Identifying complete differential varieties

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Conventions and terminology

- In the differential setting, our fields are all DCF_0. Otherwise they are algebraically closed.
- Our varieties $V, W, \ldots,$ are subsets of affine or projective space defined by appropriate polynomial equations.
- All Kolchin-closed subsets (δ-*varieties)* we consider are defined over a fixed arbitrary DCF_0 denoted by $\mathcal{F}$. We denote by $\mathcal{K}$ any DCF_0 containing $\mathcal{F}$.
- A *finite-rank* δ-*variety* is one that has dimension $< \omega$ for any/all of the usual ordinal-valued ranks on DCF_0.
- Let R be a δ-ring, $K \supseteq R$ a δ-field, L a δ-field and $\varphi : R \to L$ a δ-homomorphism . We say R is a *maximal* δ-*ring* (with respect to K) if φ does not extend to a δ-homomorphism with domain contained in K but strictly containing R and codomain a δ-field.

Compactness

A topological space X is *compact* if every open cover of X admits a finite subcover.

- *Example:* Closed, bounded intervals of $\mathbb{R}$ (by the l.u.b. axiom).
- Compact subsets of Hausdorff spaces are closed, and being compact is most valuable when a space is Hausdorff.
- *Fact:* Algebraic varieties are compact (algebraic geometers say *quasicompact*), but generally not Hausdorff.

Completeness and basic properties

Definition

An algebraic variety V is *complete* if for every variety W the projection $\pi_2 : V \times W \to W$ is a closed map (takes closed sets to closed sets).

- Completeness is a local property.
- Closed subsets of complete varieties are complete.
- Products of complete varieties are complete.
- If $\varphi : V \to W$ is a morphism of varieties and V is complete, then $\varphi(V)$ is complete.
- Morphisms of complete irreducible varieties into affine space are constant.

The fundamental theorem of elimination theory

Example

Infinite affine varieties are not complete: Consider the second projection of $Z := xy - 1 = 0$. The image $\pi_2(Z) = \{y \neq 0\}$, which is not closed in $\mathbb{A}^1$.

Fact: a complete subset of $\mathbb{P}^n$ must be projectively closed. In fact,

Theorem

Projective varieties are complete.

Proof.

Projective Nullstellensatz + linear algebra. □

δ-completeness

δ-*completeness* of an affine or projective δ-variety is defined by the obvious generalization. Most properties carry over to the differential case:

- δ-completeness is a local property.
- δ-complete varieties are projectively closed.
- Kolchin-closed subsets of δ-complete varieties are δ-complete.
- Products of δ-complete varieties are δ-complete.
- If $\varphi : V \to W$ is a morphism of δ-varieties and V is δ-complete, then $\varphi(V)$ is δ-complete.
- BUT morphisms of irreducible δ-complete varieties into affine space need not be constant.

Pong

Kolchin's example: $\mathbb{P}^1$ is not complete.

Theorem

(Pong) Only finite-rank δ-varieties can be δ-complete.

Theorem

(Pong) A δ-complete variety is isomorphic to a δ-complete variety contained in $\mathbb{A}^1$.

So it suffices to consider subsets of $\mathbb{A}^1$; for convenience we usually work with $\mathbb{P}^1$.

Freitag has generalized much of Pong's paper to the case of several commuting derivations (i.e., $DCF_{0,m}$).

Positive quantifier elimination

Positive formulas are those that contain no negation symbols.

Theorem

(van den Dries) Let T be an $\mathcal{L}$-theory and $\varphi(\bar{x})$ an $\mathcal{L}$-formula. Then φ is equivalent modulo T to a positive quantifier-free formula iff for all $\mathcal{M}, \mathcal{N} \models T$, substructures $A \subseteq \mathcal{M}$, $\bar{a} \in A$, and $\mathcal{L}$-homomorphisms $f : A \to \mathcal{N}$ we have $\mathcal{M} \models \varphi(\bar{a}) \implies \mathcal{N} \models \varphi(f(\bar{a}))$.

Proof.

Slick choice of auxiliary theory $T' +$ positive atomic diagram $+$ compactness. □

(Part of) a valuative criterion

Let $p = (p_0 : p_1) \in V \subseteq \mathbb{P}^1$, and let $\mathcal{F} \subseteq R \subseteq \mathcal{K}$ be a maximal δ-ring. We say p is in R if either p_0/p_1 or p_1/p_0 is.

Theorem

(Pong, S) Let V be a δ-subvariety of $\mathbb{P}^1$. If for every $p \in V$ and maximal δ-ring R we have $p \in R$, then V is δ-complete.

Proof.

Use van den Dries' PQE to show the formula $\varphi(x_0, x_1, \bar{y})$

$$(\exists x_0, x_1)((\wedge_i P_i(x_0, x_1, \bar{y}) = 0) \wedge (\wedge_j Q_j(x_0, x_1) = 0) \wedge (x_0 = 1 \vee x_1 = 1))$$

is equivalent to a positive quantifier-free formula. □

Ritt and P. Blum

Definition

An element a of a δ-field K is *monic* over a δ-ring $A \subseteq K$ if a satisfies a δ-polynomial equation $x^n + f(x) = 0$, where $f(x) \in A\{x\}$ and $n >$ total degree of terms in f.

Facts:

- Derivatives of monic elements are monic.
- Maximal δ-rings are local differential rings.
- If $(R, \mathfrak{m})$ is maximal, a is monic over R iff $1/a \notin \mathfrak{m}$.
- $a \in K \setminus R$ iff $1 \in \mathfrak{m}\{a\}$.

Morrison

Theorem

Maximal δ-rings are integrally closed.

Theorem

Let $(R, \mathfrak{m})$ be a maximal δ-ring. If a satisfies a linear differential equation over R and $1/a \notin R$, then $a \in R$.

Proof.

Show a is integral over R. This requires monicness of $a, a', \ldots$ and order-reducing substitutions given by the linear differential equation. □

First fruits

Proposition

(S) The projective closure in $\mathbb{P}^1$ of a linear δ-variety is δ-complete.

Proof.

Morrison's result + the valuative criterion. ☐

Corollary

When using the valuative criterion, we may suppose $a^{(n)} \neq 0$ for all $n \in \mathbb{N}$.

Some good examples

Proposition

(S) The projective closures in $\mathbb{P}^1$ of the following are δ-complete:

- $Q(x)x' = P(x)$, *where* Q, P *are ordinary polynomials*
- $x'' = x^3$
- $xx'' = x'$

Proof.

Valuative criterion + integral closedness of R + a lot of pencil lead. □

Known to be δ-complete

General classes:

- Linear: all
- $x^{(n)} = P(x^{(n-1)})$
- Equations algebraic in $\mathcal{F}[x^{(n)}]$

First-order:

- $Q(x)x' = P(x)$
- $(x')^n = x + \alpha$

Second-order:

- $x'' = x^3$ (and friends)
- $xx'' = x'$

Where to from here?

Hope/Guess

All finite-rank, projectively closed δ-varieties are δ-complete.

- Refine 1-preserving algorithms, understand integrality in this setting better
- Generalize proof of algebraic completeness or show this cannot be done
- Direct approach: analyze QE, use Rosenfeld-Groebner algorithm, etc.
- Use the preceding to look for counterexamples
- Look at interesting δ-varieties like the Manin kernel.
- Algebraic D-varieties

Thanks for listening!

References

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