Title: Algebraic group representations, and related topics

a lecture by Len Scott,
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Abstract: This lecture will survey the theory of algebraic group representations in positive characteristic, with some attention to its historical development, and its relationship to the theory of finite group representations. Other topics of a Lie-theoretic nature will also be discussed in this context, including at least brief mention of characteristic 0 infinite dimensional Lie algebra representations in both the classical and affine cases, quantum groups, perverse sheaves, and rings of differential operators. Much of the focus will be on irreducible representations, but some attention will be given to other classes of indecomposable representations, and there will be some discussion of homological issues, as time permits.

These notes cover four topics, only the first three of which were presented in the lecture;

1. Algebraic and finite groups: a historical perspective
2. Algebraic group representations, from Steinberg and Chevalley through the Lusztig conjecture and its solution for large primes; related topics from finite groups, quantum groups, classical and affine Lie algebra representations, D-modules and perverse sheaves
3. General topics in the representation theory of algebraic groups, including induced modules, injective modules, G-module structure on restricted enveloping algebra PIM (projectve indecomposable modules), tilting modules, tensor products, filtratrations by Weyl modules. Quasi-hereditary algebras.
4. Cohomoloyg and Ext for algebraic groups and related finite groups; Koszul algebras and structures.

CHAPTER VI from Borel's book Essays in the History of Lie Groups and Algebraic Groups

Linear Algebraic Groups in the 20th Century

The interest in linear algebraic groups was revived in the 1940s by C. Chevalley and E. Kolchin. The most salient features of their contributions are outlined in Chapter VII and VIII. Even though they are put there to suit the broader context, I shall as a rule refer to those chapters, rather than repeat their contents. Some of it will be recalled, however, mainly to round out a narrative which will also take into account, more than there, the work of other authors.

2.1. Chevalley's approach to algebraic groups in [C1] and [C2] uses a formal analog of the exponential mapping to set up a correspondence between Lie algebras and Lie groups with the usual properties. It is therefore tied to characteristic zero.

Completely different is the point of view of Kolchin [K1], [K2], 1948, who considered linear algebraic groups over an algebraically closed ground field of arbitrary characteristic and provided proofs insensitive to the characteristic, without any recourse to, nor even any mention of, the Lie algebra (VIII, §4). I cannot help drawing here an analogy with Lie: both were moved by the wish to establish some Galois theory of differential equations. In Lie's case, this hope, labeled Lie's "idée fixe" by Hawkins [H3], led to applications to differential equations which are interesting, but still minor in comparison with the wealth of uses of Lie groups in so many parts of mathematics. Similarly, the case of positive characteristic initiated by Kolchin found hardly any application in his Galois theory of homogeneous linear differential equations, or more generally of differential fields (his motivation), which is confined to characteristic zero; but it underwent a tremendous development in other directions.

11. In the notice on his own work [C22], written in Japan probably during the winter 1953-54, Chevalley gives the following motivation for studying algebraic groups over fields other than the complex numbers:

> The principal interest of the algebraic groups seems to me to be that they establish a synthesis, at least partial, between the two main parts of group theory, namely the theory of Lie groups and the theory of finite groups.

Whether this view was already his at the beginning I do not know, but it became foremost in his mind in the late forties. The model here was L. Dickson, who, taking advantage of the fact that the classical groups have an algebraic definition valid over general fields, had constructed new series of finite simple groups over finite fields. He had also done that for the exceptional group $\mathbf{G}_2$ (as well as for $\mathbf{E}_6$, but this was pretty much forgotten at the time and unknown to Chevalley). The task Chevalley set for himself was then to find models of the four other exceptional groups which would make sense over arbitrary fields and again lead to new simple groups. His joint paper with R.D. Schafer [C15] on $\mathbf{F}_4$ and $\mathbf{E}_6$ and his Comptes Rendus notes on $\mathbf{E}_6$ [C20, 21] are first steps in that direction. In [C22] Chevalley asserts that in the summer of 1953 he had found new algebraic definitions of $\mathbf{F}_4$, $\mathbf{E}_6$ and $\mathbf{E}_7$, by making use of the triality principle, and that these groups generate infinite series of simple groups, the first new ones in fifty years. No doubt he intended at that time to publish the proofs. In fact, he states in [C22] that in his small book on spinors [C25] he carries out a synthesis of the methods developed by Weyl and Cartan, and generalizes their results over arbitrary ground fields, a "generalization which was necessary in view of the study of the new finite groups I have discovered". But he never did. There was apparently a breakthrough shortly afterwards, and Chevalley saw how to carry this out in a uniform, classification-free manner. This leads us to the first major achievement in the second part of Chevalley's work (in the division proposed above), the very influential and justly famous "Tôhoku" paper [C28].

12. The next publication of Chevalley is the no less famous Paris Seminar [C30]. There, the framework and point of view are completely different from those of Lie II, III. The Lie algebra appears only briefly, in the last two lectures, and there is no exponential mapping. They are replaced by global arguments in algebraic geometry valid over an algebraically closed ground field of arbitrary characteristic. Since I am responsible for this change of scenery, I'll digress a little and discuss briefly my own work at that time and how it relates to Chevalley's.

I had been aware of [C9] and of Lie II early on, but from a distance. I got closer to algebraic groups during the first AMS Summer Institute in 1953, devoted to Lie groups and Lie algebras, through my joint work with G.D. Mostow [BM] and a series of lectures by Chevalley on Cartan subalgebras and Cartan subgroups of algebraic groups (the future Chapter VI of Lie III). He was not pleased with it, though, and toward the end said he felt it was too complicated and there should be a more natural approach valid in any characteristic. Another topic which came up in discussions was a claim by V.V. Morozov, to the effect that maximal solvable subalgebras of a complex semisimple Lie algebra are conjugate. Nobody understood his argument, but I found a simple global proof, using Lie's theorem and the flag variety. In 1954-55, in Chicago, I made a deliberate effort to get away from characteristic zero. Two papers by Kolchin [K1, 2], the first ones to prove substantial results on linear algebraic groups by methods insensitive to the characteristic of the ground field, led quickly to a structure theory of connected solvable groups (see VIII). Then I saw how to extend my conjugacy proof of maximal connected solvable subgroups to arbitrary characteristics. That was the decisive step. From then on, it was comparatively smooth sailing, and the other results of [B1] followed rather quickly. I lectured on this work and talked about it with Chevalley shortly afterwards, in February 1955 I think, at a conference in Urbana, Illinois. In summer 1955, before leaving the States, I gave him a copy of the manuscript of my forthcoming Annals paper. We did not discuss the subject until the Summer 1956 Bourbaki Congress, where I was told (not by him) that he had classified the simple algebraic groups over any algebraically closed ground field. When asked, he confirmed it and agreed to give us an informal lecture about this work, at which time he announced he would propose the name "Borel subgroup" for a maximal closed connected solvable subgroup of a linear algebraic group. He also told me that, after having read my paper, his first goal had been to prove the normalizer theorem: "A Borel subgroup of a connected linear algebraic group is its own normalizer", after which, "the rest followed by analytic continuation".

Part of the Chevalley bibliography from Borel's book

[C16] ———, *Algebraic Lie algebras and their invariants* (with H.F. Tuan), Acta Math. Sinica** **1** (1951), 215–242.

[C17] ———, *On a theorem of Gleason*, Proc. Amer. Math. Soc **2** (1951), 122–125.

[C18] ———, *Two proofs of a theorem on algebraic groups* (with E.R. Kolchin), Proc. Amer. Math. Soc. **2** (1951), 126–137; Kolchin, Sel. Wks., 159–167.

[C19] ———, Théorie des groupes de Lie II. Groupes algébriques, Hermann, Paris, 1951.

[C20] ———, *Sur le groupe exceptionnel* (E_6), C.R. Acad. Sci. Paris **232** (1951), 1991–1993.

[C21] ———, *Sur une variété algébrique liée à l'étude du groupe* (E_6), C.R. Acad. Sci. Paris **232** (1951), 2168 - 2170.

[C22] ———, *Notice sur les travaux scientifiques de Claude Chevalley*, (unpublished).

[C23] ———, *On algebraic group varieties*, J. Math. Soc. Japan **6** (1954), 303–324.

[C24] ———, Théorie des groupes de Lie III. Groupes algébriques, Hermann, Paris, 1954.

[C25] ———, The algebraic theory of spinors, Columbia University Press, New York, 1954; Collected Works II, 65–192.

[C26] ———, *The Betti numbers of the exceptional Lie groups* (with A. Borel), Memoirs Amer. Math. Soc **14** (1955), 1–9; Borel, Oeuvres I, 451–459.

[C27] ———, *Invariants of finite groups generated by reflections*, Amer. J. Math **77** (1955), 778–782.

[C28] ———, *Sur certains groupes simples*, Tôhoku Math. J. (2) **7** (1955), 14–66.

[C29] ———, *Plongement projectif d'une variété de groupe*, Proc. Internat. Sympos. Algebraic Number Theory (Tokyo and Nikko, 1955), Sci. Council of Japan, Tokyo, 1956, 131–138.

[C30] ———, Classification des groupes de Lie algébriques, 2 vols., Notes polycopiées, Inst. H. Poincaré, 1956-58.

[C31] ———, *Sur les décompositions cellulaires des espaces* G/B, in Algebraic Groups and Their Generalizations: Classical Methods, Proc. Sympos. Pure Math. **56**, Part 1, Amer. Math. Soc., 1994, 1–23.

[C32] ———, *La théorie des groupes algébriques*, Proc. Internat. Congr. Math. (Edinburgh, 1958), Cambridge Univ. Press, 1960, 53–68.

[C33] ———, *Une démonstration d'un théorème sur les groupes algèbriques*, J. Math. Pures Appl. (9) **39** (1960), 307–317.

[C34] ———, *Certains schémas de groupes semi-simples*, Sém. Bourbaki 1960/61, Exp. **219**, May 1961.

Steinberg, Robert

Variations on a theme of Chevalley.

Pacific J. Math. **9** 1959 875–891

The paper contains the discovery of two families of new simple groups, one of which seems to coincide with the family given independently by J. Tits [Séminaire Bourbaki, 10e année: 1957/1958, Secrétariat mathématique, Paris, 1958; MR0106247 (21 #4981)] from a different point of view. Roughly speaking, the groups are constructed by a method which generalizes that of constructing certain unitary groups.

Let $\mathfrak{g}$ be a simple Lie algebra over the complex field, and K a field. For each root r of $\mathfrak{g}$ and $t \in K$ define the automorphism $x_r(t)$ of the algebra $\mathfrak{g}_K$ as in Chevalley, Tôhoku Math. J. (2) **7** (1955), 14–66 [MR0073602 (17,457c)]. Let $\mathfrak{U}$ [resp. $\mathfrak{B}$] be the group generated by the $x_r(t)$ with $r > 0$ [resp. $r < 0$], and G the group generated by $\mathfrak{U}$ and $\mathfrak{V}$. If $\mathfrak{g}$ is of type (A_l), (D_l), or (E_6), the Dynkin diagram of $\mathfrak{g}$ shows that the root system admits an automorphism $r \to \overline{r}$ of order 2 such that $\overline{r} > 0$ whenever $r > 0$; G admits a corresponding automorphism σ which maps $x_r(t)$ upon $x_{\overline{r}}(\pm\overline{t})$, where $t \to \overline{t}$ is an automorphism of order 2 of K. Let $\mathfrak{U}^1$ [resp. $\mathfrak{V}^1$] be the group of elements in $\mathfrak{U}$ [resp. $\mathfrak{V}$] invariant by σ, and G^1 the group generated by $\mathfrak{U}^1$ and $\mathfrak{V}^1$. Then G^1 for (E_6) turn out to be new simple groups. If $\mathfrak{g}$ is of type (D_4), then the root system admits an automorphism of order 3 which, combined with an automorphism of K of order 3, again yields new simple groups. Another family of (new) infinite simple groups is also obtained from (D_4). The orders of the new finite simple groups are computed in an ingenious manner.

Reviewed by *Rimhak Ree*

Ree, Rimhak

A family of simple groups associated with the simple Lie algebra of type (G_2).

Amer. J. Math. **83** 1961 432–462

Suzuki [Proc. Nat. Acad. Sci. U.S.A. **46** (1960), 868–870; MR0120283 (22 #11038)] discovered a new class of finite simple groups. On observing the matric representation given by Suzuki, the present author and the reviewer independently noticed that each Suzuki group is made up of the fixed elements of a rather special involutary automorphism of the group of type C_2 (four-dimensional symplectic group) over a field of 2^{2n+1} elements and that similar automorphisms of groups of types G_2 and F_4 could be used to construct other classes of simple groups. In the present article, and in a comparison article devoted to groups of type F_4 [Amer. J. Math. **83** (1961), 401–420; MR0132781 (24 #A2617)], the author carries through the construction and obtains new simple groups, of which the finite ones have orders $q^3(q^3+1)(q-1)$ with $q = 3^{2n+1}$, and $q^{12}(q^6+1)(q^4-1) \times (q^3+1)(q-1)$ with $q = 2^{2n+1}$, and $n \geq 1$ in both cases. Each group of the first class has an order divisible by 8 but not by 16. First the author proves the existence of the involutary automorphism necessary for the construction, but this was already done in Séminaire C. Chevalley, 1956/58 [Secrétariat mathématique, Paris, 1958; MR0106966 (21 #5696)]. Then he obtains a Bruhat lemma and proves the simplicity of the constructed groups. Here the development closely follows that of the reviewer in his modification [Pacific J. Math. **9** (1959), 875–891; MR0109191 (22 #79)] of Chevalley's paper [Tôhoku Math. J. (2) **7** (1955), 14–66; MR0073602 (17,457c)]. In a final section the author determines the automorphisms of the new finite groups.

Reviewed by *R. Steinberg*

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Bulletin AMS article

A BRIEF HISTORY OF THE CLASSIFICATION OF THE FINITE SIMPLE GROUPS

RONALD SOLOMON

ABSTRACT. We present some highlights of the 110-year project to classify the finite simple groups.

1. THE BEGINNINGS

"Es wäre von dem grössten Interesse, wenn eine Uebersicht der sämmtlichen einfachen Gruppen von einer endlichen Zahl von Operationen gegeben werden könnte." ["It would be of the greatest interest if it were possible to give an overview of the entire collection of finite simple groups."] So begins an article by Otto Hölder in *Mathematische Annalen* in 1892 [Ho]. Insofar as it is possible to give the birthyear of the program to classify the finite simple groups, this would be it. The first paper classifying an infinite family of finite simple groups, starting from a hypothesis on the structure of certain proper subgroups, was published by Burnside in 1899 [Bu2]. As the final paper (the classification of quasithin simple groups of even characteristic by Aschbacher and S. D. Smith) in the first proof of the Classification Theorem for the Finite Simple Groups (henceforth to be called simply the Classification) will probably be published in the year 2001 or 2002, the classification endeavor comes very close to spanning precisely the 20th century.

★**The classification of quasithin groups. I.**
Structure of strongly quasithin K-groups.
Mathematical Surveys and Monographs, 111.
American Mathematical Society, Providence, RI, 2004. *xiv*+477 *pp. ISBN* 0-8218-3410-X

MR2097624 (2005m:20038b) 20D05 (20C20)
Aschbacher, Michael; **Smith, Stephen D.**
★**The classification of quasithin groups. II.**
Main theorems: the classification of simple QTKE-groups.
Mathematical Surveys and Monographs, 112.
American Mathematical Society, Providence, RI, 2004. *pp. i–xii and* 479–1221.
ISBN 0-8218-3411-8

In 1983, Danny Gorenstein announced the completion of the classification of the finite simple groups. All of the major constituent theorems were published by 1983 with one exception. This exception was at last removed and the classification has now been completed with the publication of the two monographs under review. These volumes, classifying the quasithin finite simple groups of even characteristic, are thus a major milestone in the history of finite group theory. The statement of the main theorem requires certain definitions. (omitted, as are most review details...)

The book is written with remarkable care and clarity, especially in view of its extraordinary length and depth. It is an amazing tour de force. Volume I has been read carefully by John Thompson. Volume II has been checked by a team of referees.

REVISED (October, 2005)

Current version of review. Go to earlier version.

Reviewed by *Ronald Solomon*

List of finite simple groups

From Wikipedia, the free encyclopedia

In mathematics, the classification of finite simple groups states that every finite simple group is cyclic, or alternating, or in one of 16 families of groups of Lie type (including the Tits group, which strictly speaking is not of Lie type), or one of 26 sporadic groups.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra. A linear form λ on $\mathfrak{h}$ is a *weight* of a representation $\pi\colon \mathfrak{g} \to \mathfrak{gl}(V)$ if

Work of E. Cartan 1913

$$V_\lambda = \{v \in V | \pi(h)v = \lambda(h)\ (h \in \mathfrak{h})\} \neq 0.$$

The space V is always the direct sum of the V_λ. The weights of all finite dimensional representations generate a lattice P in the smallest $\mathbb{Q}$-subspace $\mathfrak{h}^*_{\mathbb{Q}}$ of $\mathfrak{h}^*$ spanned by them, which is a $\mathbb{Q}$-form of $\mathfrak{h}^*$. A nonzero weight of the adjoint representation is a *root*. The roots generate a sublattice Q of P. Let $\mathfrak{h}^*_{\mathbb{R}}$ be the real span of P. For each root α there is a unique automorphism s_α of order 2 of $\mathfrak{h}^*_{\mathbb{R}}$ leaving the set of roots stable, transforming α to $-\alpha$ and having a fixed point set of codimension one.

Fix a set Δ of "simple roots", i.e. $l = \dim \mathfrak{h}$ linearly independent roots such that any other root is a integral combination of the $\alpha \in \Delta$ with coefficients of the same sign, and call positive those with positive coefficients. Introduce a partial ordering among the weights by saying that $\lambda \geq \mu$ if $\lambda - \mu$ is a positive linear combination of simple roots. Then $\lambda \in P$ is said to be dominant if $\lambda \geq s_\alpha \lambda$ for all simple α's. Let P^+ be the set of dominant weights. [The group W of automorphisms of $\mathfrak{h}^*_{\mathbb{R}}$ generated by the s_α is one realization of the group (S) introduced by Weyl in

Let π be irreducible. Cartan showed first that it has a unique highest weight λ_π, i.e. a weight which is greater than any other weight. This weight has multiplicity one. The main result of [**C37**] is that any $\lambda \in P^+$ is the highest weight of one and only one (up to equivalence) irreducible representation. The dominant weights are

1899 Maschke's theorem

$V \subseteq M$

$M = V \oplus V^\perp$

M has a G-invariant positive definite Hermitian form

1902 Schur

complete reducibility

1920's Schur & Hurwitz

began applying to Lie group

compact classical groups

The key point was to show that the "unitary restriction" could be applied in the general situation. This was done in two steps: first Weyl showed that a given complex semisimple Lie algebra $\mathfrak{g}$ has a "compact real" form $\mathfrak{g}_u$, i.e., a real Lie subalgebra such that $\mathfrak{g} = \mathfrak{g}_u \otimes_{\mathbb{R}} \mathbb{C}$, on which the restriction of the Killing form is negative non-degenerate.[9] For instance, if $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ one can take for $\mathfrak{g}_u$ the Lie algebra of $\mathbf{SU}_n$. To this effect, Weyl first had to outline the general theory of semisimple Lie algebras, for which the only sources until then were the papers of W. Killing and Cartan's thesis, all extremely hard to read, and then had to prove the existence of $\mathfrak{g}_u$ by a subtle argument using the constants of structure. Already this exposition, which among other things stressed the importance of a finite reflection group (S), later called the Weyl group, was a landmark, and for many years it was the standard reference. But there it was really only preliminary material. Identify $\mathfrak{g}$ to a subalgebra of $\mathfrak{gl}(\mathfrak{g})$ by the adjoint representation, which is possible since $\mathfrak{g}$, being semisimple, is in particular centerless, and let G^0 be the complex subgroup of $GL(\mathfrak{g})$ with Lie algebra $\mathfrak{g}$. It leaves invariant the Killing form $K_{\mathfrak{g}}$ defined by $K_{\mathfrak{g}}(x, y) = \mathrm{tr}(\mathrm{ad}\, x \circ \mathrm{ad}\, y)$, which is non-degenerate by a result of Cartan. Then let G^0_u be the real Lie subgroup of G^0 (viewed now as a real Lie group) with Lie algebra $\mathfrak{g}_u$. Since the restriction K_u of $K_{\mathfrak{g}}$ to $\mathfrak{g}_u$ is negative non-degenerate, it can be viewed as a subgroup of the orthogonal group of K_u, hence is compact.[10].

This is a situation to which the Schur-Hurwitz device can be applied; therefore any finite dimensional representation π of $\mathfrak{g}$ which integrates to a representation of G^0 is fully reducible. However, in general, a representation π of $\mathfrak{g}$ will integrate to a representation not of G^0, but of some covering group G_π of G^0. In the latter, there

$$A(\lambda) = \sum_{w \in W} (\det w) w \cdot \lambda \qquad (\lambda \in X(T)).$$

Weyl showed that for every continuous class function f on K we have

$$\int_K f(k)dk = \int_T f(t)\mu(t)dt, \tag{6}$$

where dk (resp. dt) is the invariant measure on K (resp. T) with mass 1 and

$$\mu = |W|^{-1}|A(\rho)|^2 \qquad (|W| \text{ the order of } W), \tag{7}$$

a point which Schur had singled out in his praise of Weyl's results. From this and the orthogonality relations for characters of K or of T, Weyl deduced that every irreducible character is of the form

rho is the half
sum of pos. roots

$$\chi_\pi = A(\lambda_\pi + \rho) \cdot A(\rho)^{-1}, \tag{8}$$

where λ_π is dominant.[12] He did not show, however, but derived from Cartan's work, that every dominant λ occurs in this way.[13] I shall soon come back to this problem. He also deduced from (8) a formula for the degree $d^\circ\pi$ of π:

$$d^\circ\pi = \prod_{\alpha>0} \frac{\langle \rho + \lambda_{\pi,\alpha} \rangle}{\langle \rho, \alpha \rangle}, \tag{9}$$

where α runs through the positive roots and $\langle\ ,\ \rangle$ is a scalar product invariant under the Weyl group. Both (8) and (9) were not at all to be seen from Cartan's construction of irreducible representations.

For any weight μ we have a Verma module $M(\mu)$ free of rank 1 over $U(\mathfrak{g}^-)$ character of $M(\mu)$ known

$$\operatorname{ch} L(\lambda) = \sum_{w \in W} (-1)^{\ell(w)} \operatorname{ch} M(w.\lambda)$$

$w.\lambda = w(\lambda+\rho)-\rho$ $\quad \rho = \frac{1}{2}$ sum of all positive roots

λ dominant

§3. Impact on É. Cartan

5. Among many things, these papers mark the birthdate of the systematic global theory of Lie groups. The original Lie theory, created in 1873, was in principle local, but during the first fifty years, global considerations were not ruled out, although the main theorems were local in character. However, a striking feature here was that algebraic statements were proved by global arguments, which moreover, seemed unavoidable at the time.[14] Weyl had not bothered to define the concepts of Lie group or of universal covering (the latter being already familiar to him in the context of Riemann surfaces).[15] He had just taken them for granted, but could of course lean on the examples of the classical groups, which had been known global objects even back in the early stages of Lie theory.

6. These papers had a profound impact on Cartan. He had first known Weyl's

This concluded the historical development of algebraic groups themselves, together with the finite groups of Lie type.

The rest of the lectures deal with the representations of algebraic groups, and of related finite groups, and other related algebraic objects. The first part of this is focused on irreducible modules.

The beginnings of this immitated Cartan's highest weight theory, though the existence proofs were different, and is in Seminaire Chevalley. It is omitted. We begin, insteady, with Steinberg's 1963 Nagoya paper.

The Steinberg tensor product theorem

1.1 THEOREM. *Let G be a semisimple algebraic group of characteristic $p \neq 0$ and rank l, and let $\Re$ denote the set of p^l irreducible rational projective representations of G in each of which the high weight λ satisfies $0 \leq \lambda(a) \leq (p-1)$ $(a \in S)$. Let α_i denote the automorphism $t \to t^{p^i}$ of the universal field as well as the corresponding automorphism (see § 5) of G, and for $R \in \Re$ let R^{α_i} denote the composition of α_i and R. Then every irreducible rational projective representation of G can be written uniquely as $\prod_{i=0}^{\infty} R_i^{\alpha_i}$ (weak tensor product, $R_i \in \Re$).*

Coversely, each such tensor product is irreducible.

Nagoya Math J 1963

Received May 21, 1962.

* This research was supported by the Air Force Office of Scientific Research.

Steinberg proves a similar formula for the finite groups of Lie type, obtaining all their irreducible modules by restriction

Example

$SL(3, \mathbb{F}_{p^2})$ $p=2$

restricted irreducibles for $SL(3, \mathbb{F}_p)$

λ one of these

$$1\omega_1 + 0\omega_2$$
$$0\omega_1 + 1\omega_2$$
$$1\omega_1 + 1\omega_2$$

Three irreducible $L(\lambda)$, λ restricted

λ_0, λ_1 be each λ as above

All irred $SL(3, \mathbb{F}_{p^2})$ modules are of the form

$$L(\lambda_0 + p\lambda_1)$$
$$\simeq L(\lambda_0) \otimes L(\lambda_1)^{\mathbb{F}_r}$$

There are 9 of them

Verma, Daya-Nand

The rôle of affine Weyl groups in the representation theory of algebraic Chevalley groups and their Lie algebras.

Lie groups and their representations (*Proc. Summer School, Bolyai János Math. Soc., Budapest*, 1971), *pp.* 653–705. *Halsted, New York*, 1975.

The author formulates a number of very interesting conjectures concerning (1) affine Weyl groups and Weyl's dimension polynomial for a root system Δ, and (2) representations of a split simply connected semisimple algebraic group G having Δ as root system, as well as representations of its Lie algebra $\mathfrak{g}_K$, when the field of definition K has prime characteristic p. The two parts of his paper are devoted to these respective subjects (and the relations between them), providing a good deal of helpful expository material as well as some original observations. Probably the author's most significant contribution is to focus attention on the unifying role of affine Weyl groups in the modular representation theory of G, where the results seemed at one time to have little coherence. The paper reached its present form in late 1972, but some footnotes added in proof take account of later developments. By now most of the author's specific conjectures have in fact been settled affirmatively, largely through the efforts of S. G. Hulsurkar, J. C. Jantzen, and the author himself, but some fundamental questions about modular representations remain unanswered (cf. the reviewer's paper [*Ordinary and modular representations of Chevalley groups*, Lecture Notes in Math., Vol. 528, Springer, Berlin, 1976], for further details).

Weyl char formula

ch $L(\lambda)$ ch $M(w\cdot\lambda)$

known

Orbits

$W\cdot\lambda$ λ integral parameterizes

the weights of irreducibles

in a given block of inf. dim

irred Lie alg modules

$L(\mu), L(\lambda)$ in same block $\Rightarrow$

$\mu \in W\cdot\lambda$

(HC using the center of the universal env. algebra)

Conj 2

When $ch L(\lambda)$

is written in terms

of ch $V(\mu)$'s (or $M(\mu)$'s)

then the coefficients

depend only on W_p

(not on p, per se)

Requires p at least

modestly large

Char p

Use the affine Weyl group

$$\mathbb{Z}\underline{\Phi} \rtimes W$$

$$\cong$$

$$p\mathbb{Z}\underline{\Phi} \rtimes W = W_p$$

<u>Conjecture</u> The blocks in char p correspond to $W_p \cdot \lambda$

λ any dominant weight

$M(\lambda) \twoheadrightarrow V(\lambda)$ Weyl module

" reduction mod p^n of a lattice in char 0 irreducible $L(\lambda)$

Conje

MR564523 (81b:14006) 14F05 (20G05 22E45)

Andersen, Henning Haahr

The strong linkage principle.

J. Reine Angew. Math. **315** (1980), 53–59.

Steve Doty has a shorter more elementary proof.

Let G denote a connected reductive algebraic group over an algebraically closed field of prime characteristic P, T a maximal torus in G, R_+ a system of positive roots for the roots of (G, T), and let $M_{\lambda_1}, M_{\lambda_2}$ be irreducible G-modules with highest weights λ_1, λ_2, relative to the ordering in R_+, which both occur as composition factors of an indecomposable G-module. Then the linkage principle (proved for some G and P) asserts that λ_1, λ_2 are linked in the sense that they lie in the same affine Weyl group orbit. Given the choice R_+ there is a natural notion of strong linkage of two given characters λ_1, λ_2 of T by a finite sequence of characters. The author proves (for all P) a strong linkage principle for the cohomology modules of induced homogeneous line bundles on G/B, where $B \supset T$ is a Borel subgroup, and deduces as a corollary a strong linkage principle for Weyl modules. It is shown moreover, by example, that for G of type B_2, some higher cohomology groups indeed can be decomposable as G-modules.

Reviewed by *Floyd L. Williams*

MR560412 (81j:20066) 20H15 (17B35 20G05 22E47)

Kazhdan, David; **Lusztig, George**

Representations of Coxeter groups and Hecke algebras.

Invent. Math. **53** (1979), *no. 2*, 165–184.

In this paper of exceptional importance, the authors have been able to "explain" many combinatorial phenomena that are associated to a Weyl group in different contexts. The key discovery is that of a set of certain polynomials $P_{x,y}$ in one variable with integral coefficients associated to a pair (x, y) of elements of a Coxeter group W. These polynomials are used extensively to (1) construct certain representations of the Hecke algebra of W thereby obtaining important information on representations of W, (2) give a formula (conjecturally) for the multiplicities in the Jordan-Hölder series of Verma modules or equivalently for the formal characters of irreducible highest weight modules (extending the celebrated Weyl character formula to "nondominant" weights), (3) Kazhdan-Lusztig conjectured formula

$$\operatorname{ch} L(w(\lambda+\rho)-\rho) = \sum_{y \le w} (-1)^{l(w)-l(y)} P_{y,w}(1) \operatorname{ch} M(y(\lambda+\rho)-\rho)$$

lambda in same "alcove" as -2 rho

where $L(\mu)$ and $M(\mu)$ denote the irreducible module and the Verma module, respectively, with highest weight μ, and $P_{y,w}(q)$ denotes the Kazhdan-Lusztig polynomial.

MR0604598 (82i:20014)

Lusztig, George

Some problems in the representation theory of finite Chevalley groups. *The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979)*, pp. 313--317, Proc. Sympos. Pure Math., 37, *Amer. Math. Soc., Providence, R.I.*, 1980.

20C15 (20C20 20D06)

Find@UVa

Author's introduction: ``In the first section of this paper, I present a classification of the unipotent (complex) representations of a finite Chevalley group and state a conjecture on their character values. The second section is a review of results of D. Kazhdan and myself [Invent. Math. **53** (1979), no. 2, 165--184; MR0560412 (81j:20066); *Geometry of the Laplace operator\/* (Honolulu, Hawaii, 1979), pp. 185--203, Amer. Math. Soc., Providence, R.I., 1980]; these lead to some questions which are formulated in the third section. In particular, I state a conjecture on the character of modular representations of a finite Chevalley group.''

Kato version: Use the Kazhdan-Lusztig formula for irreducible modules with regular high weights which are restricted. Remark: There are no regular weights at all unless p is at least the Coxeter number.

Brylinski, J.-L.; **Kashiwara, M.**

Kazhdan-Lusztig conjecture and holonomic systems.

Invent. Math. **64** (1981), *no. 3*, 387–410.

It is probably fair to say that during the last decade the single most important problem in the representation theory for a semisimple complex Lie algebra $\mathfrak{g}$ has been the problem of determining the composition factor multiplicities in Verma modules. Before the Kazhdan-Lusztig conjecture one had only been able to solve this problem in low rank and in some special cases [see J. C. Jantzen, *Modules with a highest weight* (German), Lecture Notes in Math., 750, Springer, Berlin, 1979; MR0552943 (81m:17011)]. D. Kazhdan and G. Lusztig [Invent. Math. **53** (1979), no. 2, 165–184; MR0560412 (81j:20066)] introduced certain polynomials $P_{y,w}$, one for each pair (y, w) of elements in the Weyl group W for $\mathfrak{g}$ and the Kazhdan-Lusztig conjecture states that the values at 1 of these polynomials give the above-mentioned multiplicities. These polynomials are introduced as the coefficients of certain "universal" elements in the Hecke algebra of W but they may also be described via an explicit algorithm. Moreover, Kazhdan and Lusztig in a subsequent paper [*Geometry of the Laplace operator* (Honolulu, Hawaii, 1979), pp. 185–203, Proc. Sympos. Pure Math., XXXVI, Amer. Math. Soc., Providence, R.I., 1980] showed that the polynomials also have an interpretation in terms of the so-called middle intersection cohomology developed by M. Goresky, R. MacPherson and P. Deligne. "Perverse sheaves" on G/B (Asterisque 100, BBD)

In the paper under review the Kazhdan-Lusztig conjecture is proved using this last characterization of the $P_{y,w}$'s. The authors set up an equivalence between a category consisting of certain holonomic systems with regular singularities on the flag manifold (for the group corresponding to $\mathfrak{g}$) and a subcategory of the category of $\mathfrak{g}$-modules. In this way they get associated to each simple $\mathfrak{g}$-module L a sheaf $\mathcal{L}$ and the character of L is then computable via the complex $\mathbf{R}\mathrm{Hom}_{\mathcal{D}}(\mathcal{O}, \mathcal{D} \otimes \mathcal{L})$ where $\mathcal{D}$ denotes the sheaf of differential operators on the flag manifold. The proof is then completed by proving that this complex coincides (up to a shift) with the intersection cohomology complex associated to a Schubert variety.

A. Beilinson and J. Bernstein have independently proved the Kazhdan-Lusztig conjecture using a similar method [C. R. Acad. Sci. Paris Sér. I Math. **292** (1981), no. 1, 15–18; MR0610137 (82k:14015)]. O. Gabber and A. Joseph proved [MR0644519 (83e:17009)above] that the same conjecture for Verma modules corresponding to not necessarily integral weights is a consequence of an earlier conjecture of Jantzen. Jantzen's conjecture has now been proved by J. Bernstein (unpublished).

Reviewed by *H. H. Andersen*

$$
\begin{tikzcd}
 & \mathcal{D}\text{-modules} \arrow[dr] & \\
\tilde{\mathcal{O}}_{triv} \arrow[ur] & & \text{Perverse sheaves}
\end{tikzcd}
$$

$W_{\cdot -2\rho}$

$\shortparallel$

$W \cdot 0$

block

$$M(w \cdot -2\rho) \longrightarrow \mathbb{C}_{BwB/B}[\ell(w)]$$

extended by zero

$\mathcal{D}$ = differential operators on G/B

$$M \to \mathcal{D} \otimes_U M$$

$$U \rightsquigarrow \mathcal{D}$$

Kashiwara, Masaki (J-KYOT-R); Tanisaki, Toshiyuki (J-HROSE-M2)

Kazhdan-Lusztig conjecture for affine Lie algebras with negative level.

Duke Math. J. **77** (1995), *no. 1,* 21–62.

There is a remarkable connection between the theory of D-modules on a flag manifold and characters of irreducible highest weight modules for Kac-Moody Lie algebras. The classical Weyl character formula and its generalization by V. G. Kac to symmetrizable Kac-Moody algebras determine the characters when the highest weight is dominant integral. A major contribution to understanding characters of other highest weight modules for finite-dimensional semisimple Lie algebras was made by Kazhdan-Lusztig in 1979, when they conjectured a character formula and suggested its relation to intersection cohomologies of Schubert varieties. The conjecture was independently settled by Beĭlinson-Bernstein and by Brylinski-Kashiwara using D-module theory. There are two cases involved in the generalization to Kac-Moody Lie algebras because dominant and anti-dominant weights are not conjugate under the Weyl group. The dominant case was treated by Kashiwara, Kashiwara-Tanisaki, and independently by Casian, for symmetrizable Kac-Moody Lie algebras, generalizing methods used in the finite-dimensional case to deal with infinite-dimensional flag manifolds and infinite-dimensional Schubert varieties. This paper concerns the anti-dominant case for affine Kac-Moody Lie algebras, where Lusztig gave a precise conjecture

MR1239507 (94g:17049) 17B67

Kazhdan, D. (1-HRV); **Lusztig, G.** (1-MIT)

Tensor structures arising from affine Lie algebras. IV.

J. Amer. Math. Soc. **7** (1994), *no. 2,* 383–453.

In this, the last paper of this series, the authors prove the main result, which gives an equivalence of the tensor category of the representations of an affine Kac-Moody algebra and tensor category of finite-dimensional representations of the corresponding quantum group. The proof relies on the three previous papers [Parts I and II, J. Amer. Math. Soc. 6 (1993), no. 4, 905–947, 949–1011; MR1186962 (93m:17014); Part III, J. Amer. Math. Soc.; see the preceding review].

Reviewed by *Ya. S. Soĭbel′man*

Quantum enveloping algebra at a pth root of unity representations with dominant integral weight

$\xleftarrow{\quad KL \quad}$

some restriction apply

Affine Lie algebra representations at negative level $-p-h$

$p > 0$

KT

perverse sheaves

$q \to 1$

mod p

algebraic groups

MR1272539 (95j:20036) 20G05 (17B37 17B45)

Andersen, H. H. [Andersen, Henning Haahr] (DK-ARHS-MI); **Jantzen, J. C.** (1-OR); **Soergel, W.** (D-FRBG)

$p \gg h$

Representations of quantum groups at a pth root of unity and of semisimple groups in characteristic p: independence of p. (English, French summaries)

restricted weight $L(\lambda)$'s

reduce Irreducibly mod p

Astérisque No. 220 (1994), 321 *pp*.

Fix an indecomposable finite root system R. Associate to R two families of objects: (1) For any odd integer $p > 1$ (prime to 3 if R is of type G_2) let U_p be the quantized enveloping algebra (Lusztig's version constructed via divided power) at a pth root ε of unity (with $\mathbf{Q}(\varepsilon)$ as the ground field); (2) for any prime p let G_p be the semisimple connected and simply connected algebraic group over an algebraically closed field of characteristic p.

Note also that the result (III) (or (IV)) implies that one gets for $\operatorname{char} K \gg 0$ each simple G_k-module with restricted highest weight by reduction modulo p from a simple U_p-module, as conjectured by Lusztig. It also implies that his conjecture on algebraic groups G_k follows for $\operatorname{char} k \gg 0$ from his conjecture on quantized enveloping algebras U_k. Unfortunately, the arguments in this paper do not yield reasonable bounds.

Reviewed by *Jian-Pan Wang*

Dipper, Richard (1-OK); **James, Gordon** (4-LNDIC)

The q-Schur algebra.

Proc. London Math. Soc. (3) **59** (1989), *no. 1,* 23–50.

In this paper the authors consider algebras $\mathcal{S}$ which depend on a parameter q. They call these algebras the q-Schur algebras. They study the decomposition matrices of $\mathcal{S}$ for those primes p which do not divide q. Their main theorem asserts the following: When q is a prime power and p is a prime not dividing q then the p-modular decomposition matrices of the q-Schur algebras determine the p-modular decomposition matrices for $\mathrm{GL}_n(q)$. An important consequence of this result is the fact that the decomposition matrix of $\mathrm{GL}_n(q)$ for the prime dividing q is determined by the decomposition matrices for those which do not divide q.

This is a long, technical paper but one which is full of elegant, significant results. It is well worth taking the time to read and understand.

Reviewed by *Philip Hanlon*

Schur 1901 $GL(V)$ $V^{\otimes r}$ $\mathfrak{S}(r)$

Jimbo

There is a quantum analog

$S(n,r)$ Schur algebra Dipper James

centralizer

$S_q(n,r)$ centralizer for Hecke algebra.

There was additional discussion of q-tensor space and q-Schur algebras in the afternoon discussions. The q-Schur algebras were mentioned here with the intention to illustrate how irreducible representations of a given algebra over the integers or with a root of unity adjoined can
have their irredcible representations in characterisc 0 and in characteristic p have the same dimension for large primes p of unknown size, cominging about from clearing denominators in matrix respresentations of the char. 0 algebra.

There was also additional discussion q-tensor space in the afternoon discussions, and of Schur's methods in his 1901 dissertation.

There was also an intention in the main lecture to discuss further cross-characteristic (or non-defining, or non-describing) representations of finte groups of Lie type. There was, however, no time for this, though there was some brief further discussion of the Dipper-James work in the afternoon.

MR1664998 (99k:20029) 20C33 (20C30 20E28 20G05 20G40)

Scott, Leonard (1-VA)

Linear and nonlinear group actions, and the Newton Institute program.

Algebraic groups and their representations (*Cambridge*, 1997), 1–23, *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, 517, *Kluwer Acad. Publ., Dordrecht*, 1998.

The author surveys in some detail what is known or conjectured about modular representations of finite groups of Lie type, in both defining and nondefining characteristics. He also explains how these two settings are interrelated and how representation theory is connected with the determination of maximal subgroups. The study of these finite groups involves a rich mixture of ingredients: representations of algebraic groups, q-Schur algebras, Hecke algebras, representations of symmetric groups, etc. The extensive list of references reflects the wide range of contributions to these problems.

This next part of the lecture deals with more general aspects of representation theory, especially various special indecomposalble modules, and tensor products.

The last few pages on cohomology and Ext were not included in the lecture, for lack of time,

MR0470094 (57 #9861) 20G05

Cline, Edward; **Parshall, Brian**; **Scott, Leonard**

Induced modules and affine quotients.

Math. Ann. **230** (1977), *no. 1*, 1–14.

Let H be a closed subgroup of an affine algebraic group G over an algebraically closed field k. The authors prove the equivalence of the following conditions: (a) H is exact in G (the functor $\mathrm{ind}_H{}^G$ on rational modules preserves short exact sequences); (b) the quotient variety G/H is affine; (c) H is strongly observable in G (every rational H-module V can be embedded in suitable G-module W so that $V^H = W^G$); (d) $R(G)$ is a rationally injective H-module; (e) there is a nonzero rational G-module which is rationally injective as an H-module; (f) every rationally injective G-module is rationally injective as an H-module. The principal result is the equivalence of (a) and (b). For reductive G, this means that H is also reductive.

Reviewed by *Chin-han Sah*

Induced modules,
definition:

$$H \leq G \quad \text{closed subgroup}$$

$$V \qquad H\text{-module}$$

$$V\big|^G := \mathrm{Hom}_H(\mathcal{O}(G), V)$$

There is also a vector bundle interpretation:
global sections of

$$G \times_H V$$

over G/H

· Induced modules

If $H = B^-$

λ dominant

$k(\lambda)$ 1-dim B^- module

Then

$$k(\lambda)\Big|^G$$

is the "dual" of $V(\lambda)$

(same comp factors)

The tensor identity

General H

If V is a G-module

M is an H-module

$$M\big|^G \otimes V \cong (M \otimes V)\big|^G$$

MR611465 (84e:20048) 20G05

Donkin, Stephen

A filtration for rational modules.

Math. Z. **177** (1981), *no. 1*, 1–8.

Such a filtration is now called a "good" filtration, or "costandard" filtrationl

From the introduction: “Let G be a semisimple, simply connected affine algebraic group over k, an algebraically closed field of prime characteristic p. We show that the rationally injective indecomposable G modules have a filtration of submodules such that each successive quotient is isomorphic to a module induced from a one-dimensional representation of a Borel subgroup. Moreover, we show that if I is the rationally injective envelope of the simple module L and Y an induced module of the above type, then the number of successive quotients isomorphic to Y in such a filtration is equal to the composition multiplicity of L in Y. This reciprocity is analogous to the reciprocity of I. N. Bernsteĭn, I. M. Gel′fand and S. I. Gel′fand in the category O [Funct. Anal. Appl. **10** (1976), 87–92; MR0407097 (53 #10880)]. The proof given here of the existence of the filtration is valid for an arbitrary prime p and is largely a matter of bookkeeping using G. Hochschild’s rational cohomology [Illinois J. Math. **5** (1961), 492–519; MR0130901 (24 #A755)]. An invaluable aid in its application is a result of E. Cline et al. [Invent. Math. **39** (1977), 143–163; MR0439856 (55 #12737); MR errata, EA **57**].”

MR510324 (80b:20053) 20G05 (17B45)

Review by J.P. Wang

Ballard, John W.

Injective modules for restricted enveloping algebras.

Math. Z. **163** (1978), *no. 1*, 57–63.

Let G be a simply connected simple algebraic group, split over $\mathbf{F}_p$, and let $\mathbf{u}_n$ be the hyperalgebra of its nth Frobenius kernel; $\mathbf{u}_1$ is the restricted universal enveloping algebra of the Lie algebra of G. Under the hypothesis $p \geq 3h-3$, where h is the Coxeter number of the Weyl group of G, the author exhibits each injective indecomposable $\mathbf{u}_n$-module as a G-summand of a suitable tensor product involving the corresponding Steinberg module St_n. This was partially proved by the reviewer [*Ordinary and modular representations of Chevalley groups*, Lecture Notes in Math., Vol. 528, Springer, Berlin, 1976; MR0453884 (56 #12137)], and is still conjectured to be true for arbitrary p. {Recently the hypothesis has been weakened to $p \geq 2h-2$ by J. C. Jantzen ["Darstellungen halbeinfacher Gruppen und ihrer Frobenius-Kerne", preprint, Univ. Bonn, Bonn, Consequences for G-injectives

$$Q(\lambda) = Q_0(\lambda_0) \otimes Q_1(\lambda_1)^{Fr} \otimes Q_2(\lambda_2)^{Fr^2} \otimes \cdots$$

Socle

$L(\lambda)$

$$\lambda = \lambda_0 + p\lambda_1 + p^2\lambda_2 + \cdots$$

each λ_i restricted

Good and cogood filtrations

$Q_0(\lambda_0)$ has a good filtration
(Induced $k_{\lambda}|_{B}$ $\big[\,{}^{G}_{B}\,\big)$

and a cogood filtration
by $V(\mu)$'s

first observed
"Tilting modules"

high weight:

$Q_0(\lambda_1)$ has a unique high weig

$2(p-1)y + w_0 > 0$

MR933417 (89c:20062a) 20G05 (16A64 18E30)

Scott, Leonard L. (1-VA)

Simulating algebraic geometry with algebra. I. The algebraic theory of derived categories.

The Arcata Conference on Representations of Finite Groups (*Arcata, Calif.*, 1986), 271–281, *Proc. Sympos. Pure Math.*, 47, *Part* 2, *Amer. Math. Soc.*, *Providence*, *RI*, 1987.

MR933416 (89c:20062b) 20G05 (18E30)

Parshall, Brian J. (1-IL)

Simulating algebraic geometry with algebra. II. Stratifying representation categories.

MR933407 (89c:20062c) 20G05 (18E30)

Cline, Ed (1-CLRK)

Simulating algebraic geometry with algebra. III. The Lusztig conjecture as a TG_1-problem.

The Arcata Conference on Representations of Finite Groups (*Arcata, Calif.*, 1986), 149–161, *Proc. Sympos. Pure Math.*, 47, *Part* 2, *Amer. Math. Soc.*, *Providence*, *RI*, 1987.

In paper I, after describing the motivation and background for their work, Scott surveys the algebraic theory of derived categories. As an example he illustrates how the so-called tilting modules give rise to equivalences of derived categories of module categories for certain finite-dimensional algebras. 1st appearance of "quasi-hereditary" algebras.

In paper II Parshall discusses a stratification of certain derived categories associated to the category of G-modules. This—as in fact the whole program—is based on a fundamental work by A. Beĭlinson, J. N. Bernstein and P. Deligne [in *Analyse et topologie sur les espaces singuliers*, *I* (Luminy, 1981), 5–171, Astérisque, 100, Soc. Math. France, Paris, 1982; MR0751966 (86g:32015)], although the authors' point of view is less geometric and more algebraic (this being the reason for the title of I–III). A surprising application gives (in the special case $G = \mathrm{GL}_n$) that the classical Schur algebras have finite global dimension (this result was obtained independently via a less fancy technique by S. Donkin [J. Algebra **111** (1988), no. 2, 354–364; MR0916172 (89b:20084b)]).

In paper III the derived categories are only present in the distant background. Here the objects of

Example of a quasi-hereditary algebra

PIMs of A $\quad \begin{array}{ccc} a & & \\ b & & b \\ a & & a \end{array}$

a, b irreducible

$\Longrightarrow$ A has finite global dimension

MR961165 (90d:18005) 18E30 (16A46 17B10 20G05 32C38)

Cline, E. (1-CLRK); **Parshall, B.** (1-IL); **Scott, L. [Scott, Leonard L.]** (1-VA)

Finite-dimensional algebras and highest weight categories.

J. Reine Angew. Math. **391** (1988), 85–99.

Let A be a finite-dimensional k-algebra (k a commutative field) and let $D^b(A)$ be the category of bounded complexes over mod-A, the category of finitely generated right A-modules. The authors continue their investigations for "recollement" of $D^b(A)$. For example they show that for an idempotent $e \in A$ and $f = 1 - e$ with $eAf = 0$ and $\operatorname{gl}\dim A < \infty$ one obtains a recollement $D^b(eAe) \underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}} D^b(A) \underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}} D^b(fAf)$. Furthermore they introduce by formalizing the category $\mathcal{O}$ the notion of a highest weight category, say with weight set Λ, and show that for an algebra A the conditions that A is quasi-hereditary and mod A is a highest weight category are equivalent.

$D^b(A/AeA)$

Reviewed by *Dieter Happel*

DERIVED CATEGORIES, QUASI–HEREDITARY ALGEBRAS, AND ALGEBRAIC GROUPS[1]

From Carleton University
Math Notes 3 (1988)
pp 1-104.

Now available at www.
math.virginia.edu/~LLS2L

Brian J. Parshall[2,3] and Leonard L. Scott[3]

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Two main results:
1) Perverse sheaves on G/B form a highest weight category
2) A general procedure for contructing highest weight categories inside derived categories (an alternative to the Asterisque 100 "heart" approach).

CONTENTS

MR1128706 (93c:16010) 16G10 (16D90)

Important paper on quasi-hereditary algebras and tiltling modules

Ringel, Claus Michael (D-BLF)

The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences.

Math. Z. **208** (1991), *no. 2,* 209–223.

Donkin, Stephen (4-LNDQM)

On tilting modules for algebraic groups.

Math. Z. **212** (1993), *no. 1,* 39–60.

Let G be a reductive affine algebraic group over an algebraically closed field K of characteristic $p > 0$. The category of rational G-modules which are bounded in an appropriate sense is equivalent to the category of modules of an associated quasi-hereditary algebra.

Let A be a finite-dimensional quasi-hereditary algebra with simple modules $L(\lambda)$, standard modules $\Delta(\lambda)$ and co-standard modules $\nabla(\lambda)$, and let $\mathcal{T}$ denote the class of modules which have both Δ-filtrations and ∇-filtrations. It was proved by C. M. Ringel [Math. Z. **208** (1991), no. 2, 209–223; MR1128706 (93c:16010)] that there is a one-to-one correspondence $M(\lambda) \leftrightarrow L(\lambda)$ between the indecomposable modules in $\mathcal{T}$ and the simple modules. The direct sum $T := \bigoplus M(\lambda)$ is a (generalized) tilting and cotilting module. Moreover, if $A' := \mathrm{End}_A(T)$ then A' is again quasi-hereditary (with standard modules $\Delta_{A'}(\lambda) = \mathrm{Hom}_A(T, \nabla(\lambda))$).

In this paper, the author studies the implications of these results for reductive groups; these include the following. The class $\mathcal{T}$ is closed under tensor products (which was proved earlier). The modules $M(\lambda)$ behave well on truncation to Levi factors. Further, the author studies the relationship between the $M(\lambda)$ and injective modules for the infinitesimal subgroups G_n of G. He conjectures that the injective indecomposables for G_n are always restrictions of certain $M(\lambda)$. This is a refinement of an older conjecture (the injectives of G_n have extensions to G-modules;

Let $G = \mathrm{GL}_n(K)$; then the modules $M(\lambda)$ are precisely the indecomposable summands of tensor products of exterior powers of the natural module. It is known that one gets the Schur algebra $S(n,r)$ as an associated quasi-hereditary algebra. The author proves that its conjugate $S(n,r)'$ is a generalized Schur algebra, in the sense of earlier work [S. Donkin, J. Algebra **104** (1986), no. 2, 310–328; MR0866778 (89b:20084a); J. Algebra **111** (1987), no. 2, 354–364; MR0916172 (89b:20084b)]; in particular, it is isomorphic to $S(n,r)$ if $r \leq n$. From this he obtains that the filtration multiplicity $[M(\lambda) : \nabla(\mu)]$ is equal to the decomposition number $d_{\mu'\lambda'} = [\nabla(\mu') : L(\lambda')]$ (where τ' is the transpose of the partition τ).

Reviewed by *Karin Erdmann*

Donkin, Stephen (4-YORK)

Tilting modules for algebraic groups and finite dimensional algebras.

Handbook of tilting theory, 215–257, *London Math. Soc. Lecture Note Ser.*, 332, *Cambridge Univ. Press, Cambridge*, 2007.

This survey article is aimed at describing the relationship between representation theories of algebraic groups and quasi-hereditary algebras, with emphasis on tilting modules.

The paper under review includes 10 sections. In Section 1, the author quickly reviews the theory of quasi-hereditary algebras, due to Cline, Parshall and Scott. In particular, tilting modules for a quasi-hereditary algebra are defined. Section 2 introduces notions of coalgebras and comodules which connect representations of reductive algebraic groups and quasi-hereditary algebras. Sections 3, 4 and 5 deal with representations of linear algebraic groups and reductive groups. The interaction between the tilting theory for a reductive group and the representation theory of its Lie algebra is also considered. In particular, the author gives a character formula for certain tilting modules which are projective as modules for the restricted enveloping algebra. In Section 6, he makes a conjecture describing the support variety of tilting modules for general linear groups in characteristic 2. Section 7 briefly presents the application of tilting modules in the study of invariant theory. In Sections 8 and 9, the author gives a description of polynomial tilting modules for general linear groups and makes a connection between representations of general linear groups and representations of symmetric groups via the Schur functor. The final section provides some

Theorem (Steinberg). *Let $\lambda', \lambda'' \in \Lambda^+$. Then the number of times $V(\lambda)$, $\lambda \in \Lambda^+$, occurs in $V(\lambda') \otimes V(\lambda'')$ is given by the formula*

$$\sum_{\sigma \in \mathscr{W}} \sum_{\tau \in \mathscr{W}} sn(\sigma\tau) p(\lambda + 2\delta - \sigma(\lambda' + \delta) - \tau(\lambda'' + \delta)). \quad \square$$

delta here is half the sum of all positive roots

This formula (like Kostant's) is very explicit, but not at all easy to apply when the Weyl group is large. A formula which is often more practical is developed in Exercise 9.

Humphreys, Introduction to Lie algebras and representation theory

Mathieu

MR1072820 (92a:20044) 20G05 (14M15 14M17)
Mathieu, Olivier (F-ENS)
Filtrations of G-modules.
Ann. Sci. École Norm. Sup. (4) **23** (1990), *no. 4,* 625–644.

The tensor product of modules with a good filtration has a good filtration!

Let k be an algebraically closed field, G be a connected semisimple affine algebraic group over k, and B a Borel subgroup of G. For a one-dimensional rational B-module λ, there is a G-equivariant line bundle $\mathfrak{L}(\lambda)$ on the homogeneous projective variety G/B associated to it. The G-module $\Gamma(G/B, \mathfrak{L}(\lambda))$ of global sections of $\mathfrak{L}(\lambda)$, denoted by $F(\lambda)$ in this paper and by $H^0(\lambda)$ by some other authors, is of great importance in the representation theory of algebraic groups. For example, it is known that when $\operatorname{char} k = 0$, the nonzero $F(\lambda)$'s form a complete set of representatives of irreducible G-modules (the Borel-Weyl theorem) and any rational G-module is a direct sum of $F(\lambda)$'s (the Weyl complete reducibility theorem). It is also known that in any characteristic the formal character (thus the dimension) of $F(\lambda)$, if nonzero, is given by the Weyl character formula and the dual of $F(\lambda)$ is the universal highest weight module with highest weight λ. Therefore it is natural to pay attention to G-modules with good filtrations (a filtration of a G-module is called good if its subquotients are isomorphic to some $F(\lambda)$'s). As a substitute for the Weyl complete reducibility theorem, one may ask the following questions: (1) For any λ and μ, does the G-module $F(\lambda) \otimes F(\mu)$ have a good filtration? (2) For any λ and any semisimple subgroup G' corresponding to a Dynkin subdiagram, does the G'-module $F(\lambda)$ have a good filtration?

The first work in this direction was done by the reviewer, who proved [J. Algebra **77** (1982), no. 1, 162–185; MR0665171 (84h:20032)] that the answer to question (1) is "yes", provided $\operatorname{char} k$ is not too small compared with the Coxeter number. After that S. Donkin's book *Rational representations of algebraic groups* [Lecture Notes in Math., 1140, Springer, Berlin, 1985; MR0804233 (87b:20054)], gave an almost complete solution to the above questions. More precisely, Donkin's result is that the answer to these questions is positive, provided G contains no components of type E_7, E_8 or $\operatorname{char} k \neq 2$. Donkin's proof requires long and difficult calculations. In particular, it involves a case-by-case analysis when treating exceptional groups of small characteristics. Therefore, it is natural to look for a new approach to the above questions, in order to remove Donkin's restrictive hypothesis and to avoid the case-by-case analysis.

The present paper is successful in this direction. A positive answer is given to the above questions without restrictions, and the proof is general. The answer to the above questions is obtained in this paper as a consequence of a more general result (see the theorem below).

Instead of line bundles on G/B, the author considers line bundles on a "generalized Schubert scheme". Recall that a Schubert variety is the Zariski closure of BwB/B in G/B, where w is an element of the Weyl group W of G. By a generalized Schubert variety the author means the product of n Schubert varieties for a positive integer n, viewed as a subvariety of $(G/B)^{\times n}$. A generalized Schubert scheme is simply a union of generalized Schubert varieties. Suppose $S \subseteq (G/B)^{\times n}$ is a generalized Schubert scheme. For $\lambda = (\lambda_1, \cdots, \lambda_n)$, with λ_i a one-dimensional B-

module for each i, there is a line bundle $\mathcal{L}(\lambda)$ on $(G/B)^{\times n}$ associated to it. The restriction of $\mathcal{L}(\lambda)$, also denoted by $\mathcal{L}(\lambda)$, gives a line bundle on S. If, in addition, S is $G'B$-invariant, with G' a semisimple subgroup of G corresponding to a Dynkin subdiagram, then the space of global sections $\Gamma(S, \mathcal{L}(\lambda))$ of $\mathcal{L}(\lambda)$ on S is a G'-module.

For such a G'-module there is a natural "geometrical stratification" (a stratification of a module M is simply a family $\mathcal{J}$ of submodules with $M \in \mathcal{J}$): For any $G'B$-invariant generalized Schubert schemes S' and S'' with $S' \supseteq S''$ and $S' \supseteq S$, there are restriction morphisms $a\colon \Gamma(S', \mathcal{L}(\lambda)) \to \Gamma(S'', \mathcal{L}(\lambda))$ and $b\colon \Gamma(S', \mathcal{L}(\lambda)) \to \Gamma(S, \mathcal{L}(\lambda))$. The geometrical stratification $\mathcal{J}$ of $\Gamma(S, \mathcal{L}(\lambda))$ is the set of submodules of the form $b(\mathrm{Ker}\, a)$.

The main result in this paper is the following theorem. The G'-module $\Gamma(S, \mathcal{L}(\lambda))$ has a good filtration compatible with the geometrical stratification $\mathcal{J}$ in the sense that the trace of the filtration in each $N \in \mathcal{J}$ is good. The proof of the theorem requires the notion of Frobenius splittings due to V. B. Mehta and A. Ramanathan [see, e.g., Ann. of Math. (2) **122** (1985), no. 1, 27–40; MR0799251 (86k:14038)]. According to Mehta, Ramanan and Ramanathan's theory, the variety $(G/B)^{\times n}$ has many Frobenius splittings. The author selects a "canonical" splitting and proves that the splitting is compatible with good filtrations, which is the key step to the above theorem.

Reviewed by *Jian-Pan Wang*

If $V(\lambda')$, $V(\lambda'')$ are Weyl modules

$V(\lambda') \otimes V(\lambda'')$ is filtered by Weyl modules.

Cline, E.; Parshall, B.; Scott, L. [Scott, Leonard L.]; van der Kallen, Wilberd

Rational and generic cohomology.

Invent. Math. **39** (1977), *no. 2*, 143–163.

Let G be a semisimple algebraic group defined and split over $\mathbf{F}_p$. For $q = p^m$ let $G(q)$ be the finite group of $\mathbf{F}_q$-points of G. The main objective of the authors is to relate the cohomology of the finite group $G(q)$ to the rational cohomology of G. Let V be a finite dimensional rational G-module and for an integer $e \geq 0$ let $V(e)$ be the G-module obtained by twisting the original G-action on V by the Frobenius endomorphism $x \rightsquigarrow x^{[p^e]}$, $x \in G$. The main theorem of the paper states that for sufficiently large q and e (depending on V and n) there are isomorphisms $H^n(G, V(e)) \simeq H^n(G(q), V(e)) \simeq H^n(G(q), V)$, where the first map is the restriction. The proof contains several steps. Let B be a Borel subgroup of G. In the first step the authors prove that $H^n(G, V) \simeq H^n(B, V)$ for any rational G-module V and for all $n \geq 0$. The proof is based on a suitable modification of a theorem of G. Kempf [Ann. of Math. (2) **103** (1976), no. 3, 557–591; MR0409474 (53 #13229)] on the vanishing of cohomology of certain homogeneous line bundles. Next the authors compute the cohomology ring $H^*(G_a, k)$ as a T-algebra, where T is a split torus acting on G_a with weight α, and k is the one-dimensional trivial module. With these results the proof of the main theorem follows from some purely arithmetic results on the p-adic expansions.

The previous page, on cohmology , was not included in the lecture, and the same is true for the remaining pages on Ext and on Koszul algebras.

MR1245719 (94k:20079) 20G05 (17B10 17B37 20G10)

Cline, Edward (1-OK); **Parshall, Brian** (1-VA); **Scott, Leonard** (1-VA)

Abstract Kazhdan-Lusztig theories. (English summary)

Tohoku Math. J. (2) **45** (1993), *no. 4*, 511–534.

In a series of papers, the authors have placed in a general setting many of the features shared by the representation theories of semisimple Lie algebras in characteristic 0, semisimple algebraic groups in characteristic p, quantum groups at a root of unity, etc. This involves the notion of highest weight category, as described in a paper by the authors [J. Reine Angew. Math. **391** (1988), 85–99; MR0961165 (90d:18005)]. Here they introduce further structure into such categories, centering on length functions on weight posets and associated "Kazhdan-Lusztig theories". A major theme is the explicit calculation of higher Ext groups between pairs of irreducible modules. This requires some careful work with the bounded derived category associated to a highest weight category: an "enriched" Grothendieck group is needed to keep track of degree information.

Cline, E. (1-OK); **Parshall, B.** (1-VA); **Scott, L. [Scott, Leonard L.]** (1-VA)
The homological dual of a highest weight category.
Proc. London Math. Soc. (3) **68** (1994), *no. 2,* 294–316.

This paper continues the authors' earlier extensive study of highest weight categories [J. Reine Angew. Math. **391** (1988), 85–99; MR0961165 (90d:18005)], quasihereditary rings, and "Kazhdan-Lusztig theories". The setting in which the authors work encompasses not only the characteristic 0 theory associated with the category $\mathcal{O}$, but also the more complicated (and less completely understood) modular theory of semisimple algebraic groups and their Lie algebras. Here they develop further the ideas in their paper [Tôhoku Math. J. (2) **45** (1993), no. 4, 511–534; MR1245719 (94k:20079)]. In particular, they explore the "homological dual" $\mathcal{C}^!$ of a highest weight category $\mathcal{C}$ with finite weight poset, defined as follows. Let L be the direct sum of the finitely many irreducible objects in $\mathcal{C}$, and $A^! = \mathrm{Ext}^\bullet_{\mathcal{C}}(L, L)$. Then $\mathcal{C}^!$ is the category of right $A^!$-modules. Unpublished work of A. Beĭlinson, V. Ginsburg, and W. Soergel on self-duality of the principal block of the category $\mathcal{O}$ inspires the authors to seek conditions under which $\mathcal{C}^!$ is again a highest weight category. (Examples of V. Dlab show that this is not automatic.) For example, it is sufficient that $\mathcal{C}$ have a Kazhdan-Lusztig theory. In turn, $\mathcal{C}^!$ itself will have a Kazhdan-Lusztig theory if a certain even-odd vanishing criterion is met in the derived category of $\mathcal{C}$. The graded analogues are stressed here as being perhaps more natural than the Koszul condition alone for the algebras involved.

Reviewed by *James E. Humphreys*

INTEGRAL AND GRADED QUASI-HEREDITARY ALGEBRAS, II WITH APPLICATIONS TO REPRESENTATIONS OF GENERALIZED q-SCHUR ALGEBRAS AND ALGEBRAIC GROUPS

BRIAN J. PARSHALL AND LEONARD L. SCOTT

We dedicate this paper to Ed Cline on the occasion of his 69th birthday.

Abstract. We consider a pair $(\mathfrak{a}, A)$ consisting of a quasi-hereditary algebra A and a (positively) graded subalgebra $\mathfrak{a}$. We present conditions which guarantee that the algebra gr$A$ obtained by grading A by its radical filtration is quasi-hereditary and Koszul. In such cases, we also show that the standard and costandard modules for A have a structure as graded modules for $\mathfrak{a}$. These results are applied to obtain new information about the finite dimensional algebras (e. g., the q-Schur algebras) which arise from quantum enveloping algebras. In the final section, we consider an order $\widetilde{A}$ for a quasi-hereditary algebra A_K, and consider conditions which guarantee that gr$\widetilde{A}$ is an integral quasi-hereditary algebra. This section and many other parts of the paper have consequences for representations of algebraic groups in positive characteristic. Explicit positive characteristic analogues of the above results are given, with some restrictions.

Contents

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Theorem 1. *If A is a regular block of $S_q(n, r)$, then the graded algebra* $\mathrm{gr}A$ *is a quasi-hereditary algebra with a Koszul grading.* A similar result holds in char. p>>n. (q=1)