A differential algebra sampler

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Base field and differential varieties

Unless stated otherwise, our objects are defined over a base differentially closed field $\mathcal{F} \models DCF_0$. (Most of what we say generalizes to the partial case, however.) We view $\mathcal{F}$ as being a subfield of a large differentially closed field $\mathcal{U}$, though this universal domain rarely shows up explicitly.

The closed sets in the Kolchin topology on $\mathbb{P}^n(\mathcal{U})$ that we consider are defined by δ-homogeneous polynomials in $n+1$ variables over $\mathcal{F}$; e.g., $x_1' x_0 - x_1 x_0'$. These solution sets are *projective differential algebraic varieties*.

Projective closure is tricky. As in the algebraic case, the closure may be smaller than the set defined by the δ-homogenization of the defining polynomials.

Complete differential varieties

Definition

Let V be a projective δ-variety defined over a differential field $\mathcal{F}$. We say V is *differentially complete* (or *δ-complete*) if for every projective δ-variety W over $\mathcal{F}$ and Kolchin-closed subset Z of $V \times W$, the projection map $\pi_2 : V \times W \to W$ sends Z to a Kolchin-closed set.

The key differences with the algebraic case:

- (Pong; Freitag in partial case) A complete differential variety must have differential dimension zero.
- Morphisms of irreducible complete differential varieties into affine space need not be constant.

Known complete differential varieties

General classes:

- Linear: all (single variable, at least)
- $x^{(n)} = P(x^{(n-1)})$
- Equations in $\mathcal{F}[x^{(n)}]$

First-order:

- $Q(x)x' = P(x)$ (generalization of Kolchin's and Pong's examples)
- $(x')^n = x + \alpha$

Known complete differential varieties, cont.

Second-order:

- $xx'' = x'$

Immediate relations: Projective subvarieties, images under morphisms, and products of members of the preceding list

Anomalous projective closure

Example

The projective closure in $\mathbb{P}^1$ of the affine δ-variety defined by $x'' = x^2$ is the affine variety itself.

This is surprising because the point at infinity $(1:0)$ satisfies the δ-homogeneous equation $(x'' - x^2)^{\delta h} = 0$.

Proof.

- We claim the projective closure is the set

$$V := \{(x:y) \in \mathbb{P}^1 \mid (x:y) \in (x'' - x^2)^{\delta h} \text{ and } y \neq 0\}$$

  Note that this is simply the restriction to the first affine chart, i.e., the affine variety.
- It suffices to show that the restriction of V to both charts is Kolchin closed.

Proof, continued

- The case of the first chart is immediate. The intersection of V with the second is defined by $-yy'' + 2(y')^2 - y = 0, y \neq 0$ and so is not obviously closed.
- Because we are working over DCF_0, the Kolchin closure of $-yy'' + 2(y')^2 - y = 0, y \neq 0$ in $\mathbb{A}^1$ is defined by the differential elimination ideal I_t that eliminates t from $I = [-yy'' + 2(y')^2 - y = 0, yt - 1]$.
- It suffices to find in I_t a differential polynomial that does not vanish when evaluated at 0 (i.e., has a nonzero constant term).

Proof, continued

- Differentiate $-yy'' + 2(y')^2 - y = 0$ once and compute the algebraic elimination ideal corresponding to

$$(-yy'' + 2(y')^2 - y = 0, 3y'y'' - yy''' - y', yt - 1)$$

- It turns out that $3(y'')^2 - 1 - 2y'''y' + 2y'' \in I_t$.
- Don't trust Maple? Check by hand that
  $yp_1 = (-3y'' + 1)p_2 + 2y'p_3$, where
  $p_1 = 3(y'')^2 - 1 - 2y'''y' + 2y'', p_2 = -y''y + 2(y')^2 - y$, and
  $p_3 = 3y'y'' - yy''' - y'$.

A/the bad example

Example

The affine differential varieties defined by $x'' = x^n, n \geq 2$, are projectively closed and have finite rank but are not complete.

Proof.

- We just saw that $x'' = x^2$ is projectively closed. For $n \geq 3$, the point at infinity does not lie on the δ-homogenization of $x'' = x^n$, so the projective closure is again the affine δ-variety itself.
- Consider $W = \{(x, y) \mid x'' = x^n \text{ and } y(2x^{n+1} - (n+1)(x')^2) = 1\}$. We claim that $\pi_2(W) = \{y \mid y' = 0 \text{ and } y \neq 0\}$.
- This suffices because $\{y \mid y' = 0\}$ is an irreducible Kolchin-closed set containing 0.

Proof of the bad example, continued

$\pi_2(W) \subseteq \{y \mid y' = 0 \text{ and } y \neq 0\}$:

- Given $y \in \pi_2(W)$, choose $(x, y) \in W$ and differentiate the equation $y(2x^{n+1} - (n+1)(x')^2) = 1$.
- Substituting x^n for x'', we find $y'(2x^{n+1} - (n+1)(x')^2) + y(2(n+1)x^n x' - 2(n+1)x'x^n) = y'(2x^{n+1} - (n+1)(x')^2) = 0$.
- Multiplying both sides by y and applying the relation $y(2x^{n+1} - (n+1)(x')^2) = 1$ gives $y' = 0$; clearly $y \neq 0$.

Proof of the bad example, continued

$\pi_2(W) \supseteq \{y \mid y' = 0 \text{ and } y \neq 0\}$:

- Let y be a non-zero element such that $y' = 0$. We need to find x such that $x'' = x^n$ and $2x^{n+1} - (n+1)(x')^2 = \frac{1}{y}$.
- Blum's axioms for DCF_0 imply that there exists x such that simultaneously $2x^{n+1} - (n+1)(x')^2 = \frac{1}{y}$ and $2x^{n+1} \neq \frac{1}{y}$.
- Differentiation produces $2(n+1)x^n x' - 2(n+1)x'x'' = 0$. Because $x' \neq 0$, we may divide and obtain $x'' = x^n$. Hence $y \in \pi_2(W)$.

Some remarks on the bad example

This has a similar flavor to Pong's version of Kolchin's example proving incompleteness of $\mathbb{P}^1$. Both feature differentiation of polynomials having constant term 1, as well as the use of Blum's axioms to prevent the separant from vanishing.

That earlier example: $y(x')^2 + x^4 - 1 = 0, 2yx'' + y'x' + 4x^3 = 0$. The image is $\{y \mid y \neq 0\}$.

However, our example has finite rank because of the additional constraint imposed by the relation $x'' = x^n$. What went wrong:

$(x^4 - 2(x')^2)' = 4x^3x' - 4x'x^3 = 0$, modulo $x'' = x^3$.

This seems rather special: $x'' = x^3, y(x^4 + (x')^2) = 1$ works fine.

Having too few relations (i.e., positive dimension) hurts completeness, but having too many of the wrong kind is also an obstacle.

Positive quantifier elimination and complete varieties

Theorem

(van den Dries) Let T be an $\mathscr{L}$-theory and $\varphi(\bar{x})$ an $\mathscr{L}$-formula. Then φ is equivalent modulo T to a positive quantifier-free formula iff for all $\mathcal{M}, \mathcal{N} \models T$, substructures $A \subseteq \mathcal{M}$, $\bar{a} \in A$, and $\mathcal{L}$-homomorphisms $f : A \to \mathcal{N}$ we have $\mathcal{M} \models \varphi(\bar{a}) \implies \mathcal{N} \models \varphi(f(\bar{a}))$.

Slick proof of the fund. thm. of elimination theory: (place extension theorem + image of second projection is existential and varieties are homogeneous in the first factor + divide witness by coordinate of least value) $\Rightarrow$ definition of the image is preserved.

Replacing the valuation ring: Maximal δ-rings

Definition

Let R be a δ-ring, $K \supseteq R$ a δ-field, L a δ-field and $\varphi : R \to L$ a δ-homomorphism. We say R is a *maximal δ-ring* (with respect to K) if φ does not extend to a δ-homomorphism with domain contained in K but strictly containing R and codomain a δ-field.

(Ritt, P. Blum) A maximal δ-ring R is a *local differential ring* with unique maximal differential ideal $\mathfrak{m}$. Moreover, $x \in K \setminus R$ iff $1 \in \mathfrak{m}\{x\}$.

(Morrison) K-maximal δ-rings are integrally closed. If x satisfies a linear differential equation over such an R and $1/x \notin R$, then $x \in R$.

Valuative criteria instead of fundamental theorem

Theorem

(Pong) Let V be a δ-variety in $\mathbb{A}^n$. Then V is δ-complete iff for every $K \models DCF_0$ and K-maximal δ-ring R, $V(K) = V(R)$.

In some sense this is the best you can do; Pong showed that complete δ-varieties embed in $\mathbb{A}^1$. Still, it's desirable to reformulate the criterion because...

1. Pong's criterion applies only to $\mathbb{A}^n$; we want a criterion that applies directly to varieties in $\mathbb{P}^n$.

2. In practice, verifying the valuative criterion seems to boil down to formal elimination; is there a "syntactic" version that does not refer to maximal δ-rings?

Projective valuative criterion

Definition

Let $p = (p_0 : p_1 : \cdots : p_n) \in \mathbb{P}^n$ and let S be a δ-ring. With only slight abuse of terminology, we say p is in S (denoted $p \in S$) if for some $0 \leq i \leq n$ we have $p_i \neq 0$ and $\frac{p_j}{p_i} \in S$ for all $0 \leq j \leq n$. If V is a differential subvariety of $\mathbb{P}^n$ such that $p \in V$ and $p \in S$, we write $p \in V(S)$.

Theorem

Let V be a differential subvariety of $\mathbb{P}^n$. Then V is δ-complete if and only if for every K-maximal δ-ring $(R, f, \mathfrak{m})$ and point $p \in V(K)$ we have $p \in V(R)$.

Projective valuative criterion, cont.

Theorem

Let V be a differential subvariety of $\mathbb{P}^1$. Then V is δ-complete if and only if for every K-maximal δ-ring $(R, f, \mathfrak{m})$ and point $p = (x : 1)$ or $(1 : x) \in V(K)$ we have either $x \in R$ or $\frac{1}{x} \in R$.

Proof.

- Forward direction: prove the contrapositive by using a point $p \in V(K) \setminus V(R)$ to construct a Kolchin-closed subset of $V \times \mathbb{A}^m$ whose projection is not Kolchin closed.
- The following is not equivalent to a positive q.f.f.:

$$\exists \bar{x} \, (\bar{x} \in V \wedge (\wedge_{i \notin I} x_i = 0) \wedge (\vee_{i \in I} x_i \neq 0) \wedge \\ \wedge (\wedge_{i \in I} P_i(\bar{x}, \bar{y}) = 0))$$

Proof, cont.

- Reverse direction: Extend a homomorphism $f : A \to L$ from a δ-subring A of a DCF K into a DCF L to a maximal homomorphism $\hat{f} : R \to \hat{L}$ from a K-maximal δ-ring R into a DCF $\hat{L}$.
- Write out the definition of projection as a first-order formula with parameters from A. Replace $\bar{x}$ with elements of K satisfying the formula.
- Use the property $V(K) \Rightarrow V(R)$ as a substitute for a valuation ring; then by δ-homogeneity we may assume that every parameter belongs to R. Homomorphisms preserve positive q.f.f.

More modified valuative criteria

Theorem

Let $V \subseteq \mathbb{P}^1$ be a δ-variety. Let x be the coordinate corresponding to affine chart U and denote by V_a the affine restriction $V \cap U$. Let $W \subseteq V_a \times \mathbb{A}^m$ be defined by two δ-polynomials of the form $\sum_k y_k s_k(x) - 1$ and $\sum_l z_l t_l\left(\frac{1}{x}\right) - 1$. Then V is δ-complete if and only if for every such W, $\overline{\pi_2(W)}$ does not contain the point $\overline{0} \in \mathbb{A}^m$.

Theorem

Let V and W be as above, and let I be the differential ideal in $\mathcal{F}\{x, \bar{y}, \bar{z}\}$ generated by the defining polynomials of both V_a and W. Then V is δ-complete if and only if for all such W the differential elimination ideal I_x contains a δ-polynomial in $\overline{y}, \overline{z}$ having non-zero constant term.

Proof of the "syntactic" version of the valuative criterion

Proof.

($\Rightarrow$)

- If V is δ-complete, the preceding version asserts that $\bar{0} \notin \overline{\pi_2(W)}$.
- Because $\mathbf{V}(I_x) = \overline{\pi_2(W)}$, I_x must contain a polynomial with non-zero constant term.

($\Leftarrow$)

- If V is not complete, then $\bar{0} \in \overline{\pi_2(W)}$ and no member of I_x can have a non-zero constant term.

$\square$

Transfer from DCF_0 to $\mathbb{C}$

Lemma

Let $\bar{a} = \{a_i\}_{i\in\mathbb{N}}$ be a sequence of parameters from $\mathcal{F} \models DCF_0$. Then there exists a sequence of complex numbers $\bar{b} = \{b_i\}_{i\in\mathbb{N}}$ such that $\bar{b}$ has the same (infinite) type over $\mathbb{Q}$ (restricted to the language of rings) as $\bar{a}$.

Proposition

Let $I \subseteq \mathcal{F}\{\bar{x}, \bar{y}\}$ be the differential ideal generated by δ-polynomials $f_1(\bar{x}, \bar{y}), \ldots, f_r(\bar{x}, \bar{y})$. The differential elimination ideal $I_{\bar{x}} \subseteq \mathcal{F}\{\bar{y}\}$ contains a δ-polynomial in $\bar{y}$ having non-zero constant term if and only if there exist natural numbers k, s_k such that the non-differential elimination ideal $\widetilde{I}^{(s_k)}_{\bar{x},\ldots,\bar{x}^{(s_k)}} \subseteq \mathbb{C}[\bar{y}, \ldots, \bar{y}^{(s_k)}]$ contains a polynomial having non-zero constant term.

The valuative criterion in action

Theorem

The projective closure of $xx'' = x'$ is δ-complete.

Proof.

(Sketch)

- Consider $(x : 1) \in V(K)$, where V is the projective closure of $xx'' = x'$. Suppose $x, 1/x \notin R$, so that $1 \in \mathfrak{m}\{x\}$ and $1 \in \mathfrak{m}\{1/x\}$.
- It follows by a technical elimination algorithm that actually $1 \in \mathfrak{m}[1/x]$. Thus x is integral over R.
- By Morrison, R is integrally closed, so $x \in R$. This contradicts the assumption.

□

The differential catenary conjecture

Classical theorem

Let p be a point of an irreducible affine algebraic variety V (over an ACF). If V has dimension n, then there exists a chain of irreducible algebraic subvarieties $p = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = V$.

An analogous-but-unproven statement can be made in the differential case. (Phrased in terms of prime differential ideals, finite chains are replaced by *long gap chains* having infinite descending chains of prime differential ideals in the gaps between the "main" entries.)

The following "weak catenary conjecture" is a corollary to the differential catenary conjecture:

Bounding the dimension of counterexamples to completeness

Conjecture

Let V be an irreducible positive-dimensional affine differential algebraic variety defined over a differentially closed field. Let W be an irreducible zero-dimensional differential algebraic subvariety of V. Then there exists a proper irreducible differential subvariety V_1 of V such that $V_1 \cap W \neq \emptyset$ and $V_1 \not\subseteq W$.

Conditional result

Assuming the weak catenary conjecture, a projective differential algebraic variety V (over a differentially closed field) is complete iff $\pi_2 : V \times Y \to Y$ is a closed map for every zero-dimensional differential algebraic variety Y.

Proof sketch, cont.

Sketch of proof:

- Suppose toward contradiction that there is a positive-dimensional differential variety Y such that $\pi_2 : V \times Y \to Y$ is not closed but there are no such zero-dimensional differential varieties.
- Given that assumption, use the weak catenary conjecture repeatedly to obtain a dimension-zero witness of incompleteness, which is contradictory.
- Using Freitag's differential version of Bertini's theorem, one can show that if V is not complete, then there exists a positive-dimensional irreducible affine differential algebraic variety Y_1 and irreducible closed $X \subseteq V \times Y_1$ such that not only is $\pi_2(X)$ not closed, but also $\pi_2(X)^{cl}$ is positive dimensional and $W := (\pi_2(X)^{cl} \setminus \pi_2(X))^{cl}$ is zero dimensional.

Proof sketch, cont.

- Repeatedly apply the weak catenary conjecture (starting with $\pi_2(X)^{cl}$ and W) until you obtain a zero-dimensional irreducible differential subvariety Y' of $\pi_2(X)^{cl}$ such that $W \cap Y' \neq \emptyset$ and $Y' \not\subseteq W$.
- Let $X' = X \cap (V \times Y')$. We claim that $\pi_2(X')$ is not closed.
- Suppose $\pi_2(X')$ *is* closed. Note that $\pi_2(X') = \pi_2(X) \cap Y'$. Because $\pi_2(X)$ is not closed and $Y' \cap (\pi_2(X)^{cl} \setminus \pi_2(X))^{cl}$ is nonempty, we can show that $\pi_2(X')$ is a proper closed subset of Y'.
- Since $Y' \not\subseteq W$, it follows that $W \cap Y'$ is a proper closed subset of Y'. Now review the definition of W to confirm that $Y' = \pi_2(X') \cup (W \cap Y')$.
- This contradicts irreducibility of Y'. So $\pi_2(X')$ is not closed and Y' contradicts our assumption that there are no zero-dimensional witnesses to incompleteness of V.

Constructive content of proofs in differential algebra

Many proofs in algebra use non-constructive arguments (e.g., Hilbert's basis theorem). However, often at least a partially equivalent constructive result can be systematically extracted from the non-constructive proof.

"Proof mining" relies on proof-theoretic translations of proofs in one formal theory (e.g., Peano Arithmetic) into proofs in more constructive theories (e.g., PRA^{ω}, the theory of primitive recursive functionals of finite-type).

Henry Towsner and I have started looking at proofs in differential algebra (especially the Ritt-Raudenbush basis theorem and related results) to see what the constructive analogues are and what bounds arise from the translations.

Thank you for listening!

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