Identifying complete differential varieties

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Conventions and terminology

- In the differential setting, our fields are all $DCF_0$. Otherwise they are algebraically closed.
- Our varieties $V, W, \ldots$, are subsets of affine or projective space defined by appropriate polynomial equations.
- All Kolchin-closed subsets ($\delta$-varieties) we consider are defined over a fixed arbitrary $DCF_0$ denoted by $\mathcal{F}$. We denote by $\mathcal{K}$ any $DCF_0$ containing $\mathcal{F}$.
- A finite-rank $\delta$-variety is one that has dimension $< \omega$ for any/all of the usual ordinal-valued ranks on $DCF_0$.
- Let $R$ be a $\delta$-ring, $K \supseteq R$ a $\delta$-field, $L$ a $\delta$-field and $\varphi : R \to L$ a $\delta$-homomorphism. We say $R$ is a maximal $\delta$-ring (with respect to $K$) if $\varphi$ does not extend to a $\delta$-homomorphism with domain contained in $K$ but strictly containing $R$ and codomain a $\delta$-field.
A topological space $X$ is *compact* if every open cover of $X$ admits a finite subcover.

- **Example:** Closed, bounded intervals of $\mathbb{R}$ (by the l.u.b. axiom).
- Compact subsets of Hausdorff spaces are closed, and being compact is most valuable when a space is Hausdorff.
- **Fact:** Algebraic varieties are compact (algebraic geometers say *quasicompact*), but generally not Hausdorff.
Completeness and basic properties

Definition
An algebraic variety $V$ is complete if for every variety $W$ the projection $\pi_2 : V \times W \to W$ is a closed map (takes closed sets to closed sets).

- Completeness is a local property.
- Closed subsets of complete varieties are complete.
- Products of complete varieties are complete.
- If $\varphi : V \to W$ is a morphism of varieties and $V$ is complete, then $\varphi(V)$ is complete.
- Morphisms of complete irreducible varieties into affine space are constant.
Example

Infinite affine varieties are not complete: Consider the second projection of $Z := xy - 1 = 0$. The image $\pi_2(Z) = \{y \neq 0\}$, which is not closed in $\mathbb{A}^1$.

Fact: a complete subset of $\mathbb{P}^n$ must be projectively closed. In fact,

Theorem

Projective varieties are complete.

Proof.

Projective Nullstellensatz + linear algebra.
\(\delta\)-completeness

\(\delta\)-completeness of an affine or projective \(\delta\)-variety is defined by the obvious generalization. Most properties carry over to the differential case:

- \(\delta\)-completeness is a local property.
- \(\delta\)-complete varieties are projectively closed.
- Kolchin-closed subsets of \(\delta\)-complete varieties are \(\delta\)-complete.
- Products of \(\delta\)-complete varieties are \(\delta\)-complete.
- If \(\varphi : V \rightarrow W\) is a morphism of \(\delta\)-varieties and \(V\) is \(\delta\)-complete, then \(\varphi(V)\) is \(\delta\)-complete.
- **BUT** morphisms of irreducible \(\delta\)-complete varieties into affine space need not be constant.
Kolchin’s example: \( \mathbb{P}^1 \) is not complete.

**Theorem**

(Pong) Only finite-rank \( \delta \)-varieties can be \( \delta \)-complete.

**Theorem**

(Pong) A \( \delta \)-complete variety is isomorphic to a \( \delta \)-complete variety contained in \( \mathbb{A}^1 \).

So it suffices to consider subsets of \( \mathbb{A}^1 \); for convenience we usually work with \( \mathbb{P}^1 \).

Freitag has generalized much of Pong’s paper to the case of several commuting derivations (i.e., \( DCF_{0,m} \)).
Positive quantifier elimination

Positive formulas are those that contain no negation symbols.

**Theorem**

*(van den Dries)* Let $T$ be an $\mathcal{L}$-theory and $\varphi(\bar{x})$ an $\mathcal{L}$-formula. Then $\varphi$ is equivalent modulo $T$ to a positive quantifier-free formula iff for all $\mathcal{M}, \mathcal{N} \models T$, substructures $A \subseteq \mathcal{M}$, $\bar{a} \in A$, and $\mathcal{L}$-homomorphisms $f : A \to \mathcal{N}$ we have $\mathcal{M} \models \varphi(\bar{a}) \implies \mathcal{N} \models \varphi(f(\bar{a}))$.

**Proof.**

Slick choice of auxiliary theory $T' +$ positive atomic diagram $+$ compactness.
Let \( p = (p_0 : p_1) \in V \subseteq \mathbb{P}^1 \), and let \( \mathcal{F} \subseteq R \subseteq K \) be a maximal \( \delta \)-ring. We say \( p \) is in \( R \) if either \( p_0/p_1 \) or \( p_1/p_0 \) is.

**Theorem**

(Pong, S) Let \( V \) be a \( \delta \)-subvariety of \( \mathbb{P}^1 \). If for every \( p \in V \) and maximal \( \delta \)-ring \( R \) we have \( p \in R \), then \( V \) is \( \delta \)-complete.

**Proof.**

Use van den Dries’ PQE to show the formula \( \varphi(x_0, x_1, \bar{y}) \)

\[
(\exists x_0, x_1)((\bigwedge_i P_i(x_0, x_1, \bar{y}) = 0) \land \\
(\bigwedge_j Q_j(x_0, x_1) = 0) \land (x_0 = 1 \lor x_1 = 1))
\]

is equivalent to a positive quantifier-free formula.
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Ritt and P. Blum

Definition

An element $a$ of a $\delta$-field $K$ is monic over a $\delta$-ring $A \subseteq K$ if $a$ satisfies a $\delta$-polynomial equation $x^n + f(x) = 0$, where $f(x) \in A\{x\}$ and $n >$ total degree of terms in $f$.

Facts:

- Derivatives of monic elements are monic.
- Maximal $\delta$-rings are local differential rings.
- If $(R, m)$ is maximal, $a$ is monic over $R$ iff $1/a \notin m$.
- $a \in K \setminus R$ iff $1 \in m\{a\}$. 
Theorem

Maximal $\delta$-rings are integrally closed.

Theorem

Let $(R, \mathfrak{m})$ be a maximal $\delta$-ring. If $a$ satisfies a linear differential equation over $R$ and $1/a \notin R$, then $a \in R$.

Proof.

Show $a$ is integral over $R$. This requires monicness of $a, a', \ldots$ and order-reducing substitutions given by the linear differential equation.
Proposition

(S) The projective closure in $\mathbb{P}^1$ of a linear $\delta$-variety is $\delta$-complete.

Proof.

Morrison’s result + the valuative criterion.

Corollary

When using the valuative criterion, we may suppose $a^{(n)} \neq 0$ for all $n \in \mathbb{N}$.
Some good examples

Proposition

$(S)$ The projective closures in $\mathbb{P}^1$ of the following are $\delta$-complete:

- $Q(x)x' = P(x)$, where $Q, P$ are ordinary polynomials
- $x'' = x^3$
- $xx'' = x'$

Proof.

Valuative criterion + integral closedness of $R$ + a lot of pencil lead.
## Known to be $\delta$-complete

### General classes:
- **Linear:** all
  - $x^{(n)} = P(x^{(n-1)})$
- **Equations algebraic in $\mathcal{F}[x^{(n)}]$**

### First-order:
- $Q(x)x' = P(x)$
- $(x')^n = x + \alpha$

### Second-order:
- $x'' = x^3$ (and friends)
- $xx'' = x'$

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Where to from here?

Hope/Guess

All finite-rank, projectively closed δ-varieties are δ-complete.

- Refine 1-preserving algorithms, understand integrality in this setting better
- Generalize proof of algebraic completeness or show this cannot be done
- Direct approach: analyze QE, use Rosenfeld-Groebner algorithm, etc.
- Use the preceding to look for counterexamples
- Look at interesting δ-varieties like the Manin kernel.
- Algebraic D-varieties
Thanks for listening!
References