Differential Groups and the Gamma Function

Michael F. Singer
(joint work with Charlotte Hardouin)

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Department of Mathematics
North Carolina State University
Raleigh, NC 27695-8205
singer@math.ncsu.edu

KSDA, New York

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Theorem: (Hölder, 1887) The Gamma function $\Gamma(x+1) = x\Gamma(x)$ satisfies no polynomial differential equation.

Theorem: (Hardouin, 2005; van der Put, 2006) Let $b(x) \in \mathbb{C}(x)$ and let $u(x)$ be a nonzero function, meromorphic on $\mathbb{C}$ such that

$$u(x+1) = b(x)u(x).$$

The function $u(x)$ is differentially algebraic over 1-periodic meromorphic functions if and only if there exists a nonzero homogeneous linear differential polynomial $L(Y)$ with coefficients in $\mathbb{C}$ such that

$$L(\frac{b'(x)}{b(x)}) = g(x+1) - g(x)$$

for some $g(x) \in \mathbb{C}(x)$.

Ex: For $\Gamma(x)$, $L(\frac{1}{x}) = g(x+1) - g(x)$???

Also for q-difference equations and systems $u_i(x+1) = b_i(x)u_i(x)$.

Theorem: (Ishizaki, 1998) If $a(x), b(x) \in \mathbb{C}(x)$ and $z(x) \notin \mathbb{C}(x)$ satisfies

$$z(qx) = a(x)z(x) + b(x), \ |q| \neq 1 \tag{1}$$

and is meromorphic on $\underline{\mathbb{C}}$, then $z(x)$ is not differentially algebraic over q-periodic functions.

$z(x)$ meromorphic on $\underline{\mathbb{C}\backslash\{0\}}$ and satisfies (1):

Assume distinct zeroes and poles of $a(x)$ are not q-multiples of each other.

Theorem: (H-S, 2007) $z(x)$ is differentially algebraic iff $a(x) = cx^n$ and

- $b = f(qx) - a(x)f(x)$ for some $f \in \mathbb{C}(x)$, when $a \neq q^r$, or
- $b = f(qx) - af(x) + dx^r$ for some $f \in \mathbb{C}(x), d \in \mathbb{C}$ when $a = q^r, r \in \mathbb{Z}$.

Theorem: (Roques, 2007) Let $y_1(x), y_2(x)$ be lin. indep. solutions of

$$y(q^2x) - \frac{2ax-2}{a^2x-1}y(qx) + \frac{x-1}{a^2x-q^2}y(x) = 0 \tag{2}$$

with $a \notin q^{\mathbb{Z}}$ and $a^2 \in q^{\mathbb{Z}}$. Then $y_1(x), y_2(x), y_1(qx)$ are algebraically independent.

(H-S, 2007): $y_1(x), y_2(x), y_1(qx)$ are differentially independent. Give necessary and sufficient conditions for a large class of linear differential equations.

All of these results follow from a

Differential Galois Theory of Linear Difference Equations

and an understanding of

Linear Differential Groups

- Galois Theory of Linear Difference Equations
- Linear Differential Algebraic Groups
- Differential Galois Theory of Linear Difference Equations
- Differential Relations Among Solutions of Linear Difference Equations
- Final Comments

Galois Theory of Linear Difference Equations

k - field, σ - an automorphism   Ex. $\mathbb{C}(x),\ \sigma(x) = x+1,\ \sigma(x) = qx$

Difference Equation: $\sigma(Y) = AY \quad A \in \mathrm{GL}_n(k)$

Splitting Ring: $k[Y, \frac{1}{\det(Y)}],\ Y = (y_{i,j})$ indeterminates , $\sigma(Y) = AY$,
$M =$ max σ-ideal

$$R = k[Y, \frac{1}{\det(Y)}]/M = k[Z, \frac{1}{\det(Z)}] = \sigma\text{-Picard-Vessiot Ring}$$

- M is radical $\Rightarrow$ R is reduced
- If $C = k^\sigma = \{c \in k \mid \sigma c = c\}$ is alg closed $\Rightarrow$ R is unique and $R^\sigma = C$

Ex.

$$k = \mathbb{C} \quad \sigma(y) = -y$$

$$R = \mathbb{C}[y, \frac{1}{y}]/(y^2 - 1)$$

σ-Galois Group: $\mathrm{Gal}_{\sigma(R/k)} = \{\phi : R \to R \mid \phi \text{ is a } \sigma\ k\text{-automorphism}\}$

Ex.

$$k = \mathbb{C} \quad \sigma(y) = -y \ \Rightarrow R = \mathbb{C}[y, \tfrac{1}{y}]/(y^2 - 1)$$
$$\mathrm{Gal}_\sigma(R/k) = \mathbb{Z}/2\mathbb{Z}$$

Ex.

$$k = \mathbb{C}(x), \sigma(x) = x + 1$$
$$\sigma^2 y - x\sigma y + y = 0 \Rightarrow \sigma Y = \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix} Y$$
$$R = k[Y, \tfrac{1}{\det(Y)}]/(\det(Y) - 1), \quad \mathrm{Gal}_\sigma = \mathrm{SL}_2(\mathbb{C})$$

Ex.

$$\sigma(y) - y = f, \ f \in k \Leftrightarrow \sigma \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$$
$$\phi \in \mathrm{Gal}_\sigma \Rightarrow \phi(y) = y + c_\phi, c_\phi \in C$$
$$\mathrm{Gal}_\sigma = (C, +) \text{ or } \{0\}$$

- $\phi \in \mathrm{Gal}_\sigma,\ \sigma(Z) = AZ \Rightarrow \phi(Z) = Z[\phi],\ \ [\phi] \in \mathrm{GL}_n(C)$

  $\mathrm{Gal}_\sigma \hookrightarrow \mathrm{GL}_n(C)$ and the image is Zariski closed

  $\mathrm{Gal}_\sigma = G(C)$, G a lin. alg. gp. $/C$.

- $R =$ coord. ring of a G-torsor

  $R^{\mathrm{Gal}_\sigma} = k$

  $\dim(G) =$ Krull $\dim._k R$ ($\simeq$ trans. deg. of quotient field)

The structure of $\mathrm{Gal}(K/k)$ measures algebraic relations among the solutions

.

Ex.

$$\sigma^2 y - x\,\sigma y + y = 0 \Rightarrow \sigma Y = \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix} Y$$

$$\mathrm{Gal}_\sigma = \mathrm{SL}_2(\mathbb{C})$$

$$3 = \dim \mathrm{SL}_2(\mathbb{C}) = \mathit{tr.\ deg.}_k k(y_1, y_2, \sigma(y_1), \sigma(y_2))$$

$$\Rightarrow y_1, y_2, \sigma(y_1) \text{ alg. indep. over } k$$

Ex. $f_1, \ldots, f_n \in k$, k a difference field w. alg. closed const.

$$\sigma(y_1) - y_1 = f_1$$
$$\vdots$$
$$\sigma(y_n) - y_n = f_n$$

Picard-Vessiot ring = $k[y_1, \ldots, y_n]$

Prop. $y_1, \ldots, y_n$ alg. dep. over k if and only if

$\exists g \in k$ and a const coeff. linear form L s.t. $L(y_1, \ldots, y_n) = g$
(equiv., $c_1 f_1 + \ldots + c_n f_n = \sigma(g) - g$)

Ex. $y(x+1) - y(x) = \frac{1}{x}$

$\frac{1}{x} \neq g(x+1) - g(x) \Rightarrow y(x)$ is not alg. over $\mathbb{C}(x)$.

Linear Differential Algebraic Groups

P. Cassidy-“Differential Algebraic Groups” *Am. J. Math.* 94(1972),891-954
+ 5 more papers, book by Kolchin, papers by Sit, Buium, Pillay et al., Ovchinnikov

(k, δ) = a differentially closed differential field

Definition: A subgroup $G \subset \mathrm{GL}_n(k) \subset k^{n^2}$ is a linear differential algebraic group if it is Kolchin-closed in $\mathrm{GL}_n(k)$, that is, G is the set of zeros in $\mathrm{GL}_n(k)$ of a collection of differential polynomials in n^2 variables.

Ex. Any linear algebraic group defined over k, that is, a subgroup of $\mathrm{GL}_n(k)$ defined by (algebraic) polynomials, e.g., $\mathrm{GL}_n(k), \mathrm{SL}_n(k)$

Ex. Let $C = \ker \delta$ and let $G(k)$ be a linear algebraic group defined over k. Then $G(C)$ is a linear *differential* algebraic group (just add $\{\delta y_{i,j} = 0\}_{i,j=1}^n$ to the defining equations!)

Ex. Differential subgroups of $G_a(k) = (k, +) = \{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \mid z \in k\}$

The linear differential subgroups are all of the form

$$G_a^L = \{z \in k \mid L(z) = 0\}$$

where L is a linear homogeneous differential polynomial.
For example, if $m = 1$,

$$G_a^\delta = \{z \in k \mid \delta(z) = 0\} = G_a(C)$$

Ex. Differential subgroups of $G_a^n(k) = (k^n, +)$

The linear differential subgroups are all of the form

$$G_a^{\mathcal{L}} = \{(z_1, \ldots, z_n) \in k^n \mid L(z_1, \ldots, z_n) = 0 \ \forall L \in \mathcal{L}\}$$

where $\mathcal{L}$ is a set of linear homogeneous differential polynomials.

Ex. Differential subgroups of $G_m(k) = (k^*, \cdot) = \mathrm{GL}_1(k)$
The connected linear differential subgroups are all of the form

$$G_a^L = \{z \in k^* \mid L(\frac{z'}{z}) = 0\}$$

where L is a linear homogeneous differential polynomial.

This follows from the exactness of

$$(1) \longrightarrow G_m(C) \longrightarrow G_m(k) \stackrel{z \mapsto \frac{\partial z}{z}}{\longrightarrow} G_a(k) \longrightarrow (0)$$

Ex. H a Zariski-dense proper differential subgroup of $\mathrm{SL}_n(k)$

$\Rightarrow \exists g \in \mathrm{SL}_n(k)$ such that $gHg^{-1} = \mathrm{SL}_n(C),\ C = \ker(\delta)$.

In general if H a Zariski-dense proper differential subgroup of $G \subset \mathrm{GL}_n(k)$, a simple noncommutative algebraic group defined over C

$\Rightarrow \exists g \in \mathrm{GL}_n(k)$ such that $gHg^{-1} = G(C),\ C = \ker(\delta)$.

Differential Galois Theory of Linear Difference Equations

k - field, σ - an automorphism $\quad \delta$ - a derivation s.t. $\sigma\delta = \delta\sigma$

Ex. $\mathbb{C}(x)$: $\sigma(x) = x + 1,\ \delta = \frac{d}{dx}$

$\sigma(x) = qx, \qquad \delta = x\frac{d}{dx}$

$\mathbb{C}(x, t)$: $\sigma(x) = x + 1, \quad \delta = \frac{\partial}{\partial t}$

Difference Equation: $\sigma(Y) = AY \;\; A \in \mathrm{GL}_n(k)$

Splitting Ring: $k\{Y, \frac{1}{\det(Y)}\} = k[Y, \delta Y, \delta^2 Y, \ldots, \frac{1}{\det(Y)}]$

$Y = (y_{i,j})$ *differential* indeterminates

$\sigma(Y) = AY,\ \sigma(\delta Y) = \delta(\sigma Y) = A(\delta Y) + (\delta A)Y, \ldots M =$ max $\sigma\delta$-ideal

$$R = k\{Y, \frac{1}{\det(Y)}\}/M = k\{Z, \frac{1}{\det(Z)}\} = \sigma\delta\text{-Picard-Vessiot Ring}$$

$$k \text{ - } \sigma\delta \text{ field}$$

$$\sigma(Y) = AY, A \in \mathrm{GL}_n(k)$$

$$R = k\{Z, \tfrac{1}{\det(Z)}\} \text{ - } \sigma\delta\text{-Picard-Vessiot ring}$$

- R is reduced
- If $C = k^\sigma = \{c \in k \mid \sigma c = c\}$ is *differentially* closed
  $\Rightarrow R$ is unique and $R^\sigma = C$

$\sigma\delta$-Galois Group: $\mathrm{Gal}_{\sigma\delta}(R/k) = \{\phi : R \to R \mid \phi$ is a $\sigma\delta$ k-automorphism$\}$

- $\phi \in \mathrm{Gal}_{\sigma\delta}$ $\sigma(Z) = AZ \Rightarrow \phi(Z) = Z[\phi], \quad [\phi] \in \mathrm{GL}_n(C)$

  $\mathrm{Gal}_{\sigma\delta} \hookrightarrow \mathrm{GL}_n(C)$ and the image is Kolchin closed

  $\mathrm{Gal}_{\sigma\delta} = G(C)$, G a lin. *differential* alg. gp. $/C$.

- $\mathrm{Gal}_{\sigma\delta}$ is Zariski dense in Gal_σ

- $R =$ coord. ring of a G-torsor

  - $R^{\mathrm{Gal}_{\sigma\delta}} = k$
  - Assume G connected. Then diff. $\dim._C(G) =$ diff. tr. $\deg_k F$ where F is the quotient field of R.

Ex.

$$k = \tilde{\mathbb{C}} \;\; \sigma(y) = -y \;\Rightarrow R = k[y, \tfrac{1}{y}]/(y^2 - 1)$$
$$\mathrm{Gal}_{\sigma\delta}(R/k) = \mathbb{Z}/2\mathbb{Z}$$

Ex.

$$\sigma(y) - y = f,\; f \in k, \quad \mathrm{Gal}_{\sigma\delta} \subset \mathbb{G}_a$$
$$\Rightarrow \mathrm{Gal}_{\sigma\delta} = \{c \in R^\sigma \mid L(c) = 0\} \text{ for some } L \in R^\sigma[\delta].$$

Ex.

$$k = \tilde{\mathbb{C}}(x), \sigma(x) = x + 1,\; \delta(x) = 1$$
$$\sigma^2 y - xy + y = 0 \Rightarrow \sigma Y = \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix} Y$$
$$\text{Will show: } R = k\{Y, \tfrac{1}{\det(Y)}\}/\{\det(Y) - 1\}$$
$$\mathrm{Gal}_{\delta\sigma} = \mathrm{SL}_2(\tilde{\mathbb{C}})$$

Differential Relations Among Solutions of Linear Difference Equations

Groups Measure Relations

k - $\sigma\delta$ - field, $C = k^\sigma$ differentially closed.

Differential subgroups of $G_a^n(k) = (k^n, +)$ are all of the form

$$G_a^{\mathcal{L}} = \{(z_1, \ldots, z_n) \in k^n \mid L(z_1, \ldots, z_n) = 0 \; L \in \mathcal{L}\}$$

where $\mathcal{L}$ is a set of linear homogeneous differential polynomials.

$$\Downarrow$$

Proposition. Let R be a $\sigma\delta$-Picard-Vessiot extension of k containing $z_1, \ldots, z_n$ such that

$$\sigma(z_i) - z_i = f_i, \quad i = 1, \ldots, n.$$

with $f_i \in k$. Then $z_1, \ldots, z_n$ are differentially dependent over k if and only if there is a homogeneous linear differential polynomial L over C such that

$$L(z_1, \ldots, z_n) = g \quad g \in k$$

Equivalently, $L(f_1, \ldots, f_n) = \sigma(g) - g$.

Corollary. Let $f_1, \ldots, f_n \in \mathbb{C}(x), \sigma(x) = x + 1, \delta = \frac{d}{dx}$ and let $z_1, \ldots, z_n$ satisfy

$$\sigma(z_i) - z_i = f_i, \quad i = 1, \ldots, n.$$

Then $z_1, \ldots, z_n$ are differentially dependent over $\mathcal{F}(x)$ ($\mathcal{F}$ is the field of 1-periodic functions) if and only if there is a homogeneous linear differential polynomial L over $\mathbb{C}$ such that

$$L(z_1, \ldots, z_n) = g \quad g \in \mathbb{C}(x)$$

Equivalently, $L(f_1, \ldots, f_n) = \sigma(g) - g$.

- Similar result for q-difference equations. Also for $\sigma y_i = f_i y_i$

The Gamma function is hypertranscendental.

• $z(x) = \Gamma'(x)/\Gamma(x)$ satisfies $\sigma(z) - z = \frac{1}{x}$.

• If $z(x)$ satisfies a polynomial differential equation, then

$$\exists L \in \mathbb{C}[\frac{d}{dx}], g(x) \in \mathbb{C}(x) \text{ s.t. } L(\frac{1}{x}) = g(x+1) - g(x)$$

• $L(\frac{1}{x})$ has a pole $\Rightarrow$ $g(x)$ has a pole.

• If $g(x)$ has a pole then $g(x+1) - g(x)$ has at least two poles but $L(\frac{1}{x})$ has exactly one pole.

If H a Zariski-dense proper differential subgroup of $G \subset \mathrm{GL}_n(k)$, a simple noncommutative algebraic group defined over C

$$\Rightarrow \exists g \in \mathrm{GL}_n(k) \text{ such that } gHg^{-1} = G(C),\ C = \ker(\delta).$$

$$\Downarrow$$

Proposition. Let $A \in \mathrm{GL}_n(k)$ and assume the σ-Galois group of $\sigma(Y) = AY$ to be a simple noncommutative linear algebraic group G of dimension t. Let $R = k\{Z, \frac{1}{\det Z}\}$ be the $\sigma\delta$-PV ring.

The differential trans. deg. of R over k is less than t

$$\Updownarrow$$

$$\exists\ B \in \mathrm{gl}_n(k) \text{ s.t. } \sigma(B) = ABA^{-1} + \delta(A)A^{-1}$$
$$\text{(in which case, } (\delta Z - BZ)Z^{-1} \in \mathrm{gl}_n(k^\sigma))$$

Ex.

$$k = \mathbb{C}(x), \sigma(x) = x + 1$$

$$\sigma^2 y - x\,\sigma y + y = 0 \Rightarrow \sigma Y = \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix} Y$$

$$R = k[Y, \tfrac{1}{\det(Y)}]/(\det(Y) - 1), \ \ \mathrm{Gal}_\sigma = \mathrm{SL}_2(\mathbb{C})$$

$y_1(x), y_2(x)$ linearly independent solutions.

$y_1(x), y_2(x), y_1(x+1)$ are differentialy dependent over $\mathbb{C}(x)$

$$\Updownarrow$$

$$\exists \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{gl}_2(\mathbb{C}(x)) \text{ s.t.}$$

$$\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix}' \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix}^{-1} + \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix}^{-1}$$

This 4^{th} order inhomogeneous equation has no such solutions

$\Rightarrow y_1(x), y_2(x), y_1(x+1)$ are differentialy independent over $\mathbb{C}(x)$

Final Comments

- General Theory: Consider integrable

$$\Sigma = \{\sigma_1, \ldots, \sigma_r\}, \Delta = \{\partial_1, \ldots, \partial_s\}$$

linear systems and measure dependence on auxillary derivations $\delta_1, \ldots, \delta_t$. Can show that for

$$\gamma(x,t) = \int_1^t u^x e^{-u} du$$

we have $\gamma, \gamma_x, \gamma_{xx}, \ldots$ alg. indep $/\mathbb{C}(x,t)$.

- Isomonodromic $\Leftrightarrow$ constant Galois group.

- Inverse problem

- Nonlinear equations