Basics of Dimension in Differential Algebra

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February 22 2013[1]

[1] This is a preliminary report on a joint, unfinished, study work with Phyllis Cassidy. Presented at the Kolchin Seminar in Differential Algebra on February 22, 2013.

Notations

- $\mathfrak{F}$, a differential field, characteristic zero
- $\Delta = \{ \delta_1, \dots, \delta_m \}$, a set of commuting derivations
- $Y : y_1, \dots, y_n$, set of **differential indeterminates**
- $\Theta = \{ \theta = \delta_1^{e_1} \cdots \delta_m^{e_m} \mid (e_1, \dots, e_m) \in \mathbb{N}^m \}$, set of **derivative operators**
- $\Theta Y = \{ \theta y_j \}_{\theta \in \Theta, 1 \leqslant j \leqslant m}$, set of **derivatives** of y_j, $1 \leqslant j \leqslant m$
- If $\theta = \delta_1^{e_1} \cdots \delta_m^{e_m}$, then the **order** of θ is $|\theta| = e_1 + \cdots + e_m$.
- $\mathfrak{R} = \mathfrak{F}\{ y_1, \dots, y_n \} = \mathfrak{F}[\Theta Y]$ **differential polynomial ring**
- System of PADE (**partial algebraic differential equations**)

$$F_i(y_1, \dots, y_n) = 0, \quad i = 1, \dots, k$$

- $\Phi =$ set of $F_1, \dots, F_k$
- $\mathfrak{a} = [\Phi]$, **differential ideal generated by** Φ

Ranking of Derivatives

- A **ranking** is a total order on ΘY satisfying
  $u \leqslant \theta u, \text{ and } u \leqslant v \Rightarrow \theta u \leqslant \theta v \text{ for any } u, v \in \Theta Y, \theta \in \Theta.$
- **Every ranking is a well-ordering on ΘY.**
- If $u < v$, we say u **has lower rank than** v.
- A ranking is **orderly** if

  $$|\theta| < |\theta'| \Rightarrow \theta y_i < \theta' y_j \text{ for all } \theta, \theta' \in \Theta \text{ and } 1 \leqslant i, j \leqslant n.$$

- Fix a ranking. $F \in \mathfrak{R}$, $F \notin \mathfrak{F}$, the highest ranked derivative u_F occurring in F is called its **leader**.
- $F \in \mathfrak{R}$ is **linear** if

$$F(y_1, \dots, y_n) = a_0 + \sum_{i=1}^{q} a_i \, \theta_i y_{k_i} \tag{1}$$

and **linear homogeneous** if $a_0 = 0$.

Differential Field Extensions

- $\mathfrak{G}$, $\mathfrak{F}$ differential fields
- $\mathfrak{G}$ is a (differential) **extension** of $\mathfrak{F}$ if $\mathfrak{G} \supseteq \mathfrak{F}$ and

$$\delta : \mathfrak{G} \to \mathfrak{G} \text{ restricts to } \delta : \mathfrak{F} \to \mathfrak{F}$$

- $\mathfrak{G}$ is a **finitely generated** extension of $\mathfrak{F}$ if there exist $\eta_1, \ldots, \eta_n \in \mathfrak{G}$ such that $\mathfrak{G} = \mathfrak{F}(\{\, \theta\eta_j \,\}_{\theta\in\Theta, 1\leqslant j\leqslant n})$.
  If so, we write $\mathfrak{G} = \mathfrak{F}\langle \eta_1, \ldots, \eta_n \rangle$.
- Example: $y'' - 3y' + 2y = 0$ is a linear homogeneous (ordinary) differential polynomial equation.
- e^x, e^{2x} are linearly independent solutions over $\mathbb{Q}$
- $\mathfrak{G} = \mathbb{Q}\langle e^x, e^{2x} \rangle = \mathbb{Q}(e^x)$ is a finitely generated extension of $\mathbb{Q}$, indeed, a Picard-Vessiot extension.

Differential Algebraic Dependence

- Let $\mathfrak{G}$ be a differential extension of $\mathfrak{F}$.
- Let η be a family $\{\eta_j\}_{1\leqslant j\leqslant n}$ with $\eta_j \in \mathfrak{G}$. By abuse, we also use the vector notation for η and write $\eta = (\eta_1, \ldots, \eta_n) \in \mathfrak{G}^n$.
- We say η is **Δ-algebraically dependent over $\mathfrak{F}$** if the family $\{\,\theta\eta_j\,\}_{\theta\in\Theta,1\leqslant j\leqslant n}$ is algebraic dependent over $\mathfrak{F}$
- If not, we say η is **Δ-algebraically independent over $\mathfrak{F}$**.
- Example: $(\eta_1, \eta_2) = (\tan x, \sin x)$ is Δ-algebraically dependent over $\mathbb{Q}$ since $\delta(\tan x)(1 - \sin^2 x) = 1$.
- $(\eta_1, \eta_2) = (x, J_n(x))$ is Δ-algebraically dependent over $\mathbb{Q}$.

$$\eta_1^2\delta^2\eta_2 + \eta_1\delta\eta_2 + (\eta_1^2 - n^2)\eta_2 = 0,$$

where $J_n(x)$ is the n^{th} Bessel function of the first kind, and $n \in \mathbb{N}$.

Differentially Algebraic Elements

- Let $\mathfrak{G}$ be a differential extension of $\mathfrak{F}$.
- $\alpha \in \mathfrak{G}$ is **Δ-algebraic over $\mathfrak{F}$** if it satisfies some (differential) polynomial equation with coefficients in $\mathfrak{F}$. In other words, the family $\{\theta\alpha\}_{\theta\in\Theta}$ is algebraically dependent over $\mathfrak{F}$.
- If not, say α is **Δ-transcendental over $\mathfrak{F}$**.
- e^x (resp., $\sin x$, resp. $\cos x$) is transcendental (not algebraic), but Δ-algebraic over $\mathbb{Q}$.
- $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ is Δ-transcendental (not Δ-algebraic) over $\mathbb{C}(x)$.
- $J_n(x)$ is Δ-algebraic over $\mathbb{Q}(x)$.
- $\mathfrak{G}$ is **Δ-algebraic over $\mathfrak{F}$** if every element of $\mathfrak{G}$ is.

Differential Transcendence Basis

- Let $\mathcal{G}$ be an extension of $\mathcal{F}$; Σ be a family of elements of $\mathcal{G}$.
- The following are equivalent:
  1. Σ is Δ-algebraically independent over $\mathcal{F}$, and $\mathcal{G}$ is Δ-algebraic over $\mathcal{F}\langle\Sigma\rangle$.
  2. Σ is a minimal family such that $\mathcal{G}$ is Δ-algebraic over $\mathcal{F}\langle\Sigma\rangle$.
  3. Σ is a maximal family that is Δ-algebraically independent over $\mathcal{F}$.
- If Σ satisfies the above, then Σ is called a **Δ-transcendence basis of $\mathcal{G}$ over $\mathcal{F}$**.
- Δ-transcendence basis exists (and may be the empty set).
- Any two have the same cardinal number, called the **Δ-dimension** (or **Δ-transcendence degree**) of $\mathcal{G}$ over $\mathcal{F}$.
- Let $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{H}$. Then

$$\Delta\text{-dim } \mathcal{H}/\mathcal{F} = \Delta\text{-dim } \mathcal{H}/\mathcal{G} + \Delta\text{-dim } \mathcal{G}/\mathcal{F}$$

Univariate Polynomials and the Binomial Basis

- Let k be a field of characteristic zero and let $R = k[X]$ be the polynomial ring over k in one indeterminate X.
- For $\tau \in \mathbb{N}$, let R_τ be the k-vector subspace of R consisting of all polynomials of degree $\leqslant \tau$.
- Then R_τ has a k-basis $\mathcal{P}_\tau = \{X^i \mid 0 \leqslant i \leqslant \tau\}$.
- R_τ also has the k-basis $\mathcal{B}_\tau = \{\binom{X+i}{i} \mid 0 \leqslant i \leqslant \tau\}$.
- Every $\xi(X) \in R_\tau$ can be written uniquely in the form

$$\xi(X) = \sum_{0 \leqslant i \leqslant \tau} a_i \binom{X+i}{i} \tag{2}$$

  with $a_i \in k$ for $0 \leqslant i \leqslant \tau$. Call (1) the **Binomial Form**.
- A polynomial $\xi(X) \in R_\tau$ is said to be **numerical** or called a **numerical polynomial** if $\xi(t)$ is an integer for all sufficiently large integers $t \in \mathbb{N}$.

Numerical Polynomials in Binomial Form

- **A polynomial $\xi(X) \in R_\tau$ is numerical if and only if all the a_i in Eq. (2) are integers.**
- Clearly, if all a_i in Eq. (2) are integers, then $\xi(X)$ is numerical.
- Conversely, we prove $a_i \in \mathbb{Z}$ for $i = 0, \ldots, \tau$ by induction on τ.
- The case $\tau = 0$ is trivial.
- Making use of the binomial identity

$$\binom{X+i}{i} - \binom{X+i-1}{i} = \binom{X+i-1}{i-1},$$

we see that

$$\xi(X) - \xi(X-1) = \sum_{1 \leqslant i \leqslant \tau} a_i \binom{X+i-1}{i-1}$$

is numerical, and $a_i \in \mathbb{Z}$ for $1 \leqslant i \leqslant \tau$, and hence also $a_0 \in \mathbb{Z}$.

Ordering of Numerical Polynomials

- We define an ordering relation $\leqslant$ on the set of numerical polynomials. We say $\xi \leqslant \xi'$ if $\xi(t) \leqslant \xi'(t)$ for all sufficiently large $t \in \mathbb{N}$.
- **Let** $\xi(X) = \sum_{i=0}^{m} a_i \binom{X+i}{i}$ **and** $\xi'(X) = \sum_{i=0}^{m} b_i \binom{X+i}{i}$ **be two numerical polynomials in** R_m**. Then**
$$\xi \leqslant \xi' \iff (a_m, \dots, a_0) \leqslant_{\text{lex}} (b_m, \dots, b_0).$$
- Let τ be the maximum j such that $a_j \neq b_j$. By subtracting off $\sum_{i=\tau+1}^{m} a_i \binom{X+i}{i}$ from $\xi(X)$ and $\xi'(X)$, we may suppose $\tau = m$. Then $\xi'(s) - \xi(s) = (b_m - a_m)\binom{s+m}{m} +$ lower terms, and this is positive for all sufficient large s if and only if $(b_m - a_m) > 0$, which holds if and only if $(a_m, \dots, a_0) \leqslant_{\text{lex}} (b_m, \dots, b_0)$.
- The same holds for any basis $f_0, f_1, \dots, f_m$ of R_m provided for all i, $\deg f_i = i$ and $f_i(s) > 0$ for all $s \gg 0$.
- $\leqslant$ is a total ordering on the set of all numerical polynomials.

Differential Dimension Polynomial

- Let $\eta = (\eta_1, \cdots, \eta_n) \in \mathfrak{G}^n$, $\mathfrak{G}$ being an extension of $\mathfrak{F}$.
- A finer measure of the algebraic dependence of the family $\Theta\eta$ is given by that of the finite family $\Theta(s)\eta$, where for $s \in \mathbb{N}$,
$$\Theta(s)\eta := \{\theta\eta_j\}_{\theta\in\Theta, |\theta|\leqslant s, 1\leqslant j\leqslant n}.$$
- Let $\dim(s) = \operatorname{tr.deg}_{\mathfrak{F}} \mathfrak{F}(\Theta(s)\eta)$.
- **There exists a (unique) polynomial $\xi(X) \in \mathbb{Q}[X]$ satisfying:**
  1. **For every sufficiently large $s \in \mathbb{N}$, $\dim(s) = \xi(s)$.**
  2. $\deg \xi(X) \leqslant m$, **where** $m = |\Delta|$.
  3. **If we write** $\xi(X) = \sum_{i=0}^{m} a_i \binom{X+i}{i}$, **then** $a_m = \Delta\text{-}\mathbf{dim}_{\mathfrak{F}}(\mathfrak{F}\langle\eta\rangle)$.
- $\xi(X) = \xi_{\eta/\mathfrak{F}}(X)$ is called the **differential dimension polynomial (or Kolchin polynomial) of η over $\mathfrak{F}$**.

Finitely Generated Extensions and Primes

- Let $\mathfrak{p}$ be a prime differential ideal in $\mathcal{R} = \mathcal{F}\{ y_1, \ldots, y_n \}$
- Let $\mathcal{G}$ = quotient field of $\mathcal{F}\{ y_1, \ldots, y_n \}/\mathfrak{p}$
- Let $\eta_i = y_i + \mathfrak{p} \in \mathcal{F}\{ y_1, \ldots, y_n \}/\mathfrak{p}$. Then $\mathcal{G} = \mathcal{F}\langle \eta_1, \ldots, \eta_n \rangle$.
- The kernel of the substitution homomorphism: $\mathcal{F}\{ y_1, \ldots, y_n \} \longrightarrow \mathcal{F}\{ y_1, \ldots, y_n \}/\mathfrak{p} = \mathcal{F}\{ \eta_1, \ldots, \eta_n \}$ defined by $F \mapsto F + \mathfrak{p} = F(\eta_1, \ldots, \eta_n)$ is $\mathfrak{p}$.
- More generally, the set of differential polynomials in $\mathcal{R} = \mathcal{F}\{ y_1, \ldots, y_n \}$ vanishing at any $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathcal{H}^n$, where $\mathcal{H}$ is some extension of $\mathcal{F}$, is a prime differential ideal $\mathfrak{q}$ of $\mathcal{R}$, called the **defining differential ideal of α over $\mathcal{F}$**.
- We define the **differential dimension polynomial** $\xi_{\mathfrak{p}/\mathcal{F}}(X)$ of $\mathfrak{p}$ over $\mathcal{F}$ to be $\xi_{\eta/\mathcal{F}}(X)$.
- **If $\mathfrak{p} \subseteq \mathfrak{q}$, then $\xi_{\mathfrak{p}/\mathcal{F}} \geqslant \xi_{\mathfrak{q}/\mathcal{F}}$, and equality holds if and only if $\mathfrak{p} = \mathfrak{q}$.**

Transforming Dependent Variables

- $\eta = (\eta_1, \ldots, \eta_n) \in \mathfrak{G}^n$, $\zeta = (\zeta_1, \cdots, \zeta_{n'}) \in \mathfrak{G}^{n'}$
- Suppose for some $h \in \mathbb{N}$, and for all $j, 1 \leqslant j \leqslant n$ we have $\eta_j \in \mathfrak{F}(\{\, \theta\zeta_k \,\}_{\theta \in \Theta(h), 1 \leqslant k \leqslant n'})$, then
  1. $\mathfrak{F}(\{\, \theta\eta_k \,\}_{\theta \in \Theta(s), 1 \leqslant k \leqslant n}) \subseteq \mathfrak{F}(\{\, \theta\zeta_k \,\}_{\theta \in \Theta(s+h), 1 \leqslant k \leqslant n'})$
  2. $\xi_{\eta/\mathfrak{F}}(X) \leqslant \xi_{\zeta/\mathfrak{F}}(X + h)$.
- **If $\mathfrak{F}\langle\eta\rangle = \mathfrak{F}\langle\zeta\rangle$, then there exists $h \in \mathbb{N}$ such that**

$$\xi_{\zeta/\mathfrak{F}}(X - h) \leqslant \xi_{\eta/\mathfrak{F}}(X) \leqslant \xi_{\zeta/\mathfrak{F}}(X + h).$$

- **If $\mathfrak{F}(\eta) = \mathfrak{F}(\zeta)$ ($h = 0$), then $\xi_{\eta/\mathfrak{F}} = \xi_{\zeta/\mathfrak{F}}$.**
- $\xi_{\eta/\mathfrak{F}}$ is a **birational invariant**, but **not** a **differential birational invariant**.
- $\tau = \deg \xi_{\eta/\mathfrak{F}}$, called the **differential type** (resp. the leading coefficient a_τ, called the **typical differential dimension**) **of $\mathfrak{F}\langle\eta\rangle$ over $\mathfrak{F}$** is a differential birational invariant.

Transforming Independent Variables

- Let $\mathfrak{C}$ be the field of constants of $\mathfrak{F}$.
- $C = (c_{i,i'})_{1 \leqslant i,i' \leqslant m} \in GL(m, \mathfrak{C})$
- $\delta_i = \sum_{i'=1}^{m} c_{i,i'} \delta'_{i'}$
- Then $\mathfrak{F}$ is a Δ'-field, where $\Delta' = \{ \delta'_1, \cdots, \delta'_m \}$
- **Let $\mathcal{G}$ be a finitely generated extension of $\mathcal{F}$. Let τ be the differential type and a_τ the typical differential dimension of $\mathcal{G}$ over $\mathcal{F}$. There there exists a matric C and a subset Δ^* of Δ' consisting of τ linearly independent elements such that $\mathcal{G}$ is a finitely generated Δ^* extension of $\mathfrak{F}$ of Δ^*-dimension a_τ.**
- The $m \times m$ matrix C over $\mathfrak{C}$ that gives Δ' may be chosen from a Zariski open set. This result is of interest mainly when $\tau < m$, and then a $\tau \times m$ matric suffices.

Interpreting Differential Type and Dimension

- Let $\mathfrak{G} = \mathfrak{F}\langle \eta_1, \ldots, \eta_n \rangle$ be a finitely generated extension of $\mathfrak{F}$. Let $\mathfrak{p}$ be the defining differential ideal of η over $\mathfrak{F}$.
- Then η is a **generic zero of $\mathfrak{p}$ over $\mathfrak{F}$**, or loosely, the "**general solution**" of a finite system of PADE with coefficients from $\mathfrak{F}$:

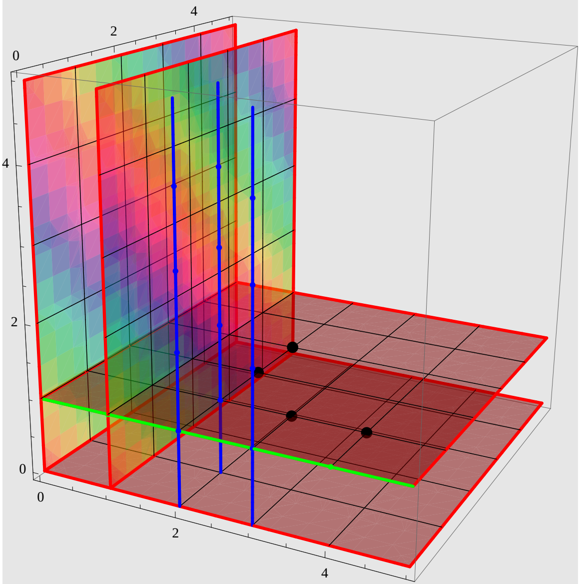
$$F_i(y_1, \ldots, y_n) = 0, \qquad i = 1, \ldots, p. \tag{3}$$

- **If $\xi_{\eta/\mathfrak{F}} = 0$, then $[\mathfrak{G} : \mathfrak{F}] < \infty$ ($\mathfrak{G}$ is algebraic over $\mathfrak{F}$).**
- Otherwise, if $\tau = \deg \xi_{\eta/\mathfrak{F}}$, then $a_\tau > 0$, and the general solution of Eq. (3) depends exactly on a_τ arbitrary functions of τ independent variables.
- The differential type τ and typical differential dimension $\mathfrak{a}_\tau$ are invariant not only under differentially birational transformation of the dependent variables ($\mathfrak{F}\langle \eta \rangle = \mathfrak{F}\langle \zeta \rangle$), but also under transformation of the independent variables (Δ to Δ').

Initial Sets in m-Dimensional Lattice

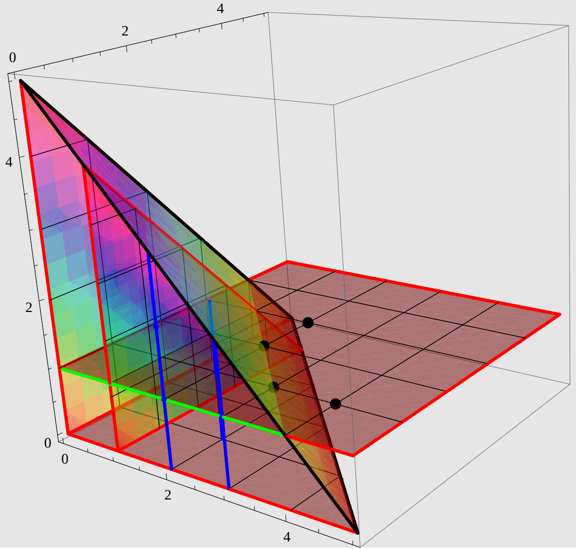
- The **product order** in $\mathbb{N}^m$ is defined by $(a_1, \ldots, a_m) \leqslant (b_1, \ldots, b_m)$ if $a_i \leqslant b_i$ for $i = 1, \ldots, m$. This is a partial order: two vectors need not be compatible, such as $(1, 2)$ and $(2, 1)$. It is reflexive, antisymmetric and transitive.
- Given a positive integer m, a subset V of $\mathbb{N}^m$ is an **initial set** if under the natural product order of $\mathbb{N}^m$, for all $a, b \in \mathbb{N}^m$, $b \in V$ and $a \leqslant b$ implies $a \in V$.
- For $m = 3$, V is built by stacking planes, then lines, then points away from the origin.
- Example:An initial set V built from 3 planes: x = 0,1; z = 0;
  4 lines based at $(0, 0, 1)$ in the direction X (green);
  at $(2, 0, 0)$ in the direction Z (blue);
  at $(3, 0, 0)$ in the direction Z (blue);
  at $(2, 1, 0)$ in the direction of Z (blue);
  and 4 points at $(2, 2, 1), (2, 3, 1), (3, 1, 1), (4, 1, 1)$.

Example of an Initial Set

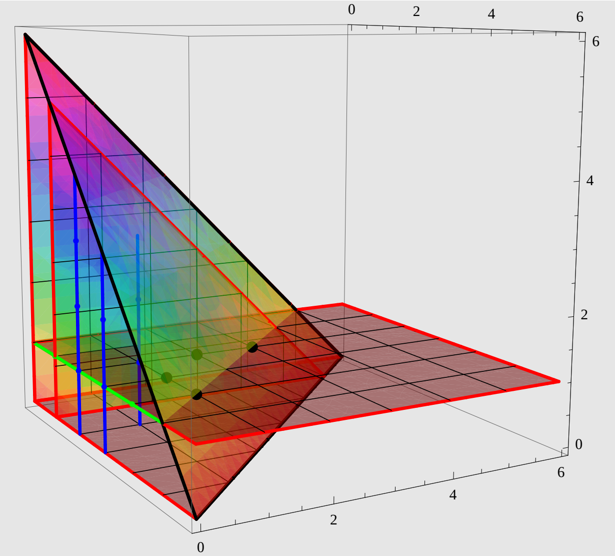


Three planes (x=0,1; z=0), 4 lines (3 blue 1 green), 4 points on z=1

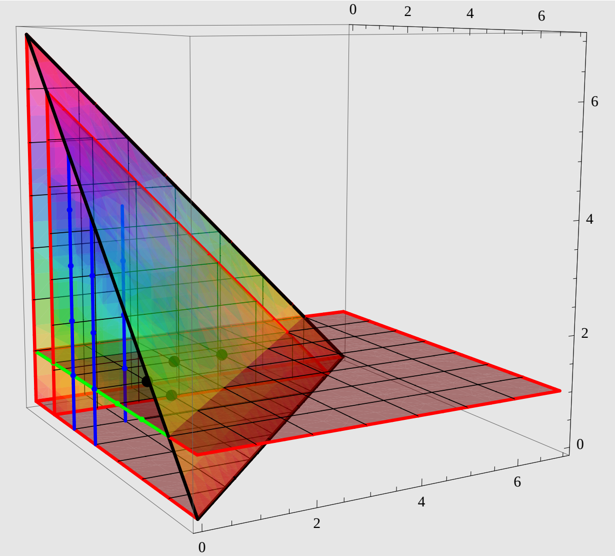
Cut Off at Norm $\leqslant 5$



Cut Off at Norm $\leqslant 6$



Cut Off at Norm $\leqslant 7$



Lattice points and Initial Sets

- For $u \in \mathbb{N}^m$, define **the m-dimensional cone based at u** to be the set the set $u + \mathbb{N}^m = \{ u + e \mid e \in \mathbb{N}^m \}$.
- For an initial set $V \subseteq \mathbb{N}^m$, let $E(V)$ be the set of minimal elements with respect to the product order in $\mathbb{N}^m \backslash V$. Elements of $E(V)$ are called **the minimal cogenerators of V**. **$E(V)$ is finite.**[2]
- $\mathbb{N}^m \backslash V = \bigcup_{e \in E(V)} e + \mathbb{N}^m$.
- **Conversely, given a finite set $E \subset \mathbb{N}^m$, the complement $V(E)$ of $\bigcup_{e \in E(V)} e + \mathbb{N}^m$ is an initial set, and consists of all points $v \in \mathbb{N}^m$ that are not greater than or equal to any point in E under the product order.**
- For any $a = (a_1, \ldots, a_m) \in V$, we denote the sum of its components by $|a|$ and called it the **norm** of a.

[2] Dickson's Lemma or Hilbert Basis Theorem.

Dimension Sequence of Initial Sets

- Let $\mathbb{N}_m = \{1, \ldots, m\}$, $u \in \mathbb{N}^m$, and $J \subseteq \mathbb{N}_m$.
- Define the **J-cone at u** to be the set u_J of all points $v \in \mathbb{N}^m$ such that $v_j = u_j$ for $j \notin J$. An m-dimensional cone at u is a $\mathbb{N}_m$-cone at u. Call J the **(free) direction of u_J**.
- A subset $K \subseteq \mathbb{N}^m$ is **k-dimensional** if $K = u_J$ for some $J \subset \mathbb{N}_m$ with $\mathrm{Card}(J) = k$. K is **properly k-dimensional in an initial set V** if K is a k-dimensional subset of V but is not contained in any $(k+1)$-dimensional subset of V.
- Let V be an initial subset of $\mathbb{N}^m$. **The number $d_k(V)$ of subsets properly k-dimensional in V is finite for all $k \in \mathbb{N}$.**
- **$d_k(V) = 0$ for $k > m$; $d_m(V) \leqslant 1$, with equality if and only if $V = \mathbb{N}^m$, in which case, $d_k(V) = 0$ for all $k \neq m$.**
- The sequece $\{d_k(V)\}_{k \in \mathbb{N}}$ is called the **dimension sequence of V**.

Dimension Polynomials

- **(Peeling Lemma) Let V be an initial subset of $\mathbb{N}^m$ with dimension sequence $\{ d_k(V) \}_{k\in\mathbb{N}}$. Then for any h, ($0 \leqslant h \leqslant m$) such that $d_h(V) \neq 0$, there exists an initial subset $V_1 \subset V$ such that $d_k(V_1) = d_k(V)$ if $k > h$, $d_h(V_1) = d_h(V) - 1$, and $d_k(V_1) = 0$ if $k < h$.**
- Let $\{V_j\}_{j=1,\ldots,n}$ be a finite sequence of n initial sets $V_j \subseteq \mathbb{N}^{m_j}$ and let V be the disjoint union $\cup_{j=1}^n V_j$.
- **The number of lattice points $a \in V$ with $|a| \leqslant t$ as a function of t is given by a numerical polynomial $\xi_V(X)$.**
- A numerical polynomial obtained this way is called a **dimension polynomial**.
- Examples: Hilbert polynomials, Kolchin polynomials.

Prime (Linear) Differential Ideals

- For simplicity, we restrict ourselves to **linear differential ideals** in $\mathfrak{R} = \mathfrak{F}\{y_1, \ldots, y_n\}$.
- Recall that $F \in \mathfrak{R}$ is **linear** if
  $$F(y_1, \ldots, y_n) = a_0 + \sum_{i=1}^{q} a_i\, \theta_i y_{k_i}.$$
  and **linear homogeneous** if $a_0 = 0$.
- Let $\mathfrak{R}_1 = \mathfrak{F}\{\, y_1, \ldots, y_n \,\}_1$ be the differential vector space consisting of all linear homogeneous differential polynomials.
- A differential ideal $\mathfrak{p}$ is **linear** if $\mathfrak{p} = [\Lambda]$, $\Lambda \subset \mathfrak{R}_1$.
- **$\mathfrak{p}$ linear $\Rightarrow$ $\mathfrak{p}$ prime and homogeneous.**
- $\mathcal{L} = \mathfrak{p} \cap \mathfrak{R}_1 = \sum_{\theta \in \Theta, L \in \Lambda} \mathfrak{F} \cdot \theta L$ **is a Δ-$\mathfrak{F}$-subspace of $\mathfrak{R}_1$.**
- **The mapping from the set of linear Δ-ideals of $\mathfrak{R}$ to the set of Δ-$\mathfrak{F}$-subspace of $\mathfrak{R}_1$ given by $\mathfrak{p} \mapsto \mathfrak{p} \cap \mathfrak{R}_1$ is bijective; with inverse $\mathcal{L} \mapsto [\mathcal{L}] = (\mathcal{L})$.**

Leaders of Characteristic Sets

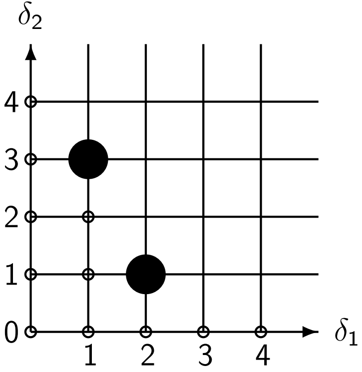
- Fix an orderly ranking. Let $\mathcal{L}_{\text{leader}}$ be set of all θy_j such that it is a leader of some $L \in \mathcal{L}$
- If $u \in \mathcal{L}_{\text{leader}}$, there exists a *unique* $L_u \in \mathcal{L}$ of the form

$$u + \sum a_{uv} v \quad (a_{uv} \in \mathfrak{F}) \tag{4}$$

  where $v \in \Theta Y$, $v \notin \mathcal{L}_{\text{leader}}$, and v is of lower rank than u.
- Every linear differential ideal $\mathfrak{p}$ has a unique, finite, generating set $\mathbf{A}$ with elements of the form (4). $\mathbf{A}$ is called the **canonical characteristic set** of $\mathfrak{p}$.
- For each j ($1 \leqslant j \leqslant n$), let E_j be the set of points $(e_1, \cdots, e_m)$ such that $\delta_1^{e_1} \cdots \delta_m^{e_m} y_j$ is a leader of an element of $\mathbf{A}$.
- For each j, $V_j := V(E_j) = \mathbb{N}^m \setminus \bigcup_{e \in E_j} (e + \mathbb{N}^m)$ is an initial set.
- Then $\xi_{\mathfrak{p}/\mathfrak{F}} = \sum_{j=1}^{m} \xi_{V(E_j)}$. (The derivatives (when evaluated at the generic zero η) are algebraically independent if they lie outside the m-dimensional cones of elements of E_j's).

Example

- The complement of $\mathcal{L}_{\mathrm{leader}}$ in ΘY is an $\mathfrak{F}$-basis of $\mathfrak{R}_1/\mathcal{L}$. Furthermore, if the ranking is orderly, and if $\mathbf{A} : A_1, \ldots, A_k$ is the canonical characteristic set of $\mathfrak{p}$, where the leader of A_i is u_i, then the complement of $\mathcal{L}_{\mathrm{leader}}$ is the set of derivatives $v \in \Theta Y$ that is not a derivative of any u_i, $i = 1, \ldots, k$.



Leaders $\delta_1^2\delta_2 y, \delta_1\delta_2^3 y$

Finite Combinatorics of Stacking and Well-order

- **Let $d = \{ d_k \}_{k \in \mathbb{N}}$ be a sequence of natural numbers such that $d_k = 0$ for all sufficiently large $k \in \mathbb{N}$. Then**
  1. **For any fixed $m \in \mathbb{N}$, there esist only finitely many initial subsets of $\mathbb{N}^m$ with dimension sequence d.**
  2. **There exist only finitely many numerical polynomials of the form $\xi_V(X)$, where V is an initial subset of $\mathbb{N}^m$ for some m, with dimension sequence d.**
- (1): There are only a finite number of ways to "stack" the d_h properly h-dimensional subsets once the properly k-dimensional subsets ($k > h$) have been stacked.
- (2); There is a bound m_0 depending on d alone such that any dimension polynomial ξ_V with $V \subseteq \mathbb{N}^m$ with $m \geqslant m_0$ and dimension sequence d can be **realized** in $\mathbb{N}^{m_0}$ already.
- **The set of dimension polynomials is well-ordered by the ordering on numerical polynomials.**

Minimal Coefficient Vector

- Given a (not necessarily numerical) polynomial $\xi(X) \in R_\tau$, we define a polynomial $\lambda(X)$ called the **derived lower bound** (DLB) of $\xi(X)$ by

$$\lambda(X) = \xi(X + a_\tau) - \binom{X + a_\tau + \tau + 1}{\tau + 1} + \binom{X + \tau + 1}{\tau + 1}. \quad (5)$$

- **$\xi(X)$ is numerical if and only if $\lambda(X)$ is.**
- Let d be the degree of $\lambda(X)$; then $d < \tau$.
- We associate to $\xi(X)$ by induction on τ a vector $\kappa(\xi) \in k^{\tau+1}$ called the **minimal coefficient vector** of ξ as follows:

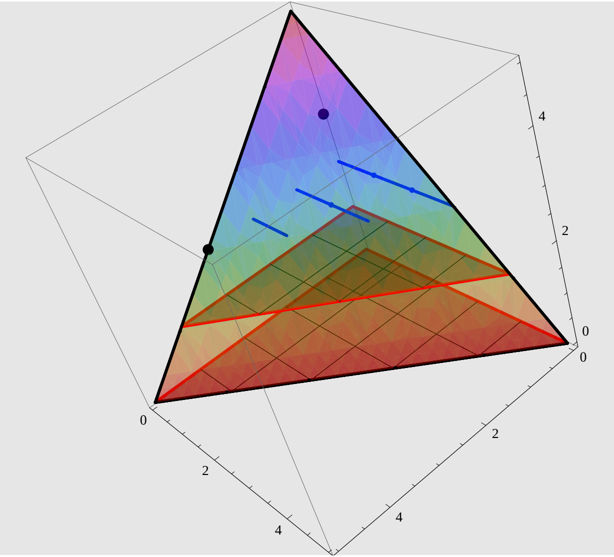
$$\kappa(\xi) = \begin{cases} (a_0) & \text{if } \tau = 0, \\ (a_\tau, 0, \ldots, 0, \kappa(\lambda)) & \text{if } \tau > 0, \end{cases} \quad (6)$$

where a_τ is followed by $\tau - d - 1$ zeros.

Dimension Sequence and Dimension Polynomial

- Let $m = 3$ and $n = 1$. Consider the dimension sequence $d = (2, 3, 2)$, that is, two planes, 3 lines and 2 points. We want to define two initial sets V_1, V_2 with this d as their dimension sequence but different dimension polynomials.
- Define V_1 to consist of the xy planes at floors 0 and 1, the three lines at $(0, 0, 2), (1, 0, 2), (2, 0, 2)$ in the y direction, and the two points $(3, 0, 2)$ and $(0, 0, 3)$. Its dimension polynomial is $2\binom{X+2}{1} + 2\binom{X+1}{1} - 7$. The minimum coefficient vector is $(2, 3, 2)$.
- Observe that the minimum coefficient vector is the same as the dimension sequence. This is the case when the dimension polynomial is minimal for d.

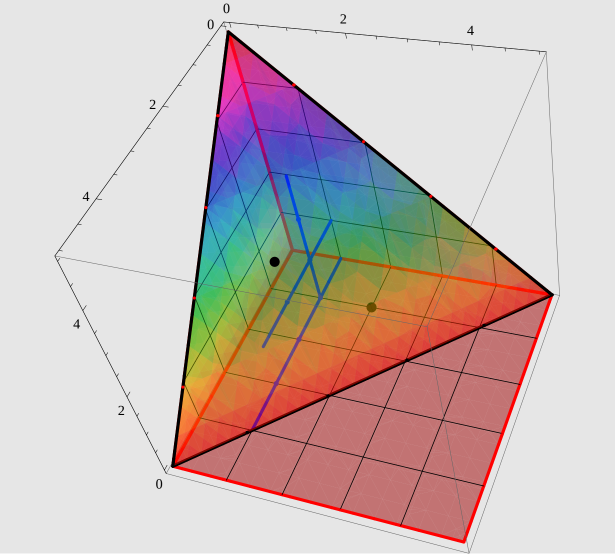
2P3L2PV1



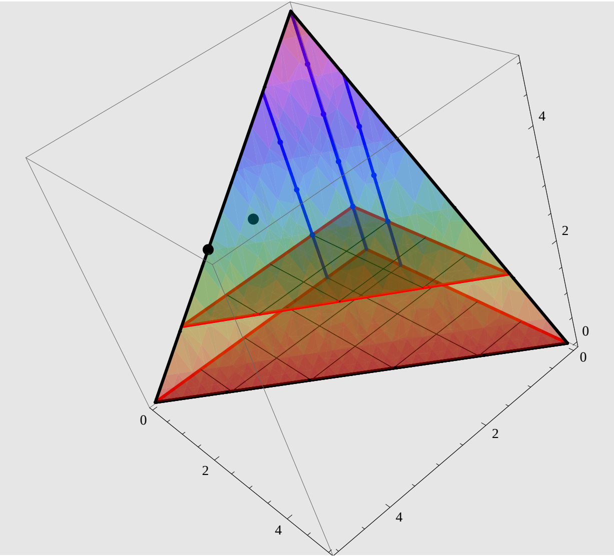
Another try and Some Conjectures

- We define V_2 to consist of the xz-plane, yz-plane, lines at $(1,1,0)$ in the direction z, $(0,1,0)$ in the direction x, and $(0,1,1)$ in the direction x, and two points $(2,1,2)$ and $(1,2,0)$. This has the same dimension polynomial as V_1. This is the minimal dimension polynomial for d.
- Define V_3 to consist to consist of the xy planes at floors 0 and 1, the three lines at $(0,0,0),(0,1,0),(1,0,0)$ in the y direction, and the two points $(2,0,2)$ and $(3,0,2)$. This has the dimension polynomial $2\binom{X+2}{2}+2\binom{X+1}{1}-6$. The minimal coefficient vector is $(2,3,3)$.
- Conjecture: Given a dimension sequence d, let $\xi_{\mathcal{V}}(t)$ be the dimension polynomial for a finite set $\mathcal{V}$ of initial sets V_i, with dimension sequence $d_k(\mathcal{V})=d$. If $\xi(t)$ is minimal among all such $\xi_{\mathcal{V}}(t)$, then the minimal coefficient vector $\kappa(\xi)$ of $\xi(t)$ is the dimension sequence d.

2P3L2PV2



2P3L2PV3



Model Theory, n-types[3]

- Let $\mathcal{T}$ be a theory with language $\mathcal{L}(\mathcal{T})$. Let φ be a formula in the language $\mathcal{L}(\mathcal{T})$. Let $\mathcal{M}$ be a (first order) structure $\mathcal{M}$ and underlying set M.
- Let $A \subseteq M$ and let $\mathcal{L}(A) = \mathcal{L}(\mathcal{T}) \cup \{c_a \mid a \in A\}$ where every c_a is a new constant symbol. A is called a parameter set.
- $\mathcal{M} \models \varphi$ means $\mathcal{M}$ models or satisfies φ.
- $\mathcal{M} \models \mathcal{T}$ means $\mathcal{M} \models \varphi$ for all sentences φ in $\mathcal{L}(\mathcal{T})$.
- An ***n*-type of $\mathcal{M}$ over *A*** is a set p of formulas $\varphi(x_1, \ldots, x_n)$ with parameters from A in at most n free variables $x_1, \ldots, x_n$ such that every finite subset $p_0 \subset p$ can be realized in M^n, that is, there exists $b \in M^n$ and $\mathcal{M} \models \varphi(b)$ for all $\varphi \in p_0$.
- An n-type p is **complete** if it is maximal with respect to inclusion. Equivalently, for every $\varphi(x_1, \ldots, x_n)$ with parameters from A, either $\varphi \in p$ or $\neg\varphi \in p$.

[3] The introduction here should not be considered accurate or correct.

Definable Sets

- A subset $X \subset M^n$ is ***A*-definable** if there is a formula $\varphi(x_1, \ldots, x_n)$ with parameters from A such that $X = \{\, u = (u_1, \ldots u_n) \in M^n \mid \mathcal{M} \models \varphi(u_1, \ldots, u_n) \,\}$. For a differential closed field M, an A-definable subset of M^n is a Kolchin constructible set defined by differential equations and inequations with coefficients from A.
- The ***n*-type of *u* over *A*** is the totality of all A-definable sets $X \subset M^n$ such that $u \in X$. Identifying X with φ, the n-type of u is a set p of formulas $\varphi(x_1, \ldots, x_n)$ with parameter from A in at most n free variables, such that $\mathcal{M} \models p(u)$. Thus the n-type of u is an n-type, denoted by $\mathrm{tp}_n^{\mathcal{M}}(u/A)$, or simply $\mathrm{tp}(u/A)$. The n-type of u is complete.
- An n-type p is **realized** in $\mathcal{M}$ if there exists $b \in M^n$ and $\mathcal{M} \models p(b)$. The n-type of u over A is of course realized (by $u \in M^n$).

Stone Space

- The set $S_n(M/A)$ of all complete n-types of $\mathcal{M}$ over A can be equipped with a topology generated by the family of clopen sets $[X]$ where X is an A-definable subset of M^n, and

$$[X] = \{p \in S_n(M/A) \mid X \in p\}.$$

  Here $X \in p$ means the formula φ defining X belongs to p. The topological space $S_n(M/A)$ is called the **Stone space**.

- The Stone space is compact (that is, every open cover has a finite subcover).

Lasker Rank (or U-rank)

- In Model Theory, the Lasker rank or U-rank is a measure of complexity of a (complete) type, in the context of stable theories.
- For any (complete) n-type $p \in S(A)$, the Lasker Rank of p is an ordinal, denoted $\mathrm{RU}(p)$. For any ordinal α, "$\mathrm{RU}(p) \geqslant \alpha$" is defined recursively as follows:
  1. $\mathrm{RU}(p) \geqslant 0$.
  2. If α is a limit ordinal, then $\mathrm{RU}(p) \geqslant \alpha$ precisely when $\mathrm{RU}(p) \geqslant \alpha'$ for all $\alpha' < \alpha$.
  3. Otherwise, $\mathrm{RU}(p) \geqslant \alpha + 1$ precisely when there is a forking extension q of p with $\mathrm{RU}(q) \geqslant \alpha$.
- We say $RU(p) = \alpha$ when $RU(p) \geqslant \alpha$ but not $RU(p) \geqslant \alpha + 1$.
- If $RU(p) \geqslant \alpha$ for all ordinals α, we say $RU(p) = \infty$ (or is unbounded, undefined).

Expanding the Definition

- Equivalence. $\mathrm{RU}(p) \geqslant \beta \iff \exists\{\,(B_\alpha, q_\alpha)\,\}$ such that $B_0 = A$, $q_0 = p$, $B_\alpha \subseteq \mathfrak{U}$, $\mathfrak{q}_\alpha \subset B_\alpha\{y\}$, and for all $\alpha, \alpha' < \beta; \alpha' < \alpha$, the following holds:

  (1_α) $B_{\alpha'} \subseteq B_\alpha$

  (2_α) $q_{\alpha'} \subseteq q_\alpha$

  (3_α) $\xi_{q_{\alpha'}|B_{\alpha'}} > \xi_{q_\alpha|B_\alpha}$

- $\mathrm{RU}(p) = \beta$ if and only if $\mathrm{RU}(p) \geqslant \beta$ and for all (B_α, q_α) with $\alpha < \beta$ such that $(1_\alpha), (2_\alpha), (3_\alpha)$ hold, but there does not exist (B_β, q_β) $(\beta < \beta + 1)$ $(1_\beta), (2_\beta), (3_\beta)$ hold. In other words, for all (B_β, q_β) such that (1_β) and (2_β) hold, (3_β) fails.

Example Computation (not verified)

- Lasker Rank. Let $p \in S(A)$. Define the Lasker Rank of p as $\mathrm{RU}(p) = Sup\{\mathrm{RU}(q) + 1 \mid \exists B, A \subseteq B \subset \mathfrak{U}, q \in S(B), p \subset q\}$??
- Example. $A = k$. $\mathfrak{p} = [y''] \subset k\{y\}$. Claim: $\mathrm{RU}(p) \geqslant 2$. $\alpha = 1$. Need $B_1, q_1, \mathrm{RU}(q_1) \geqslant 1, B_1 \supset A$. Let $c_1 \in \mathfrak{U}^\delta$ be a constant. Let $B_1 = k(c_1)$. Let $\mathfrak{q}_1 = [y' - c_1] \subset k(c_1)\{y\} = B\{y\}$. Then $\mathfrak{q}_1 \supset \mathfrak{p}$.
- Now need $B_0, q_0, \mathrm{RU}(q_0) \geqslant 0, B_0 \supset B_1$ and $q_0 \supset q_1 \supset \mathfrak{p}$. Let $B_0 = B_1(t)$ aassuming $\delta t = 1$. Let $c_0 \in \mathfrak{U}^\delta$ and let $\mathfrak{q}_0 = [y - c_1 t - c_0] \subset k(c_1)(c_0, t)\{y\}$. So $\mathrm{RU}(q_0) \geqslant 0$.
- $\mathrm{RU}(q_1) = Sup_q\{\mathrm{RU}(q) + 1\} \geqslant \mathrm{RU}(q_0) + 1 \geqslant 0 + 1$.
- $\mathrm{RU}(p) \geqslant \mathrm{RU}(q_1) + 1 \geqslant 2$.