What Initial Values Guarantee Existence and Uniqueness In Algebraic Differential Systems?

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Part I

Introduction

A quotation, more questions, and references

An an example and main results

A Quotation from Gear, 2006

Interestingly, electrical networks were originally modelled with ODEs and a lot of sophisticated techniques were developed to reduce a network to ODEs. ... Initially these were index one problems, so did not present the DAE difficulties of higher index problems, but newer modelling approaches have lead [sic] to index two problems (e.g., [19]) and the problems have become so large and non-linear that some aspects of them, such as *finding consistent initial conditions, are extremely challenging*.

Mechanical systems with constraints usually lead to **index three problems** that cannot be solved directly. *The constraints in a DAE restrict the solution to a manifold and usually we cannot easily find the ODE on that manifold*.

Questions

- Who have been studying these problems?
- Where are the difficulties?
- What determines the set of consistent initial values?
- When is the solution unique?
- When does an explicit form $\dot{\mathbf{z}} = r(\mathbf{z})$ exist?
- How can symbolic methods help?

Theoretical Developments

- Campbell (1980,1985,1987), Campbell and Gear (1995),
  Campbell and Griepentrog (1995)
  Gear and Petzold (1983,1984),
  Gear (1988, 2006), Reich (1988, 1989):
  G. Thomas (1996, 1997), J. Tuomela (1997, 1998)
  singularities, constant rank conditions, linear and
  differentiation index
- Rabier and Rheinboldt (1991, 1994, 1996):
  general existence and uniquenss theory for differential-algebraic systems on π-submanifolds
- Kunkel and Mehrmann (1994, 1996, 2006):
  local invariants, strangeness index, and canonical forms for linear systems with variable coefficients, numerical solutions

Numerical Methods

- difficulties with implicit, unprocessed, high index systems: constant rank condition, and stability
- Campbell (1987):
reduce index through differentiations, drift-off
- Kunkel and Mehrmann (1996a, 1996b):
numerical methods requiring *a priori* knowledge of local and/or global invariants

Other Approaches

- Campbell and Griepentrog (1995):
  combining symbolic with numerical methods
- Thomas (1996):
  symbolic computation of differential index for quasi-linear systems based on algebraic geometry and prolongation
- Thomas (1997), Rabier and Rheinboldt (1994b) :
  singularities, impasse points
- Tuomela (1997a):
  regularizing singular systems with jet spaces

And Some More Other Approaches

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A Quasi-Linear Example Illustrating Our Method

- $z_1(t), z_2(t), z_3(t)$ are functions of t
- $\dot{z}_1(t), \dot{z}_2(t), \dot{z}_3(t)$ are their derivatives with respect to t

$$\begin{array}{rcrcrcr} & - & z_2\dot{z}_2 & + & z_1\dot{z}_3 & = & z_1^4 \\ -z_2\dot{z}_1 & & & + & 2\dot{z}_3 & = & 5z_3 \\ z_3\dot{z}_1 & + & z_1^2\dot{z}_2 & & & = & 3z_2^2 \\ & - & z_1\dot{z}_2 & + & \dot{z}_3 & = & z_3 \end{array}$$

- Algebraic constraints found by symbolic computation:

$$z_1^2 = z_2, z_1^3 = z_3.$$

- Explicit representation found by symbolic computation:

$$\dot{z}_1 = \frac{z_3}{z_2}, \quad \dot{z}_2 = 2z_2, \quad \dot{z}_3 = 3z_3, \quad z_2 \neq 0.$$

Existence and Uniquenss Theorems Summary

- Given **arbitrary** system of first order DAEs (or an ideal J), we can **compute** a **Zariski-closed subset M of $\mathbb{C}^n$** and an **open subset M^0 of M as a finite irredundant union of non-empty Zariski open sets U_k**.
- For all k and any $\mathbf{x}_0 \in U_k$, the initial value problem $(J, \mathbf{x}_0)$ is solved by the unique solution of a **dynamical system**:

$$\dot{z}_i = \mathbf{r}_i(z_1, \ldots, z_n), \quad 1 \leqslant i \leqslant n; \qquad z(0) = \mathbf{x}_0$$

  where $\mathbf{r}_i$ are rational functions (depending on k) which we can compute in $\mathbb{C}(\mathbf{X})^n$ and are everywhere defined on U_k.
- For any **$\mathbf{x} \notin M$, the initial value problem $(J, \mathbf{x})$ does not admit a solution** on interval $(-\epsilon, \epsilon)$ in $\mathbb{C}^n$ for any $\epsilon > 0$.
- Due to its generality, this result **allows for degenerate situations** such as M^0 (or even M) is empty.

The Goals and a Four Part Outline

- Goals: To answer the questions and provide an alternative approach to attack the problems Gear mentioned. We compute algebraic constraints on the initial conditions, and when possible, find an explicit representation of the ODE system as a dynamical system which can be solved uniquely either numerically or symbolically by quadrature.
- Part I: (This brief) Introduction
- Part II: Gerasimova and Razmyslov: Convergence
- Part III: Sit and Pritchard: Basics Theory
- Part IV: Sit and Pritchard: E & U Results

Part II

Gerasimova and Razmyslov

Convergence of Taylor Map

An intuitive discussion

A 2016 Result of Gerasimova and Razmyslov

- Let $\mathcal{A}$ be an arbitrary finitely generated differential commutative-associative $\mathbb{C}$-algebra without divisors of zero and transcendence degree 1 over $\mathbb{C}$.
- The **spectrum $\mathrm{Spec}_{\mathbb{C}}\mathcal{A}$ of $\mathcal{A}$** is the set of *maximal ideals* of $\mathcal{A}$. Let $M \in \mathrm{Spec}_{\mathbb{C}}\mathcal{A}$.
- Let ψ_M be the $\mathbb{C}$-homomorphism $\mathcal{A} \to \mathcal{A}/M \cong \mathbb{C}$. Let $\widetilde{\psi}_M : \mathcal{A} \to \mathbb{C}[[z]]$ be defined by the "Taylor" map:
- $$\widetilde{\psi}_M(a) := \sum_{r=0}^{\infty} \psi_M(a^{(r)}) \frac{z^r}{r!}, \qquad a \in \mathcal{A}.$$
- **Main Theorem. For every $a \in \mathcal{A}$, $\widetilde{\psi}_M(a)$ is convergent in a neighborhood of zero.**

Connecting Differential Algebra with Algebra

- Let $\mathcal{A} = \mathbf{k}\{f_1, \ldots, f_m\}$ be a finitely generated ordinary differential $\mathbf{k}$-algebra without zero divisors with $\mathbf{k}$ an algebraically closed field of constants and char 0.

- Let $\mathfrak{p}$ be the defining (prime) differential ideal for $f_1, \ldots, f_m$ in $\mathbf{k}\{y_1, \ldots, y_m\}$, the differential polynomial ring on n indeterminates. Then we have an exact sequence:

$$0 \to \mathfrak{p} \to \mathbf{k}\{y_1, \ldots, y_m\} \to \mathcal{A} = \mathbf{k}\{f_1, \ldots, f_m\} \to 0.$$

- **Theorem 1 (G & R). Suppose $\mathcal{A}$ has tr. deg. 1 over k. Then $\mathcal{A}$ is a finitely generated k-algebra.**

- The **Main Theorem** earlier is **Corollary 1**, where $\mathbf{k} = \mathbb{C}$.

Intuitive Justification

- Indeed $\mathcal{A} = \mathbf{k}[g_1, \ldots, g_n, g_1', \ldots, g_n']$ by reduction of the system generating $\mathfrak{p}$ to first order, where $g_1, \ldots, g_n$ belong to a finite subset of $f_1, \ldots, f_m$ and their derivatives, and g_i' is the derivative of g_i.
- Questions:
  1. Do we need **k** to be algebraically closed?
  2. Do we need **k** be a field of constants?
  3. What if $\mathcal{A}$ just have finite tr. deg. over **k**?
- Let I be the defining polynomial ideal for $g_1, \ldots, g_n, g_1', \ldots g_n'$ in $\mathbf{k}[u_1, \ldots, u_n, v_1, \ldots, v_n]$. Then we have an exact sequence of **k**-homomorphisms:

$$0 \to I \to \mathbf{k}[u, v] \to \mathcal{A} = \mathbf{k}[g, g'] \to 0.$$

Parameterized Curves Interpretation

- Suppose that $f_1, \ldots, f_m$ (and hence also $g_1, \ldots, g_n, g_1', \ldots, g_n'$) are analytic functions $\mathbb{C} \to \mathbb{C}$ of a complex variable z in a neighborhood of $z = 0$ and that differentiation is d/dz.
- Then each $a \in \mathcal{A}$ is also an analytic function $a : \mathbb{C} \to \mathbb{C}$ and has a convergent Taylor Series in a neighborhood U_a of $z = 0$:

$$a(z) = a(0) + \sum_{r=0}^{\infty} a^r(0) \frac{z^r}{r!}, \qquad z \in U_a.$$

- The **graph** of $a \in \mathcal{A}$ would be a complex plane curve.
- We may interpret any tuple $\mathbf{h} := (h_1, \ldots, h_s) \in \mathcal{A}^s$ as an analytic curve in $\mathbb{C}^s$, or more precisely, a **parameterized space curve in the complex parameter** z. At $z = 0$, $\mathbf{h}(0) =: (c_1, \ldots, c_s)$ is the "starting point" of the curve $\mathbf{h}$.

A Dynamical System Setting for $\mathcal{A}$

- Suppose the defining ideal $I \subset \mathbf{k}[u, v]$ for $g_1, \dots, g_n, g_1', \dots, g_n'$ contains n polynomials of the form $v_i - R_i(u_1, \dots, u_n)$.
- Then $\mathcal{A} = \mathbf{k}[g_1, \dots, g_n]$. Let J be the defining polynomial ideal of $(g_1, \dots, g_n)$. We have an exact sequence:

$$0 \to J \to \mathbf{k}[u_1, \dots, u_n] \xrightarrow{\sigma} \mathcal{A} = \mathbf{k}[g_1, \dots, g_n] \to 0.$$

- In general, if we can solve for $v_i = R_i(u_1, \dots, u_n)$ $(1 \leqslant i \leqslant n)$ from the algebraic system defined by I (say by the Implicit Function Theorem), then R_i may be analytic or rational.
- A system of ODEs of the form $y_i' = R_i(y_1, \dots, y_n)$ for $1 \leqslant i \leqslant n$, where differentiation is d/dz in terms of a parameter z, is known as a **dynamical system**. A **solution** $g_1(z), \dots, g_n(z)$ is a parameterized differentiable curve in $\mathbb{C}^n$.

Initial Values and $\mathrm{Spec}_{\mathbf{k}}\mathcal{A}$

- The **initial conditions** for a dynamical system are specified by $(g_1(0), \ldots, g_n(0)) \in \mathbb{C}^n$.
- By the exact sequence, there is a natural bijection between $\mathrm{Spec}_{\mathbf{k}}\mathcal{R}/J$ and $\mathrm{Spec}_{\mathbf{k}}\mathcal{A}$.
- The maximal ideals N of $\mathcal{R}/J$ are the maximal ideals of $\mathcal{R}$ containing J. Since $\mathbf{k}$ is algebraically closed, N has the form $(u_1 - c_1, \ldots, u_n - c_n)$ for some $c_1, \ldots, c_n \in \mathbf{k}$.
- Under the bijection induced by $\sigma : \mathcal{R} \to \mathcal{A}$ (with $u_i \mapsto g_i$), the maximal ideals $M \in \mathrm{Spec}_{\mathbf{k}}\mathcal{A}$ has the form $(g_1 - c_1, \ldots, g_n - c_1)$.
- Thus $\mathcal{A}/M \cong \mathbf{k}[c_1, \ldots, c_n] = \mathbf{k}$ and $\mathrm{Spec}_{\mathbf{k}}\mathcal{A} \cong \mathbf{k}^n$. When $\mathbf{k} = \mathbb{C}$, and g_i are functions, these are the set of initial values.

Dynamical Systems and Picard Differential Algebra

- Let $\mathcal{R} := \mathbf{k}[u_1, \ldots, u_n]$ be the polynomial ring in $u_1, \ldots, u_n$ and let $\mathbf{R} := (R_1, \ldots, R_n)$ be any n-tuple in $\mathcal{R}^n$.
- We associate to $\mathbf{R}$ a **polynomial dynamical system (also denoted by R)**: $y_i' - R_i(y_1, \ldots y_n)$, $(1 \leqslant i \leqslant n)$. Let $\mathcal{J}$ be the differential ideal generated by $y_i' - R_i(y_1, \ldots, y_n)$, $(1 \leqslant i \leqslant n)$.
- We associate $\mathbf{R}$ with a **derivation** $D_{\mathbf{R}} : \mathcal{R} \to \mathcal{R}$ by defining, for $P \in \mathcal{R}$:

$$D_{\mathbf{R}}(P) := \sum_{i=0}^{n} R_i \cdot \frac{\partial P}{\partial X_i}.$$

  Then $\mathcal{R}$ is an ordinary differential ring called a **Picard differential algebra**, and is isomorphic with $\mathbf{k}\{y_1, \ldots, y_n\}/\mathcal{J}$.

- In this setting (polynomial dynamical system), $\mathcal{A} = \mathbf{k}[g_1, \ldots, g_n] = \mathbf{k}\{g_1, \ldots, g_n\}$ is both the differential algebra and algebra.

Summary: System Transformations

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{p} & \longrightarrow & \mathbf{k}\{y_1, \ldots, y_m\} & \longrightarrow & \mathcal{A} = \mathbf{k}\{f_1, \ldots, f_m\} & \longrightarrow & 0 \\
 & & \Downarrow & & \Downarrow & \text{reduction to order 1} & \Downarrow & & \\
0 & \longrightarrow & I & \longrightarrow & \mathbf{k}[u, v] & \longrightarrow & \mathcal{A} = \mathbf{k}[g, g'] & \longrightarrow & 0 \\
 & & \Downarrow & & \Downarrow & \text{Picard reduction} & \Downarrow & & \\
0 & \longrightarrow & J & \longrightarrow & \mathbf{k}[u_1, \ldots, u_n] & \xrightarrow{\sigma} & \mathcal{A} = \mathbf{k}[g_1, \ldots, g_n] & \longrightarrow & 0
\end{array}
$$

Figure: 1. System Transformations

- (G & R) Any finitely generated commutative associative $\mathbf{k}$-algebra $\mathcal{A}$ with a fixed derivation $D : \mathcal{A} \to \mathcal{A}$ is a homomorphic image of some Picard algebra $\mathcal{R}$ for an appropriate choice of n and $g_1, \ldots, g_n$.

Summary: Initial Values Diagram

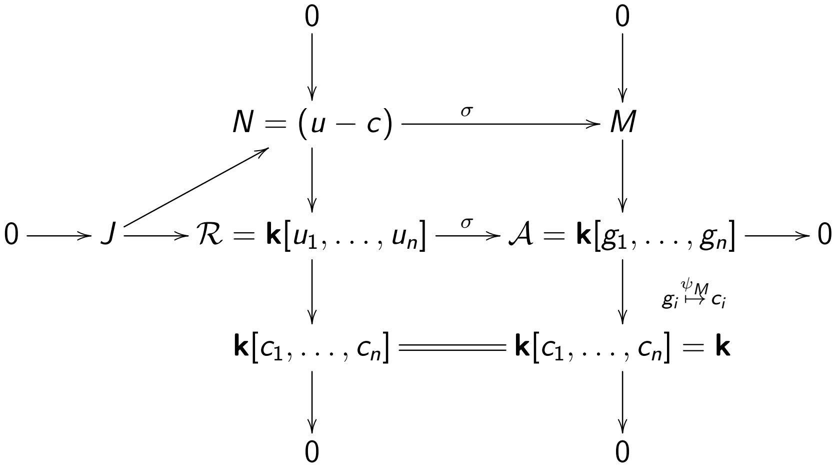

Figure: 2. Initial Conditions

Remarks on the Diagrams

- The maximal ideal N in $\mathcal{R}$ is given by

$$N = (u_1 - c_1, \ldots, u_n - c_n) = \{P \in \mathcal{R} \mid P(c_1, \ldots, c_n) = 0\}.$$

- For any $a \in \mathcal{A}$, $a = P(g_1, \ldots, g_n)$ for some $P \in \mathcal{R}$ and

$$\psi_M(a) = P(\psi_M(g_1), \ldots, \psi_M(g_n)) = P(c_1, \ldots, c_n).$$

- When the elements of $\mathcal{A}$ are functions, we can also think of $\psi_M(a) = a(0)$.
- The maximal ideal M in $\mathcal{A}$ is given by:

$$M = \{a \in \mathcal{A} \mid a = P(g_1, \ldots, g_n), P \in \mathcal{R}, P(c_1, \ldots, c_n) = 0\},$$

and $M = (g_1 - c_1, \ldots, g_n - c_n)$.

Taylor Series

- Recall that ψ_M is a $\mathbb{C}$-homomorphism $\mathcal{A} \to \mathcal{A}/M \cong \mathbb{C}$ and that $\widetilde{\psi}_M : \mathcal{A} \to \mathbb{C}[[z]]$ is defined by the "Taylor" map:
- $$\widetilde{\psi}_M(a) := \sum_{r=0}^{\infty} \psi_M(a^{(r)}) \frac{z^r}{r!}, \qquad a \in \mathcal{A}.$$
- Note that ψ_M is not a differential homomorphism and so in general $\psi_M(a^{(r)}) \neq (\psi_M(a))^{(r)}$ (which would be 0 for $r > 0$). Indeed $a^{(r)} = D_{\mathbf{R}}^{(r)}(P)(g_1, \ldots, g_n)$. Nonetheless, $\psi_M(a^{(r)}) = a^{(r)}(0)$, and hence $\widetilde{\psi}_M(a)$ is the Taylor series of $a(z)$ at $z = 0$.
- The **Main Theorem** of G & R says for any initial conditions $(c_1, \ldots, c_n)$, the algebra generated by a solution to the system defined by $\mathcal{A}$ is analytic (every $a \in \mathcal{A}$ has a convergent power series in a neighborhood U_a of $z = 0$).

Convergence

- To show that $\widetilde{\psi}_M(a)$ is convergent for any $a \in \mathcal{A}$, let $a = P(g_1, \ldots, g_n)$ where $P \in \mathcal{R}$. We have $a^{(r)}(0) = D_{\mathbf{R}}^{(r)}(P)(c_1, \ldots, c_n)$.
- It suffices to show convergence for g_i for all i. We have $g_i^{(r)}(0) = D_{\mathbf{R}}^{(r)}(u_i)(c_1, \ldots, c_n)$.
- Let b be the maximum of absolute values of the polynomials $P, R_1, \ldots, R_n$ and all their derivatives of all orders with respect to $D_{\mathbb{R}}$ (in other words, all the partial derivatives as polynomials in $u_1, \ldots, u_n$), evaluated at $(c_1, \ldots, c_n)$.
- Claim: $|\frac{\psi_M(a^{(r)})}{r!}| \leqslant n^r b^{r+1}$.

An Example, Painlevé I

- Painlevé I: $y'' - 6y^2 - x = 0$.
- The first order reduction gives:

$$x' = 1, y' = z, z' = 6y^2 + x.$$

- Let $D = D_{\mathbf{R}}$ and D^r for the r-fold composition of D with itself. For any $a \in \mathcal{A} = \mathbb{C}[x, y, z]$, with $a = P(x, y, z)$, we have $a^{(r)} = D^r(P) \mid_{u_1=x, u_2=y, u_3=z}$. Special cases when $P = u_i$. We have: $D(x) = 1; D(y) = z; D(z) = 6y^2 + x$.
- So $D^r(x) = 0$ for $r > 1$; So $\psi_M(x^{(r)}) = 0$ for $r > 0$.
- $D^2(y) = 6y^2 + x$; $D^3(y) = 12yz + 1 = D^2(z)$;
- Let $a = 12yz + 1$. $D(a) = 12z^2 + 12y(6y^2 + x)$; $D^2(a) = 24z(6y^2 + x) + 3!6^2y^2z + 12(zx + y) = 360y^2z + 36xz + 12y$;

Classical Existence and Uniqueness Theorem

Let $\mathcal{D}$ be an open subset of $\mathbb{C}^n$, and let the system $\mathbf{v}$ on $\mathcal{D}$ be given by $\dot{\mathbf{z}}(t) = \mathbf{r}(\mathbf{z}(t))$ for $t \in \mathbb{R}$, where $\mathbf{r} : \mathcal{D} \longrightarrow \mathbb{C}^n$ is some analytic map. Then for any $\mathbf{x}_0 \in \mathcal{D}$, there exist an interval $B_\epsilon = (-\epsilon, \epsilon)$ some $\epsilon > 0$, some open neighborhood $\mathcal{O}$ of $\mathbf{x}_0$, and an analytic map $\psi : \mathcal{B}_\epsilon \times \mathcal{O} \longrightarrow \mathcal{D}$ such that $\mathcal{O} \subseteq \mathcal{D}$, and for every $\mathbf{x} \in \mathcal{O}$, we have

- $\psi(0, \mathbf{x}) = \mathbf{x}$
- the map $\psi_{\mathbf{x}} : \mathcal{B}_\epsilon \longrightarrow \mathcal{D}$ defined by $t \mapsto \psi(t, \mathbf{x})$ is the unique solution $\mathbf{z}$ defined on $\mathcal{B}_\epsilon$ satisfying the system $\mathbf{v}$ and the initial condition $\mathbf{z}(0) = \mathbf{x}$.

Part III

Sit and Pritchard

Basic Theory

Concepts, Properties, Algorithms

Main Steps of Our Approach

We combined and modified approaches of **Thomas, Rabier and Rheinboldt** but **with no restrictions** on input form.

- transformations to quasi-linear systems
- the concepts of *essential degree* and *algebraic index* and algorithms to compute these
- algorithms for *prolongation and completion*
- **generalized concepts** of quasi-linearity
- **sufficient conditions for existence and uniqueness theorem**
- **algorithm** to compute constraints on initial conditions
- **algorithm** to compute explicit vector field
- examples and **implementation** in *Axiom*

Transformations to Quasi-linear Systems

By adding new variables and the chain rule, we can convert:

- an explicit system with rational right hand sides to a explicit system **polynomial** system right hand sides.
- a non-autonomous system to an **autonomous** system
- a high order system to a **first order** system
- an analytic system (with some limitation) to a **differential algebraic system**
- a non-linear system to a **quasi-linear system** (**quasi-linearization**)

(Non-linear) First Order DAE System

- Algebraic Indeterminates:

$$\mathbf{X} = (X_1, \cdots, X_n),$$
$$\mathbf{P} = (P_1, \cdots, P_n).$$

Polynomials $f_i(\mathbf{X}, \mathbf{P}) \in \mathbb{C}[\mathbf{X}, \mathbf{P}]$ for $1 \leqslant i \leqslant m$.

- Dependent Variables: $z = (z_1, \ldots, z_n)$
First Order Derivatives: $\dot{z} = \dot{z}_1, \ldots, \dot{z}_n$ (with respect to t)

Any system of first order ordinary DAE:

$$f_i(z_1, \cdots, z_n, \dot{z}_1, \cdots, \dot{z}_n) = 0, \quad 1 \leqslant i \leqslant m$$

Initial conditions: $z(0) = \mathbf{x}_0$ where $\mathbf{x}_0 \in \mathbb{C}^n$.

Essential P-degree Basis and **P**-Strong Basis

- The **P**-*degree* $\deg_{\mathbf{P}}(f)$ of a polynomial $f \in \mathbb{C}[\mathbf{X}, \mathbf{P}]$ is the total degree of f in the variables **P**. The **P**-*degree* of a finite set $F \subset \mathbb{C}[\mathbf{X}, \mathbf{P}]$ is the maximum of $\deg_{\mathbf{P}}(f)$ for $f \in F$.
- The *essential* **P**-*degree* d of a non-zero ideal J of $\mathbb{C}[\mathbf{X}, \mathbf{P}]$ is the least **P**-degree of a finite set F generating J. Such an F is an *essential* **P**-*degree basis*.
- A subset F of J is **P**-*strong* if it generates all polynomials f in J of **P**-degree $\leqslant d$ without involving cancellations of terms of **P**-degree higher than the $\deg_{\mathbf{P}}(f)$.
- Specifically, $f = \sum_{j=1}^{N} h_j f_j$ with $h_j \in \mathbb{C}[\mathbf{X}, \mathbf{P}]$, $h_j \neq 0$, $f_j \in F$ and $\deg_{\mathbf{P}}(h_j f_j) \leqslant \deg_{\mathbf{P}}(f)$.

Algorithm for Essential P-Degree P-Strong Basis

- Using a **P**-degree compatible elimination term ordering where $\mathbf{X} < \mathbf{P}$, compute a Gröbner basis G of the ideal $J = (F)$.

- **The essential P-degree d is the least k such that the elements P-degree $\leqslant k$ in G generates J.**

- **The set E_d of those elements of G is a P-strong essential P-degree basis of J.**

Prolongation of an Ideal

- *prolongation*: For arbitary $h \in \mathbb{C}[\mathbf{X}]$, let $\nabla h = \sum_{j=1}^{n} \frac{\partial h}{\partial X_j} P_j$ in $\mathbb{C}[\mathbf{X}, \mathbf{P}]$.

- The *prolongation ideal* J^* of an ideal J is the ideal generated by J, R, and ∇R, where $R = \sqrt{J \cap \mathbb{C}[\mathbf{X}]}$

- **Algorithm for Prolongation**:
  The prolongation ideal can be computed from any generating set of J. We need only to prolong generators of R, which can be computed.

- Prolongation only introduces polynomials of **P**-degree $\leqslant 1$.

Completion Ideal and Algebraic Index

- An ideal J is *complete* if $J = J^*$.
  The intersection of complete ideals of $\mathbb{C}[\mathbf{X}, \mathbf{P}]$ is complete.
  The *completion ideal* of J is the smallest complete ideal $\widetilde{J}$ containing J.

- **Algorithm for Completion**: Just keep prolonging till it stops. The last one is $\widetilde{J}$.
  The *algebraic index* p is the smallest number of prolongation to obtain a complete ideal.

- Use of an essential $\mathbf{P}$-degree basis for J keeps $\mathbf{P}$-degree low.

Geometric Property

- *first jet domain* V = algebraic set of zeros of J
  initial domain W = algebraic set of zeros of $J \cap \mathbb{C}[\mathbf{X}]$
  = algebraic set of zeros of R
  projection $\pi : V \longrightarrow W$
  an open subset $W^0 = \{\mathbf{x} \mid \pi^{-1}(\mathbf{x})$ **is finite**$\}$

- *tangent variety* $T(W)$ = algebraic set of zeros in $\mathbb{C}^{2n}$ of
  $(R \cup \nabla R)$

- J **complete implies** $V \subseteq T(W)$

Quasi-Linearities and Associated Quasi-linear Ideal

- An ideal J of $\mathbb{C}[\mathbf{X}, \mathbf{P}]$
  (a) is (essentially) *quasi-linear* if $\mathrm{edeg}_{\mathbf{P}}(J) \leqslant 1$.
  (b) is *eventually quasi-linear* if its completion $\widetilde{J}$ is quasi-linear.

- Every J has an *associated quasi-linear ideal* J^{ℓ}, which is generated by the set of all polynomials of **P**-degree at most 1 in J.
  If E a P-strong subset of J, then J^{ℓ} is generated by E_1, the subset of E of P-degree $\leqslant 1$.

Properties

- **Properties**:(i) J quasi-linear implies $\widetilde{J}$ quasi-linear;
(iii) $V \subseteq V^\ell$ and $W = W^\ell$ (hence $T(W) = T(W^\ell)$)
(iii) J is complete if and only if J^ℓ is complete
(iv) ind $J^\ell \leqslant$ ind J.

- All these concepts: essential **P**-degree, prolongation, completion, quasi-linearities are ideal-theoretic, yet all algorithms are simple in an intuitive way, providing flexibility in implementation.

Primary Decomposition

- J complete, but not necessarily quasi-linear
- $R(J) = \sqrt{J \cap \mathbb{C}[\mathbf{X}]}$
- $R = Q_1 \cap \cdots \cap Q_r$: irredundant primary (prime) decomposition
- $K_i = (J \cup Q_i \cup \nabla Q_i)$
- J quasi-linear implies K_i quasi-linear
- J complete implies K_i complete and $K_i \cap \mathbb{C}[\mathbf{X}] = Q_i$.

Part IV

Sit and Pritchard

Algebraic formulation, E & U Theorem

Algorithms, Examples, Conclusion

Algebraic Setting for Existence & Uniqueness

- J be an ideal in $\mathbb{C}[\mathbf{X}, \mathbf{P}]$
  $\mathbf{x} \in \mathbb{C}^n$, $\mathcal{B}_\epsilon = (-\epsilon, \epsilon)$ be an open interval in $\mathbb{R}$
  M be a constructible subset of $\mathbb{C}^n$, for example, M may be:
  $W^0 = \{\, \mathbf{x} \in W \mid \pi^{-1}(\mathbf{x}) \text{ is finite} \,\}$
- A *differentiable map* $\varphi : \mathcal{B}_\epsilon \longrightarrow M$ is a differentiable map $\varphi : \mathcal{B}_\epsilon \longrightarrow \mathbb{C}^n$ whose image is contained in M.
  A *solution to the initial value problem* $(J, \mathbf{x})$ *on* $\mathcal{B}_\epsilon$ *in* M is a differentiable map $\varphi : \mathcal{B}_\epsilon \longrightarrow M$ such that $\varphi(0) = \mathbf{x}$ and $f(\varphi(t), \dot{\varphi}(t)) = 0$ for all $t \in \mathcal{B}_\epsilon$ and for all $f \in J$.

  We also say:
- φ *satisfies the initial value problem* $(J, \mathbf{x})$
  $(J, \mathbf{x})$ *admits a solution in* M
  the image of φ is an *integral curve of* J *through* $\mathbf{x}$.

Existence and Uniqueness (Theorem 6.2.7)

Let $J = (g_1, \ldots, g_m)$ be an ideal in $\mathbb{C}[\mathbf{X}, \mathbf{P}]$, and consider the system of differential algebraic equations

$$\begin{aligned} g_1(z_1, \ldots, z_n, \dot{z}_1, \ldots, \dot{z}_n) &= 0, \\ &\vdots \\ g_m(z_1, \ldots, z_n, \dot{z}_1, \ldots, \dot{z}_n) &= 0. \end{aligned}$$

Then we can effectively compute

- (1) a Zariski-closed subset M of $\mathbb{C}^n$ and some integer $\nu \geqslant 0$;
- (2) for each k, $1 \leqslant k \leqslant \nu$, a non-empty Zariski open subset U_k of M;
- (3) for each k, $1 \leqslant k \leqslant \nu$, an n-dimensional vector $\mathbf{r}_k = (r_{k,1}, \ldots, r_{k,n})$ of rational functions in $\mathbb{C}(\mathbf{X})^n$, everywhere defined on U_k such that

- (4) the union $M^0 = \cup_{k=1}^{\nu} U_k$ is irredundant;

- (5) for every $\epsilon > 0$ and for every $\mathbf{x} \in M^0$, the image of a differentiable map $\psi_{\mathbf{x}} : \mathcal{B}_\epsilon \longrightarrow M^0$ is an integral curve of J through $\mathbf{x}$ if and only if $\psi_{\mathbf{x}}(0) = \mathbf{x}$ and for every k, $1 \leqslant k \leqslant \nu$, such that $\mathbf{x} \in U_k$, we have $\dot{\psi}_{\mathbf{x}}(t) = \mathbf{r}_k(\psi_{\mathbf{x}}(t))$;

- (6) for every $\mathbf{x}_0 \in M^0$, there exist some $\epsilon > 0$, some open neighborhood $\mathcal{U}$ of $\mathbf{x}_0$ in M^0 and a map $\varphi : \mathcal{B}_\epsilon \times \mathcal{U} \longrightarrow M^0$ such that for every $\mathbf{x} \in \mathcal{U}$, the image of the map $\varphi_{\mathbf{x}} : \mathcal{B}_\epsilon \longrightarrow M^0$ defined by $t \mapsto \varphi(t, \mathbf{x})$ is an integral curve of J through $\mathbf{x}$; and

- (7) for any $\mathbf{x} \notin M$, the initial value problem $(J, \mathbf{x})$ does not admit a solution on $\mathcal{B}_\epsilon$ in $\mathbb{C}^n$ for any $\epsilon > 0$.

Methods for E&U Theorems

- The theorem for E&U of analytic solutions is **first proved** for any given initial value problem defined by a **complete quasi-linear ideal**, using a classical E&U theorem and a result on parametric linear systems.
- The computations of algebraic constraints and equivalent vector fields are made effective by introducing the *linear rank at a point*, proving its relation to matrix rank, obtaining an algorithm to compute this rank, and characterizing the **set of points with maximum linear rank** as precisely W^0 for quasi-linear ideals.
- The theorem is then **generalized to a quasi-linear ideal**, and by passing to the associated quasi-linear ideal, **further to an arbitrary ideal**, again "**effectively**."

Algorithm for the General Case

- Given J, an ideal in $\mathbb{C}[\mathbf{X}, \mathbf{P}]$.
  Compute its completion ideal $\widetilde{J}$.
- Compute a $\mathbf{P}$-strong essential $\mathbf{P}$-degree basis $f_1, \ldots, f_m$ of the associated quasi-linear ideal $\widetilde{J}^\ell$ of $\widetilde{J}$.
- Compute an irredundant set of Zariski open sets $U_1, \ldots, U_\nu$ whose union is $M^0 = \widetilde{W}^0 = (\widetilde{W}^\ell)^0$.
- For $1 \leqslant k \leqslant \nu$, compute the **vectors of rational functions** $\mathbf{r}_k$ on U_k using Cramer's Rule (or other algorithms for parametric linear system).
- For any initial condition $\mathbf{x}_0$, use any U_k containing $\mathbf{x}_0$ to (numerically) **integrate the vector field** defined by $\mathbf{r}_k$.
- The symbolic part of the algorithm is implemented in Axiom.

Non Quasi-Linear Example

- $x(t), y(t)$ functions of t
- $p(t), q(t)$ their derivatives with respect to t

$$\begin{array}{rcrcl} & & pq & = & xy \\ -yp & + & 3xq & = & 3x^2+6 \\ & & 4q^2 & = & 9x^2 \\ & & p^2 & = & x^2-4 \end{array}$$

- The ideal J corresponding to this system is complete and has essential **P**-degree 2.

Illustration of the Algorithm

- An **essential P-degree basis** gives another presentation:

$$\begin{array}{rcrcl} & & q^2 & = & y^2+9, \\ 27p & + & 6xyq & = & 4y^3+54y, \\ & & (4y^2+54)q & = & 6xy^2+81x, \\ & & 0 & = & 9x^2-4y^2-36. \end{array}$$

- *Associated quasi-linear ideal*: Retaining only the quasi-linear equations: $\operatorname{rank}(J, \mathbf{x}) = 2$ whenever $27(4y^2+54) \neq 0$.
- The **explicit system** is

$$\mathbf{v} : p = \frac{2y}{3}, \qquad q = \frac{3x}{2}.$$

- The **integral curve** for $\mathbf{v}$ satisfying $x(0) = x_0, y(0) = y_0$ is

$$x = x_0 \cosh(t) + \frac{2}{3} y_0 \sinh(t), \quad y = y_0 \cosh(t) + \frac{3}{2} x_0 \sinh(t).$$

Comments on Example

- This solution exists and lies on the (complex) **hyperbola** $9x^2 - 4y^2 - 36 = 0$ whenever (x_0, y_0) does.
- The solution satisfies $q^2(t) = y^2(t) + 9$ for all t.
- When $2y_0^2 + 27 = 0$, we have $x_0^2 + 2 = 0$.
- The 4 points $(\pm\sqrt{-2}, \pm 3\sqrt{-3/2})$ are **equilibrium solutions** of J^ℓ.
- They are *not* solutions of J, nor are equilibrium points of $\mathbf{v}$.
- At each of these 4 initial conditions, $(J^\ell, \mathbf{x})$ **does not have unique solutions**, but $(J, \mathbf{x})$ **does**.
- The sets of solutions for J^ℓ and J are not the same.

Revisiting the Quasi-Linear Example

- $z_1(t), z_2(t), z_3(t)$ are functions of t
  $\dot{z}_1(t), \dot{z}_2(t), \dot{z}_3(t)$ are their derivatives with respect to t

$$\begin{array}{rcrcrcr} & - & z_2\dot{z}_2 & + & z_1\dot{z}_3 & = & z_1^4 \\ -z_2\dot{z}_1 & & & + & 2\dot{z}_3 & = & 5z_3 \\ z_3\dot{z}_1 & + & z_1^2\dot{z}_2 & & & = & 3z_2^2 \\ & - & z_1\dot{z}_2 & + & \dot{z}_3 & = & z_3 \end{array}$$

- **Algebraic constraints** found by symbolic computation:

$$z_1^2 = z_2, z_1^3 = z_3.$$

Explicit representation found by symbolic computation:

$$\dot{z}_1 = \frac{z_3}{z_2}, \quad \dot{z}_2 = 2z_2, \quad \dot{z}_3 = 3z_3, \quad z_2 \neq 0.$$

Some Statistics on the Example

- J contains an algebraic constraint of total degree 7 in $\mathbf{X}$. An essential $\mathbf{P}$-degree basis of the completion ideal consists of 7 binomials of $\mathbf{P}$-degree 1 and 4 binomial algebraic constraints.
- The ideal has index 3.

$X_1 > X_2 > X_3$	first prolongation	second prolongation
max deg in algebraic constraints	36	10
max coefficient in constraints	95 digits	30 digits
max $\mathbf{P}$-degree in system	4	4

Maple (dsolve) ran out of memory on a PC with 2GB DRAM. Mathematica (DSolve) does not accept overdetermined systems. The same PC runs the Axiom algorithm.

Unconstrained and Underdetermined Ideals

- An ideal J in $\mathbb{C}[\mathbf{X}, \mathbf{P}]$ is *unconstrained* if $J \cap \mathbb{C}[\mathbf{X}] = (0)$. Unconstrained implies complete.

- J is *underdetermined* if $(\widetilde{W^{\ell}})^0 = \emptyset$.
Intuitively, underdetermined means there is no initial conditions $\mathbf{x}_0$ that will guarantee a unique solution. Either some dependent variable will be arbitrary, or there are multiple integral curves through $\mathbf{x}_0$.

- **These properties are algorithmically decidable.**

Underdetermined Quasi-Linear Example

- System (complete system, index 0):

$$\begin{array}{rcrcrcl} (-x+y)\dot{x} & + & x\dot{y} & + & (x^2-1)\dot{z} & = & 0 \\ y\dot{x} & + & (x^2+1)\dot{y} & + & x^3\dot{z} & = & 0 \end{array}$$

- Solutions: Every constant point is an **equilibrium solution**. Explicit System: **No algebraic constraints**. k arbitrary.

$$\begin{array}{rcl} \dot{x} & = & k \\ \dot{y} & = & k(x^4-(x^3+x^2-1)y) \\ \dot{z} & = & k(-x^3-x+(x^2-x+1)y) \end{array}$$

We can obtain unique solution by adding any quasi-linear equation $g(\mathbf{X},\mathbf{P})=0$, for example $\dot{x}=1$.

Conclusions

- Approach is ideal theoretic, providing maximum flexibility in implementation
- Applies to all eventually quasi-linear systems without any transformation
- Applies to overdetermined as well as underdetermined systems
- Applies to non-linear systems either by a transformation or by dropping some non-linear equations
- Any system may be completed with no *a priori* conditions.
- Existence and Uniqueness theorem holds for computed initial conditions
- Provides equivalent explicit form ready for numerical methods and dynamical analysis

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