivalently (snake lemma), $H_2(K_\bullet(\underline{\xi}, M_l)(p)) = 0$ for $l >> 0$, where M is defined by:

$$0 \to N \to B/\mathscr{I}_p \to M/ \to 0$$

References

[1] B. Malgrange, Systèmes différentiels involutifs, *Panorama et Synthèses*, **19**, Société Mathématique de France, 2005, Paris.