Algebraic Constraints on Initial Values of Differential Equations

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Abstract

- Objects: Systems of ordinary differential equations which are polynomial in the unknown functions and their derivatives.
- Method: Study the differential consequences of algebraic constraints on the initial value domain by an inductive but finite process, and exploiting the quasi-linearity of such consequences.
- Goals: To compute algebraic constraints on the initial conditions, and when possible, an explicit representation of the vector field, such that on the set determined by the constraint equations (and inequations), the initial value problem of the original system of differential equations can be solved uniquely by solving the explicit system.

A Quasi-Linear Example

- $z_1(t), z_2(t), z_3(t)$ are functions of t
- $\dot{z}_1(t), \dot{z}_2(t), \dot{z}_3(t)$ are their derivatives with respect to t

$$\begin{array}{rcrcrcr} & - & z_2\dot{z}_2 & + & z_1\dot{z}_3 & = & z_1^4 \\ -z_2\dot{z}_1 & & & + & 2\dot{z}_3 & = & 5z_3 \\ z_3\dot{z}_1 & + & z_1^2\dot{z}_2 & & & = & 3z_2^2 \\ & - & z_1\dot{z}_2 & + & \dot{z}_3 & = & z_3 \end{array}$$

- Algebraic constraints found by symbolic computation:

$$z_1^2 = z_2, z_1^3 = z_3.$$

- Explicit representation found by symbolic computation:

$$\dot{z}_1 = \frac{z_3}{z_2}, \quad \dot{z}_2 = 2z_2, \quad \dot{z}_3 = 3z_3, \quad z_2 \neq 0.$$

Questions

- Who have been studying these problems?
- Where are the difficulties?
- What determines the set of consistent initial values?
- When is the solution unique?
- When does an explicit form $\dot{\mathbf{z}} = r(\mathbf{z})$ exist?
- How can symbolic methods help?

Theoretical Developments

- Campbell (1980,1985,1987), Campbell and Gear (1995), Campbell and Griepentrog (1995)
  Gear and Petzold (1983,1984),
  Gear (1988, 2006), Reich (1988, 1989):
  G. Thomas (1996, 1997), J. Tuomela (1997, 1998)
  singularities, constant rank conditions, linear and differentiation index
- Rabier and Rheinboldt (1991, 1994, 1996):
  general existence and uniquenss theory for differential-algebraic systems on π-submanifolds
- Kunkel and Mehrmann (1994, 1996, 2006):
  local invariants, strangeness index, and canonical forms for linear systems with variable coefficients, numerical solutions

Numerical Methods

- difficulties with implicit, unprocessed, high index systems: constant rank condition, and stability
- Campbell (1987):
  reduce index through differentiations, drift-off
- Kunkel and Mehrmann (1996a, 1996b):
  numerical methods requiring *a priori* knowledge of local and/or global invariants

Recent Approaches

- Campbell and Griepentrog (1995):
  combining symbolic with numerical methods
- Thomas (1996):
  symbolic computation of differential index for quasi-linear systems based on algebraic geometry and prolongation
- Thomas (1997), Rabier and Rheinboldt (1994b) :
  singularities, impasse points
- Tuomela (1997a):
  regularizing singular systems with jet spaces

Contents

- transformations to quasi-linear systems
- the concepts of *essential degree* and *algebraic index* and algorithms to compute these
- algorithms for prolongation and completion
- generalized concepts of quasi-linearity
- sufficient conditions for existence and uniqueness theorem
- algorithm to compute constraints on initial conditions
- algorithm to compute explicit vector field
- examples and implementation in *Axiom*

Set Up

- $z = (z_1, \ldots, z_n)$: indeterminate functions of t
- $\dot{z} = (\dot{z}_1, \ldots, \dot{z}_n)$: their derivatives with respect to t
- $\mathbf{x}_0$: a point in $\mathbf{C}^n$, complex n-space
- $(\mathbf{X}, \mathbf{P}) = (X_1, \ldots .X_n, P_1, \ldots, P_n)$: algebraic indeterminates
- $f_1, \ldots, f_m$: polynomials in $\mathbf{X}, \mathbf{P}$ over $\mathbf{C}$
- Initial value problem:

$$\begin{aligned} f_1(z_1, \ldots, z_n, \dot{z}_1, \ldots, \dot{z}_n) &= 0, \\ &\vdots \\ f_m(z_1, \ldots, z_n, \dot{z}_1, \ldots, \dot{z}_n) &= 0, \end{aligned}$$

$$z(0) = \mathbf{x}_0.$$

Explicit Representation, Rational Form

A system is *explicit* or *explicitly given* if $m \geqslant n$ and $f_1, \ldots, f_m$ have the form

$$\begin{array}{rcll} f_i & = & P_i - r_i(\mathbf{X}), & \text{where } r_i(\mathbf{X}) \in \mathbf{C}(\mathbf{X}) \text{ for } 1 \leqslant i \leqslant n, \\ f_{n+k} & \in & \mathbf{C}[\mathbf{X}], & \text{for } 1 \leqslant k \leqslant m-n. \end{array}$$

- Write $r_i(\mathbf{X}) = R_i(\mathbf{X})/S(\mathbf{X})$ with a common denominator $S(\mathbf{X})$

Explicit Representation, Polynomial Form

- Introduce new indeterminates X_0, P_0.
- Equivalent system: $g_i \in \mathbf{C}[X_0, \mathbf{X}, P_0, \mathbf{P}]$ for $1 \leqslant i \leqslant m+2$

$$\begin{aligned}
g_i &= P_i - X_0 R_i(\mathbf{X}) && \text{for } 1 \leqslant i \leqslant n, \\
g_{n+k} &= f_{n+k} && \text{for } 1 \leqslant k \leqslant m-n, \\
g_{m+1} &= X_0 S(\mathbf{X}) - 1, \\
g_{m+2} &= P_0 + X_0^3 \sum_{i=1}^{n} \frac{\partial S(\mathbf{X})}{\partial X_i} R_i(\mathbf{X}).
\end{aligned}$$

Basic Transformations

- Non-autonomous to autonomous
- High order to first order
- Analytic to differential algebraic
- Non-linear to quasi-linear

Analytic To Differential-Algebraic

- Composite functions $f \circ U \circ z$
- $f(t)$ satisfies a LODE with constant coefficients and specific initial conditions. For example, $f(t) = x^r e^{\alpha t} \cos \beta t$.
- Or: $f(t)$ satisfies a quasi-linear polynomial ODE that is easily integrable. For example, $f(t) = 1/t$ or $f(t) = \log(t)$.
- $U(X_1, \ldots, X_n)$ is a polynomial in $\mathbf{C}[\mathbf{X}]$
- Add new dependent variable $u(t) = U(z_1(t), \cdots, z_n(t))$
- Add new dependent variables $w_1(t), w_2(t), \ldots$
- Apply Chain Rule

Example

- $\sin(U(\mathbf{z})), \cos(U(\mathbf{z}))$ may be replaced by $w_1(t), w_2(t)$
- Adding quasi-linear ODE's

$$
\begin{aligned}
0 &= \dot{w}_1(t) - w_2(t)\dot{u}(t), \\
0 &= \dot{w}_2(t) + w_1(t)\dot{u}(t), \\
u(t) &= U(z_1(t), \ldots, z_n(t)), \\
\dot{u}(t) &= \sum_{i=1}^{n} \frac{\partial U}{\partial X_i}(z_1(t), \ldots, z_n(t))\dot{z}_i(t)
\end{aligned}
$$

- Adding initial conditions

$$
\begin{aligned}
u(0) &= U(z_1(0), \ldots, z_n(0)), \\
w_1(0) &= \sin(u(0)), \\
w_2(0) &= \cos(u(0)).
\end{aligned}
$$

Non-linear Polynomial Systems

- Algebraic Indeterminates:

$$\begin{aligned}\mathbf{X} &= (X_1, \cdots, X_n),\\ \mathbf{P} &= (P_1, \cdots, P_n).\end{aligned}$$

- Polynomials $g_i(\mathbf{X}, \mathbf{P}) \in \mathbf{C}[\mathbf{X}, \mathbf{P}]$ for $1 \leqslant i \leqslant m$.
- Dependent Variables: $z = (z_1, \ldots, z_n)$
- First Order Derivatives: $\dot{z} = \dot{z}_1, \ldots, \dot{z}_n$
- System of ODE:

$$g_i(z_1, \cdots, z_n, \dot{z}_1, \cdots, \dot{z}_n) = 0, \quad 1 \leqslant i \leqslant m$$

- Initial conditions: $z(0) = \mathbf{x}_0$

Transformed to Quasi-Linear System

- New algebraic indeterminates:

$$\begin{aligned}\mathbf{Y} &= (Y_1, \cdots, Y_{2n}),\\ \mathbf{Q} &= (Q_1, \cdots, Q_{2n}).\end{aligned}$$

- New polynomial system: $f_k(\mathbf{Y}, \mathbf{Q}) \in \mathbf{C}[\mathbf{Q}, \mathbf{P}]$ for $1 \leq k \leqslant n + m$

$$\begin{aligned}f_k &= Q_k - Y_{n+k} && (1 \leqslant k \leqslant n),\\ f_{n+k} &= g_k(Y_1, \ldots, Y_{2n}) && (1 \leqslant k \leqslant m),\end{aligned}$$

- New system of ODE:

$$f_k(w_1, \cdots, w_{2n}, \dot{w}_1, \cdots, \dot{w}_{2n}) = 0, \quad 1 \leqslant k \leqslant 2n$$

- New initial conditions:

$$\begin{aligned}w_k(0) &= z_k(0), && 1 \leqslant k \leqslant n\\ w_{n+k}(0) &= \dot{z}_k(0), && 1 \leqslant k \leqslant n\end{aligned}$$

Proposition

Let I be some interval on the real line. There is a bijection between

- the set of twice differentiable curves

$$\varphi : I \longrightarrow \mathbf{C}^n$$

such that $(\varphi(t), \dot{\varphi}(t))$ satisfies the system of algebraic equations

$$g_i(\varphi(t), \dot{\varphi}(t)) = 0 \qquad (t \in I, 1 \leqslant i \leqslant m)$$

and

- the set of differentiable curves

$$\sigma : I \longrightarrow \mathbf{C}^{2n}$$

such that $(\sigma(t), \dot{\sigma}(t))$ satisfies the system of algebraic equations

$$f_k(\sigma(t), \dot{\sigma}(t)) = 0, \qquad (t \in I, 1 \leqslant k \leq n + m).$$

Essential P-degree

- $\mathbf{C}[\mathbf{X}, \mathbf{P}] = \mathbf{C}[X_1, \ldots, X_m, P_1, \ldots, P_n]$
- F is a finite subset and J a non-zero ideal of $\mathbf{C}[\mathbf{X}, \mathbf{P}]$
- Define the **P**-*degree* of F: $\deg_{\mathbf{P}} F = \max\{\deg_{\mathbf{P}} f \mid f \in F\}$
- Define the *essential* **P**-*degree* of J:

$$\operatorname{edeg}_{\mathbf{P}}(J) = \min\{\deg_{\mathbf{P}} F \mid J = (F), F \text{ finite}\}$$

- For the zero ideal, define essential **P**-degree to be $-\infty$.
- Essential **P**-degree basis: finite set F such that

$$(F) = J, \text{ and } \deg_{\mathbf{P}} F = \operatorname{edeg}_{\mathbf{P}}(J).$$

Example

- $f_1 = X_1 P_2 + P_2$, $f_2 = X_1 P_1$, and $f_3 = P_1 P_2$
- $J = (f_1, f_2, f_3)$, essential **P**-degree $= 1$
- $F = \{ f_1, f_2 \}$ is an essential **P**-degree basis.
- The differential system is given by

$$\begin{aligned} z_1 \dot{z}_2 + \dot{z}_2 &= 0 \\ z_1 \dot{z}_1 &= 0 \\ \dot{z}_1 \dot{z}_2 &= 0 \end{aligned}$$

but we can replace it by the quasi-linear system

$$\begin{aligned} z_1 \dot{z}_2 + \dot{z}_2 &= 0 \\ z_1 \dot{z}_1 &= 0 \end{aligned}$$

Essential P-degree Basis Algorithm

- F, a finite subset of $\mathbf{C}[\mathbf{X}, \mathbf{P}]$
- Choose any term ordering on $\mathbf{X}$
- Choose any degree-compatible term-ordering on $\mathbf{P}$
- Combine into an elimination term ordering $\mathbf{X} < \mathbf{P}$
- Compute a Gröbner basis G of the ideal $J = (F)$
- Select the least d such that the elements E_d of $\mathbf{P}$-degree $\leqslant d$ in G generates J
- Then d is essential $\mathbf{P}$-degree and
- E_d is an essential $\mathbf{P}$-degree basis of J.

Remarks on Essential **P**-degree Basis

- An essential **P**-degree basis of an ideal J *presents* J using the lowest degree in **P** possible.
- An essential **P**-degree basis in general has fewer elements than a Gröbner basis.
- Computation of essential **P**-degree basis may be built into the Buchberger algorithm for efficiency.
- Concept may be applied with $|X| \neq |P|$, in particular, with $|X| = 0$
- The **P**-degree of a Gröbner basis may be higher than the essential **P**-degree.

P-strong Generators

- J an ideal in $\mathbf{C}[\mathbf{X}, \mathbf{P}]$ of edeg$_{\mathbf{P}}$ d
- F a subset of J
- F is **P**-*strong* if it has the following property:

Every $f \in J$ of **P**-*degree* $\leqslant d$ *has a representation*

$$f = \sum_{j=1}^{N} h_j f_j$$

for some N, where for each $j = 1, \cdots, N$,

  - $h_j \in \mathbf{C}[\mathbf{X}, \mathbf{P}]$, $h_j \neq 0$,
  - $f_j \in F$ *and*
  - **P**-*deg* $h_j f_j \leqslant$ **P**-*deg* f.

- In the essential **P**-degree algorithm, E_d is a **P**-strong essential **P**-degree basis.

Example

Not every essential **P**-degree basis is **P**-strong.

- $F = \{ f_1, f_2 \} \subset \mathcal{R} = \mathbf{C}[X_1, X_2, P_1, P_2]$
- $f_1 = X_1 P_1 - X_2,\ f_2 = X_1 P_2 - X_1$
- $J = (F)$
- F is an essential **P**-degree basis of J but not **P**-strong.
- $f = P_2 f_1 - P_1 f_2 = X_1 P_1 - X_2 P_2$
- $f \in J$, has **P**-degree 1, but cannot be represented as $h_1 f_1 + h_2 f_2$ for any $h_1, h_2 \in \mathcal{R}$ such that the **P**-degrees of $h_1 f_1$ and $h_2 f_2$ are at most 1.

Prolongation of an Ideal

- J ideal in $\mathbf{C}[\mathbf{X}, \mathbf{P}]$
- $R = R(J) = \sqrt{J} \cap \mathbf{C}[\mathbf{X}] = \sqrt{J \cap \mathbf{C}[\mathbf{X}]}$
- For arbitary $h \in \mathbf{C}[\mathbf{X}]$, let

$$\nabla h = \sum_{j=1}^{n} \frac{\partial h}{\partial X_j} P_j .$$

- $\nabla R = \{\nabla q \mid q \in R\}$
- $J^* = (J \cup R(J) \cup \nabla R(J))$ is called the *prolongation ideal* of J.

Proposition

- J an ideal in $\mathbf{C}[\mathbf{X}, \mathbf{P}]$
- J^* its prolongation
- I some interval on the real line
- $\varphi : I \longrightarrow \mathbf{C}^n$ any smooth curve Then

$$f(\varphi(t), \dot{\varphi}(t)) = 0, \quad f \in J, \; t \in I$$

if and only if

$$f(\varphi(t), \dot{\varphi}(t)) = 0, \quad f \in J^*, \; t \in I.$$

Algorithm for Prolongation

- J generated by $f_1, \cdots, f_m \in \mathbf{C}[\mathbf{X}, \mathbf{P}]$
- Compute $J \cap \mathbf{C}[\mathbf{X}]$ and generators $q_1, \ldots, q_N \in \mathbf{C}[\mathbf{X}]$ of its radical $R(J)$. Then

$$J^* = (f_1, \ldots, f_m, q_1, \ldots, q_N, \nabla q_1, \ldots, \nabla q_N).$$

- Prolongation of J can be effectively computed from any set of generators.
- Concept is independent of generators or term-ordering, thus permits flexibility in implementation.
- Prolongation only introduces polynomials of **P**-degree at most one.

Completeness and Completion Ideal

- An ideal J is *complete* if $J = J^*$.
- The *completion ideal* of J is the smallest complete ideal $\widetilde{J}$ containing J.
- The zero ideal and $\mathbf{C}[\mathbf{X}, \mathbf{P}]$ are complete.
- $J \cap \mathbf{C}[\mathbf{X}] = 0$ implies J complete.
- The intersection of an arbitrary family of complete ideals of $\mathbf{C}[\mathbf{X}, \mathbf{P}]$ is complete.
- The completion ideal $\widetilde{J}$ of J exists and is unique.

Geometric Property

- *first jet domain* V = algebraic set of zeros of J
- *initial domain* W = algebraic set of zeros of $J \cap \mathbf{C}[\mathbf{X}]$
  = algebraic set of zeros of $R(J)$
- $\pi : V \longrightarrow W$ implies $\overline{\pi(V)} = W$ and
  W contains a non-empty open set if $V \neq \emptyset$ (Closure Theorem)
- *tangent variety* $T(W)$ = algebraic set of zeros in $\mathbf{C}^{2n}$ of
  $(R(J) \cup \nabla R(J))$
- for $\mathbf{x} \in W$, the *tangent space* to W at $\mathbf{x}$ is

$$T_{\mathbf{x}}(W) = \{\, \mathbf{p} \in \mathbf{C}^n \mid (\mathbf{x}, \mathbf{p}) \in T(W) \,\}.$$

- J complete implies $V \subseteq T(W)$

Algorithm for Completion

- J an ideal in $\mathbf{C}[\mathbf{X}, \mathbf{P}]$.
- The sequence of prolongation ideals defined by

$$J^0 = J \subseteq J^1 = J^* \subseteq \cdots \subseteq J^k = (J^{k-1})^* \subseteq \cdots$$

  is stationary.
- The *algebraic index* p is the smallest index k such that $J^k = J^{k+1}$
- The completion ideal can be effectively computed: $\widetilde{J} = J^p$.
- Algebraic index and completion concepts are ideal theoretic.
- Total flexibility in implementation
- Use of an essential $\mathbf{P}$-degree basis for J keeps $\mathbf{P}$-degree low.

Quasi-Linearities

- J an ideal of $\mathbf{C}[\mathbf{X}, \mathbf{P}]$
- J is (essentially) *quasi-linear* if $edeg_{\mathbf{P}}(J) \leqslant 1$.
- J is *eventually quasi-linear* if $\widetilde{J}$ is quasi-linear.
- Quasi-linearity is effectively decidable.
- Eventual quasi-linearity is effectively decidable.
- J quasi-linear implies $\widetilde{J}$ quasi-linear.
- Polynomial version of an explicit system is quasi-linear.

Associated Quasi-Linear Ideal

- J an ideal of $\mathbf{C}[\mathbf{X}, \mathbf{P}]$ of essential $\mathbf{P}$-degree d
- E a $\mathbf{P}$-strong subset of J
- $L(J)$, set of all polynomials of $\mathbf{P}$-degree at most 1 in J
- The *associated quasi-linear ideal* of J is $J^\ell = (L(J))$.
- $J^\ell = (L(J) \cap E)$, hence effectively computable.
- $V \subseteq V^\ell$ and $W = W^\ell$ (hence $T(W) = T(W^\ell)$)
- J is complete if and only if J^ℓ is complete.
- $\operatorname{ind} J^\ell \leqslant \operatorname{ind} J$.

Linear Rank at a Point

- J an ideal of $\mathbf{C}[\mathbf{X}, \mathbf{P}]$
- $L(J)$, set of all polynomials of $\mathbf{P}$-degree at most 1 in J
- $\mathbf{x} \in \mathbf{C}^n$, $f \in L(J)$
- $\mathbf{P}$-homogeneous form: $f^1 = \sum_{i=1}^{n} P_i \frac{\partial f}{\partial P_i}$
- $H(\mathbf{x}) = \{\, f^1(\mathbf{x}, \mathbf{P}) \mid f \in L(J) \,\}$ (vector space)
- The (linear) *rank* of J at $\mathbf{x}$ is defined by

$$\operatorname{rank} J(\mathbf{x}) = \dim_{\mathbf{C}} H(\mathbf{x})$$

- $\operatorname{rank} J(\mathbf{x}) = \operatorname{rank} J^{\ell}(\mathbf{x})$
- $J_1 \subseteq J_2$ implies $\operatorname{rank} J_1(\mathbf{x}) \leqslant \operatorname{rank} J_2(\mathbf{x})$

Quasi-linear SubSystems

- J an ideal of $\mathbf{C}[\mathbf{X}, \mathbf{P}]$
- $F = \{ f_1, \ldots, f_m \} \subset L(J)$
- Write $f_i = \sum_{j=1}^{n} \alpha_{i,j}(\mathbf{X}) P_j - \gamma_i(\mathbf{X})$, $\qquad \alpha_{i,j}(\mathbf{X}), \gamma_i(\mathbf{X}) \in \mathbf{C}[\mathbf{X}]$.
- The system $f_i = 0, \quad 1 \leqslant i \leqslant m$ in matrix notation is

$$\begin{bmatrix} \alpha_{1,1}(\mathbf{X}) & \alpha_{1,2}(\mathbf{X}) & \ldots & \alpha_{1,n}(\mathbf{X}) \\ \alpha_{2,1}(\mathbf{X}) & \alpha_{2,2}(\mathbf{X}) & \ldots & \alpha_{2,n}(\mathbf{X}) \\ \vdots & \vdots & \ldots & \vdots \\ \alpha_{m,1}(\mathbf{X}) & \alpha_{m,2}(\mathbf{X}) & \ldots & \alpha_{m,n}(\mathbf{X}) \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{bmatrix} = \begin{bmatrix} \gamma_1(\mathbf{X}) \\ \gamma_2(\mathbf{X}) \\ \vdots \\ \gamma_m(\mathbf{X}) \end{bmatrix}$$

- Or simply: $L(\mathbf{X}) : A(\mathbf{X})\mathbf{P}^T = c(\mathbf{X})$
- For $\mathbf{x} \in \mathbf{C}^n$, let $\rho_F(\mathbf{x}) = \operatorname{rank} A(\mathbf{x})$.

Computing Linear Rank at $\mathbf{x}$

- J an ideal of $\mathbf{C}[\mathbf{X}, \mathbf{P}]$
- F a finite subset of $L(J)$, $\mathbf{x} \in \mathbf{C}^n$
- Then $\operatorname{rank} J(\mathbf{x}) \geqslant \rho_F(\mathbf{x})$.
  Equality if either
  - *$\mathbf{x} \in W$ and F is $\mathbf{P}$-strong for J^ℓ or*
  - *$\mathbf{x} \in \pi(V)$ and F is an essential $\mathbf{P}$-degree basis of J^ℓ*
- $\operatorname{rank} J(\mathbf{x})$ is effectively computable for any $\mathbf{x} \in W$, since we can compute a $\mathbf{P}$-strong essential $\mathbf{P}$-basis F for J^ℓ and $\rho_F(\mathbf{x})$.

Maximum Rank Lemmas

- J an ideal of $\mathbf{C}[\mathbf{X}, \mathbf{P}]$
- $\mathbf{x} \in W$ with $\operatorname{rank} J(\mathbf{x}) = n$
- Then there exists a unique $\mathbf{p} \in \mathbf{C}^n$ such that $(\mathbf{x}, \mathbf{p}) \in V$.
- If J is quasi-linear, then $\operatorname{rank} J(\mathbf{x}) = n$ if and only if the fiber $\pi^{-1}(\mathbf{x})$ is finite and non-empty.
- Notation: $W^0 = \{\, \mathbf{x} \in W \mid \pi^{-1}(\mathbf{x}) \text{ is finite} \,\}$

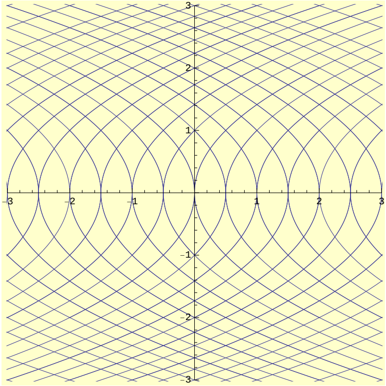
Example

- $J = (P_1 - X_2, P_2^2 - 1)$
- J is complete, but not quasi-linear.
- $J^\ell = (P_1 - X_2)$
- $\mathbf{x} = (x_1, x_2) \in \mathbf{C}^2$ implies
  $\pi^{-1}(\mathbf{x}) = \{\, (x_1, x_2, x_2, 1), (x_1, x_2, x_2, -1) \,\}$
- $(\pi^\ell)^{-1}(\mathbf{x}) = \{\, (x_1, x_2, x_2, p_2) \mid p_2 \in \mathbf{C} \,\}$
- $W = W^0 = W^\ell = \mathbf{C}^2$ and $(W^\ell)^0 = \emptyset$.
- $\operatorname{rank} J(\mathbf{x}) = 1$ for any $\mathbf{x} \in W^0$

Solutions to Example

- System of ODEs: $\dot{x} = y, \quad \dot{y}^2 = 1$
- Two solutions for each initial condition $(x, y) = (x_0, y_0)$

$$x = \pm \frac{t^2}{2} + y_0 t + x_0, \quad y = \pm t + y_0,$$



Theorem for Quasi-Linear Ideal

Let J be a quasi-linear ideal in $\mathbf{C}[\mathbf{X}, \mathbf{P}]$. There exist a computable non-negative integer ν, and ν non-empty, affine, effectively computable, Zariski basic open subsets $U_1, \ldots, U_\nu$ of the initial domain W of J with these properties:

- $W^0 = \cup_{k=1}^{\nu} U_k$; in particular, W^0 is an effectively computable constructible subset of $\mathbf{C}^n$.
- For each k, $1 \leqslant k \leqslant \nu$, the set $Y_k = \pi^{-1}(U_k)$ is an affine, non-empty, Zariski basic open subset of the jet domain V of J.
- For each k, $1 \leqslant k \leqslant \nu$, the restriction π_k of π to Y_k is an isomorphism from Y_k to U_k as affine sets.
- For each k, $1 \leqslant k \leqslant \nu$, the inverse isomorphism $\eta_k : U_k \longrightarrow Y_k$ is an everywhere defined rational map.
- There is an unique isomorphism $\eta : W^0 \longrightarrow \pi^{-1}(W^0)$ of affine schemes such that $\pi(\eta(\mathbf{x})) = \mathbf{x}$ for $\mathbf{x} \in W^0$. Morever, $\eta|_{U_k} = \eta_k$ for $1 \leqslant k \leqslant \nu$.

Classical Existence and Uniqueness Theorem

Let $\mathcal{D}$ be an open subset of $\mathbf{C}^n$, and let the system $\mathbf{v}$ on $\mathcal{D}$ be given by $\dot{\mathbf{z}}(t) = \mathbf{r}(\mathbf{z}(t))$ for $t \in \mathbf{R}$, where $\mathbf{r} : \mathcal{D} \longrightarrow \mathbf{C}^n$ is some analytic map. Then for any $\mathbf{x}_0 \in \mathcal{D}$, there exist an interval $B_\epsilon = (-\epsilon, \epsilon)$ some $\epsilon > 0$, some open neighborhood $\mathcal{O}$ of $\mathbf{x}_0$, and an analytic map $\psi : \mathcal{B}_\epsilon \times \mathcal{O} \longrightarrow \mathcal{D}$ such that $\mathcal{O} \subseteq \mathcal{D}$, and for every $\mathbf{x} \in \mathcal{O}$, we have

- $\psi(0, \mathbf{x}) = \mathbf{x}$
- the map $\psi_{\mathbf{x}} : \mathcal{B}_\epsilon \longrightarrow \mathcal{D}$ defined by $t \mapsto \psi(t, \mathbf{x})$ is the unique solution defined on $\mathcal{B}_\epsilon$ satisfying the system $\mathbf{v}$ and the initial condition $\mathbf{z}(0) = \mathbf{x}$.

Algebraic Setting

- J be an ideal in $\mathbf{C}[\mathbf{X}, \mathbf{P}]$
- $\mathbf{x} \in \mathbf{C}^n$, $\mathcal{B}_\epsilon = (-\epsilon, \epsilon)$ be an open interval in $\mathbf{R}$
- M be a constructible subset of $\mathbf{C}^n$
- A *differentiable map* $\varphi : \mathcal{B}_\epsilon \longrightarrow M$ is a differentiable map $\varphi : \mathcal{B}_\epsilon \longrightarrow \mathbf{C}^n$ whose image is contained in M.
- A *solution to the initial value problem* $(J, \mathbf{x})$ *on* $\mathcal{B}_\epsilon$ *in* M is a differentiable map $\varphi : \mathcal{B}_\epsilon \longrightarrow M$ such that $\varphi(0) = \mathbf{x}$ and $f(\varphi(t), \dot{\varphi}(t)) = 0$ for all $t \in \mathcal{B}_\epsilon$ and for all $f \in J$. We also say:
- φ *satisfies the initial value problem* $(J, \mathbf{x})$
- $(J, \mathbf{x})$ *admits a solution in* M
- the image of φ is an *integral curve of* J *through* $\mathbf{x}$.

Existence and Uniquenss Theorem I

Let J be a complete quasi-linear ideal in $\mathbf{C}[\mathbf{X}, \mathbf{P}]$, let V and W be respectively the jet and initial domain of J, and let $\mathbf{x}_0 \in W^0$. Then there exist some Euclidean open subset $\mathcal{U}$ of W^0 containing $\mathbf{x}_0$, some interval $B_\epsilon = (-\epsilon, \epsilon)$ for some $\epsilon > 0$, and a mapping $\varphi : \mathcal{B}_\epsilon \times \mathcal{U} \longrightarrow W^0$, such that for every $\mathbf{x} \in \mathcal{U}$, we have

- $\varphi(0, \mathbf{x}) = \mathbf{x}$;
- the map $\varphi_{\mathbf{x}}(t)$ defined by $t \mapsto \varphi(t, \mathbf{x})$ is the unique solution on $\mathcal{B}_\epsilon$ in W^0 to the initial value problem $(J, \mathbf{x})$; and
- the map φ is the restriction of an analytic map $\psi : \mathcal{B}_\epsilon \times \mathcal{O} \longrightarrow \mathbf{C}^n$ where $\mathcal{O}$ is a Euclidean open subset of $\mathbf{C}^n$ containing $\mathbf{x}_0$ such that $\mathcal{U} = W^0 \cap \mathcal{O}$.

Sketch of Proof

- $\mathbf{x}_0 \in W^0$ implies for some $n \times n$ determinant $\det \Delta(\mathbf{x}_0) \neq 0$.
- Let $f_1, \ldots, f_n$ define the matrix for Δ.
- On $\mathcal{D} = \{\, \mathbf{x} \in \mathbf{C}^n \mid \Delta(\mathbf{x}) \neq 0 \,\}$, define $\mathbf{v}_1 : \dot{\mathbf{z}}(t) = \mathbf{r}(\mathbf{z}(t))$.
- Let $q_1, \ldots, q_\ell$ generate $J \cap \mathbf{C}[\mathbf{X}]$.
- Completeness implies $\Delta \cdot \nabla q_i = \sum_{i'=1}^{\ell} h_{ii'} q_{i'} + \sum_{j=1}^{n} q_{ij} f_j$.
- On $\mathcal{D} \times \mathbf{C}^\ell$ define $\mathbf{v}_2$:

$$\dot{\mathbf{z}}(t) = \mathbf{r}(\mathbf{z}(t)),\ \ \dot{w}_i(t) = \sum_{i'=1}^{m} \frac{h_{ii'}(\mathbf{z}(t))}{\Delta(\mathbf{z}(t))} w_{i'}(t) \quad 1 \leqslant i \leqslant \ell,\ \ t \in \mathbf{R}.$$

- Let $\varphi : \mathcal{B}_\epsilon \times \mathcal{U} \longrightarrow \mathcal{D}$ be a solution to $\mathbf{v}_1$.
- For any $\mathbf{x} \in \mathcal{U}$, $(\varphi(t, \mathbf{x}), \mathbf{0})$ and $(\varphi(t, \mathbf{x}), q_1(\varphi(t, \mathbf{x})), \ldots, q_\ell(\varphi(t, \mathbf{x})))$ are solutions to $\mathbf{v}_2$.
- $\varphi(t, \mathbf{x}) \in W^0$.

Algorithm for the General Case

- J an ideal in $\mathbf{C}[\mathbf{X}, \mathbf{P}]$
- Compute its completion ideal $\widetilde{J}$
- Compute a $\mathbf{P}$-strong essential $\mathbf{P}$-degree basis $f_1, \ldots, f_m$ of the associated quasi-linear ideal $\widetilde{J}^\ell$ of $\widetilde{J}$
- Compute an irredundant representation $U_1, \ldots, U_\nu$ of $M^0 = \widetilde{W}^0$
- For $1 \leqslant k \leqslant \nu$, compute the vector field $\mathbf{r}_k$ on U_k using Cramer's Rule.
- For any initial condition $\mathbf{x}_0$, use any U_k containing $\mathbf{x}_0$ and integrate the vector field $\mathbf{r}_k$.

A Quasi-Linear Example Revisited

- $z_1(t), z_2(t), z_3(t)$ are functions of t
- $\dot{z}_1(t), \dot{z}_2(t), \dot{z}_3(t)$ are their derivatives with respect to t

$$\begin{array}{rcrcrcr} & - & z_2\dot{z}_2 & + & z_1\dot{z}_3 & = & z_1^4 \\ -z_2\dot{z}_1 & & & + & 2\dot{z}_3 & = & 5z_3 \\ z_3\dot{z}_1 & + & z_1^2\dot{z}_2 & & & = & 3z_2^2 \\ & - & z_1\dot{z}_2 & + & \dot{z}_3 & = & z_3 \end{array}$$

- Ideal version: J generated by polynomials

$$\begin{array}{rcrcrcr} & - & X_2P_2 & + & X_1P_3 & - & X_1^4, \\ -X_2P_1 & & & + & 2P_3 & - & 5X_3, \\ X_3P_1 & + & X_1^2P_2 & & & - & 3X_2^2, \\ & - & X_1P_2 & + & P_3 & - & X_3. \end{array}$$

Illustration of Algorithm

- J contains an algebraic constraint of total degree 7 in $\mathbf{X}$.
- The ideal has index 3.
- an essential $\mathbf{P}$-degree basis of the completion ideal is

$$\begin{array}{rrcrcrrcr}
- & X_1^2 & + & X_2, & \quad & - & X_1X_2 & + & X_3, \\
- & X_1X_3 & + & X_2^2, & & - & X_2^3 & + & X_3^2, \\
 & 2X_1P_1 & - & P_2, & & & X_2P_1 & - & X_3, \\
 & X_3P_1 & - & X_2^2, & & & X_2P_2 & - & 2X_2^2, \\
 & X_1P_2 & - & 2X_3, & & & P_3 & - & 3X_3. \\
 & X_3P_2 & - & 2X_2X_3, & & & & &
\end{array}$$

- $M^0 = \widetilde{W}^0 = \{ (x_1, x_2, x_3) \in \mathbf{C}^3 \mid x_2 \neq 0, x_2 = x_1^2, x_3 = x_1^3 \}$
- an explicit system on M^0 is:

$$P_1 = \frac{X_3}{X_2}, \quad P_2 = 2X_2, \quad P_3 = 3X_3, \quad X_2 \neq 0.$$

Statistics

	first prolongation	second prolongation
max deg in algebraic constraints	36	10
max coefficient in constraints	95 digits	30 digits
max **P**-degree in system	4	4

Non Quasi-Linear Example Revisited

- $x(t), y(t)$ functions of t
- $p(t), q(t)$ their derivatives with respect to t

$$\begin{array}{rcl} & pq & = & xy \\ -yp & + \; 3xq & = & 3x^2+6 \\ & 4q^2 & = & 9x^2 \\ & p^2 & = & x^2-4 \end{array}$$

- The ideal J corresponding to this system is complete and has essential **P**-degree 2.

Illustration of the Algorithm

- An essential **P**-degree basis gives another presentation:

$$\begin{array}{rcrcl} & & q^2 & = & y^2+9, \\ 27p & + & 6xyq & = & 4y^3+54y, \\ & & (4y^2+54)q & = & 6xy^2+81x, \\ & & 0 & = & 9x^2-4y^2-36. \end{array}$$

- Retaining only the quasi-linear equations: $\mathrm{rank}\,(J,\mathbf{x})=2$ whenever $27(4y^2+54)\neq 0$.
- The explicit system is

$$\mathbf{v}: p=\frac{2y}{3}, \qquad q=\frac{3x}{2}.$$

- The integral curve $\mathbf{v}$ satisfying $x(0)=x_0, y(0)=y_0$ is

$$x=x_0\cosh(t)+\frac{2}{3}y_0\sinh(t), \quad y=y_0\cosh(t)+\frac{3}{2}x_0\sinh(t).$$

Comments on Example

- This solution exists and lies on the hyperbola $9x^2 - 4y^2 - 36 = 0$ whenever (x_0, y_0) does.
- The solution satisfies $q^2(t) = y^2(t) + 9$ for all t.
- When $2y_0^2 + 27 = 0$, we have $x_0^2 + 2 = 0$.
- The four points $(\pm\sqrt{-2}, \pm 3\sqrt{-3/2})$ are equilibrium solutions J^ℓ.
- They are *not* solutions to J, nor are equilibrium points of $\mathbf{v}$.
- At these 4 initial conditions, $(J^\ell, \mathbf{x})$ does not have uniqueness solutions, but $(J, \mathbf{x})$ does.
- The sets of solutions for J^ℓ and J are not the same.

Existence and Uniqueness Theorem II

Let $J = (g_1, \ldots, g_m)$ be an ideal in $\mathbf{C}[\mathbf{X}, \mathbf{P}]$, and consider the system of differential algebraic equations

$$\begin{aligned} g_1(z_1, \ldots, z_n, \dot{z}_1, \ldots, \dot{z}_n) &= 0, \\ &\vdots \\ g_m(z_1, \ldots, z_n, \dot{z}_1, \ldots, \dot{z}_n) &= 0. \end{aligned}$$

Then we can effectively compute

- (1) a Zariski-closed subset M of $\mathbf{C}^n$ and some integer $\nu \geqslant 0$;
- (2) for each k, $1 \leqslant k \leqslant \nu$, a non-empty Zariski open subset U_k of M;
- (3) for each k, $1 \leqslant k \leqslant \nu$, an n-dimensional vector $\mathbf{r}_k = (r_{k,1}, \ldots, r_{k,n})$ of rational functions in $\mathbf{C}(\mathbf{X})^n$, everywhere defined on U_k such that

- (4) the union $M^0 = \cup_{k=1}^{\nu} U_k$ is irredundant;

- (5) for every $\epsilon > 0$ and for every $\mathbf{x} \in M^0$, the image of a differentiable map $\psi_{\mathbf{x}} : \mathcal{B}_\epsilon \longrightarrow M^0$ is an integral curve of J through $\mathbf{x}$ if and only if $\psi_{\mathbf{x}}(0) = \mathbf{x}$ and for every k, $1 \leqslant k \leqslant \nu$, such that $\mathbf{x} \in U_k$, we have $\dot{\psi}_{\mathbf{x}}(t) = \mathbf{r}_k(\psi_{\mathbf{x}}(t))$;

- (6) for every $\mathbf{x}_0 \in M^0$, there exist some $\epsilon > 0$, some open neighborhood $\mathcal{U}$ of $\mathbf{x}_0$ in M^0 and a map $\varphi : \mathcal{B}_\epsilon \times \mathcal{U} \longrightarrow M^0$ such that for every $\mathbf{x} \in \mathcal{U}$, the image of the map $\varphi_{\mathbf{x}} : \mathcal{B}_\epsilon \longrightarrow M^0$ defined by $t \mapsto \varphi(t, \mathbf{x})$ is an integral curve of J through $\mathbf{x}$; and

- (7) for any $\mathbf{x} \notin M$, the initial value problem $(J, \mathbf{x})$ does not admit a solution on $\mathcal{B}_\epsilon$ in $\mathbf{C}^n$ for any $\epsilon > 0$.

Algorithm for Explicit Form

- **Input**: (1) An essential $\mathbf{P}$-degree basis $f_1, \ldots, f_m$ for a complete quasi-linear ideal J in $\mathbf{C}[\mathbf{X}, \mathbf{P}]$, or in matrix form

$$L: \quad A(\mathbf{X})\mathbf{P}^T = c(\mathbf{X})$$

- (2) Polynomials $q_1, \ldots, q_\ell$ generating $J \cap \mathbf{C}[\mathbf{X}]$ (defining W)
- **Output**: An irredundant representation of $W^0 = \cup_{k=1}^{\nu} U_k$ and for each U_k, a vector of rational functions $\mathbf{r}_k(\mathbf{X})$ for the vector field $\mathbf{v} : \dot{\mathbf{z}}(t) = \mathbf{r}_k(\mathbf{z}(t))$
- May use in (1) only those f_i which has $\mathbf{P}$-degree 1, and in (2) those f_i which has $\mathbf{P}$-degree 0.
- Each U_k will be given by defining equations and inequations.

Subroutines

Adapted from Sit's (1992) algorithm for parametric linear equations.

- $\textsc{Determinants}(A(\mathbf{X}))$ returns the set Δ of all non-zero determinants of $n \times n$ submatrices of $A(\mathbf{X})$ *if none of these determinants are constants*; otherwise, it returns $\{\Delta\}$ where Δ is one such non-zero, constant determinant. We assume that any determinant returned by this routine carries with it the row index set a.
- $\textsc{MinGenerator}(h)$, where $h = (h_1, \ldots, h_s)$ is a family of polynomials. This procedure returns a minimal subfamily $H = (h_{i_1}, \ldots, h_{i_k})$ of h such that for every j, $1 \leqslant j \leqslant s$, h_j belongs to the radical of (H).
- $\textsc{Solve}(L, a)$ returns the unique solution vector $\mathbf{r}_a(\mathbf{X})$ of rational functions in $\mathbf{C}(\mathbf{X})$ to the linear system $A^a(\mathbf{X})\mathbf{P}^T = c^a(\mathbf{X})$.

Algorithm for Explicit Form

begin
  $S \longleftarrow \emptyset$
  if $m < n$ **then**
    return $W^0 = \emptyset$
  else
    $h \longleftarrow \textsc{Determinants}(A(\mathbf{X}))$
    $H \longleftarrow \textsc{MinGenerator}(h)$
    for $\Delta_a \in H$ **do**
      $\mathcal{D}_a \longleftarrow \{\, \mathbf{x} \in \mathbf{C}^n \mid \Delta_a(\mathbf{x}) \neq 0 \,\}$
      $U_a \longleftarrow \mathcal{D}_a \cap W$
      $\mathbf{r}_a \longleftarrow \textsc{Solve}(L(\mathbf{X}), a)$
      $S \longleftarrow S \cup \{\, a \,\}$
    end
  return $\{\, (U_a, \mathbf{r}_a) \mid a \in S \,\}$
end

Axiom Implementation

- [makeSytem(F):] creates internal representation S for the ideal J given by polynomials $F = \{ f_1, \ldots, f_m \} \subset \mathbf{Q}[\mathbf{X}]$
- [matrixView(S):] displays system S in matrix form if the ideal represented by S is quasi-linear.
- [algebraicSystem(S):] displays a generating set of polynomials in $J \cap \mathbf{Q}[\mathbf{X}]$, when the ideal J has representation S.
- [linearize(S):] computes a system S^ℓ representing the associated linear ideal J^ℓ of the ideal J, where J is represented by the system S.
- [prolong(S):] computes the system S^* representing the prolongation J^* of the ideal J where J is represented by S.
- [complete?(S):] returns `true` if the ideal J represented by S is complete, and `false` otherwise.

- [completion(S):] computes the representation $\widetilde{S}$ for the completion $\widetilde{J}$ of J, when J is represented by S.
- [index(S):] computes index of J, when J is represented by S.
- [parSolve(S):] computes the algebraic conditions for the matrix $A(\mathbf{X})$ associated with a quasi-linear ideal J represented by S to have rank n and computes the rational functions $r_i(\mathbf{X})$ which represent coordinates of $\mathbf{P}$ such that when $\mathbf{x} \in W$ and satisfies the algebraic conditions, $\mathbf{p} = (r_1(\mathbf{x}), \ldots, r_n(\mathbf{x}))$ satisfies the linear system $A(\mathbf{X}) = c(\mathbf{X})$. Adapted from the `ParametricLinearEquations` package.
- The data structure S allows caching of all prolongation and completion computations. Routines that cache results: `prolong`, `index`, `complete?` and `completion`.
- An essential **P**-degree basis is kept for all ideals.

Unconstrained and Underdetermined Ideals

- J an ideal in $\mathbf{C}[\mathbf{X}, \mathbf{P}]$
- J is *unconstrained* if $J \cap \mathbf{C}[\mathbf{X}] = (0)$.
- unconstrained implies complete
- J is *underdetermined* if $(\widetilde{W^\ell})^0 = \emptyset$
- Intuitively, underdetermined means there is no initial conditions $\mathbf{x}_0$ that will guarantee a unique solution. Either some dependent variable will be arbitrary, or there are multiple integral curves through $\mathbf{x}_0$.
- These properties are algorithmically decidable.

Example of Underdetermined Ideal

- $J = (X_1X_2 - 1,\ P_1^2 - X_1^2,\ P_2^2 - X_2^2)$
- Index 1 with $\widetilde{J} = (P_1^2 - X_1^2,\ P_2 + X_2^2P_1,\ X_1X_2 - 1)$,
- $\widetilde{W}$ is the hyperbola $X_1X_2 = 1$.
- $\mathrm{rank}\,\widetilde{J}^{\ell}(\mathbf{x}) = 1$ for any $\mathbf{x} \in \widetilde{W}$.
- J is underdetermined, $\widetilde{W^{\ell}}^{0} = \emptyset$.
- $\mathbf{z} = (\varphi_1, \varphi_2)$, φ_2 arbitrary, $\varphi_2(t) \neq 0$ for all t, and $\varphi_1 = 1/\varphi_2$.
- Differential algebraic system: $z_1z_2 = 1,\ \dot{z}_1^2 = z_1^2,\ \dot{z}_2^2 = z_2^2$
- Initial conditions $\mathbf{z}(0) = (x_1, x_2)$ where $x_1x_2 = 1$
- Two solutions $(x_1e^{\pm t}, x_2e^{\mp t})$

Quasi-linearization of Ideals

- J be an ideal in $\mathbf{C}[\mathbf{X}, \mathbf{P}]$.
- $\mathbf{Y} = (Y_1, \cdots, Y_{2n})$ and $\mathbf{Q} = (Q_1, \cdots, Q_{2n})$ indeterminates
- $\lambda : \mathbf{C}[\mathbf{X}, \mathbf{P}] \longrightarrow \mathbf{C}[\mathbf{Y}]$ where $\lambda(g) = g(Y_1, \cdots, Y_{2n})$
- *Quasi-linearization* of J is the ideal in $\mathbf{C}[\mathbf{Y}, \mathbf{Q}]$ given by

$$q\ell(J) = (\{\, Q_k - Y_{n+k} \mid 1 \leqslant k \leqslant n \,\} \cup \{\, \lambda(g) \mid g \in J \,\})$$

- $J = (g_1, \ldots, g_m)$ implies

$$\{\, Q_k - Y_{n+k} \mid 1 \leqslant k \leqslant n \,\} \cup \{\, \lambda(g_i) \mid 1 \leqslant i \leqslant m \,\}$$

  is a $\mathbf{Q}$-strong essential $\mathbf{Q}$-degree basis for $q\ell(J)$.
- $q\ell(J^*) \subseteq q\ell(J)^*$.

Quasi-linearization Helps

- $J = (X_1X_2 - 1,\ P_1^2 - X_1^2,\ P_2^2 - X_2^2)$ (underdetermined ex.)
- $q\ell(J) = (Q_1 - Y_3,\ Q_2 - Y_4,\ Y_1Y_2 - 1,\ Y_3^2 - Y_1^2,\ Y_4^2 - Y_2^2)$
- $\widetilde{q\ell(J)} = (Q_4 - Y_2,\ Q_3 - Y_1,\ Q_2 + Y_3Y_2^2,\ Q_1 - Y_3,$
  $Y_1Y_2 - 1,\ Y_4 + Y_3Y_2^2,\ Y_3^2 - Y_1^2)$
- $\widetilde{q\ell(W)} : Y_1Y_2 - 1 = 0, \quad Y_4 + Y_3Y_2^2 = 0, \quad Y_3^2 - Y_1^2 = 0$
- $\operatorname{rank} \widetilde{q\ell(J)}(\mathbf{y}) = 4$ for any $\mathbf{y} \in \widetilde{q\ell(W)}$
- Translation:

Y_1	Y_2	Y_3	Y_4	Q_1	Q_2	Q_3	Q_4
z_1	z_2	$\dot{z}_1$	$\dot{z}_2$	$\dot{z}_1$	$\dot{z}_2$	$\ddot{z}_1$	$\ddot{z}_2$

- Second Order System: $\ddot{z}_1 = z_1, \qquad \ddot{z}_2 = z_2$
- Constraints: $z_1z_2 = 1, \quad \dot{z}_1^2 = z_1^2, \quad \ddot{z}_2 = -\dot{z}_1z_2^2$
- It is necessary to decide the branch by giving $\dot{z}_1(0)$ only.

Underdetermined Quasi-Linear Example

- System:

$$\begin{array}{rcrcrcl} (-x+y)\dot{x} & + & x\dot{y} & + & (x^2-1)\dot{z} & = & 0 \\ y\dot{x} & + & (x^2+1)\dot{y} & + & x^3\dot{z} & = & 0 \end{array}$$

- Solutions: Every constant point is an equilibrium solution.
- Complete System (Index 0)
- Explicit System: No algebraic constraints. k arbitrary.

$$\begin{array}{rcl} \dot{x} & = & k \\ \dot{y} & = & k(x^4-(x^3+x^2-1)y) \\ \dot{z} & = & k(-x^3-x+(x^2-x+1)y) \end{array}$$

- We can obtain unique solution by adding any quasi-linear equation $g(\mathbf{X},\mathbf{P})=0$, for example $\dot{x}=1$.

Underdetermined Quasi-linear Systems

- J complete, quasi-linear ideal, underdetermined
- Assume W is irreducible.
- $\rho = \max\{\ \mathrm{rank}\, J(\mathbf{x}) \mid \mathbf{x} \in W\ \} < n$
- $g_1, \ldots, g_\nu$ ($\nu \leqslant n - \rho$) polynomials of **P**-degree 1
- $J_1 = (J, g_1, \ldots g_\nu)$
- Suppose W_1 contains a point $\mathbf{x}_0$ with $\mathrm{rank}\, J_1(\mathbf{x}_0) = \rho + \nu$.
- Then $J_1 \cap \mathbf{C}[\mathbf{X}] = J \cap \mathbf{C}[\mathbf{X}]$.

Primary Decomposition

- J complete, but not necessarily quasi-linear
- $R(J) = \sqrt{J \cap \mathbf{C}[\mathbf{X}]}$
- $R = Q_1 \cap \cdots \cap Q_r$: irredundant primary (prime) decomposition
- $K_i = (J \cup Q_i \cup \nabla Q_i)$
- J quasi-linear implies K_i quasi-linear
- J complete implies K_i complete and $K_i \cap \mathbf{C}[\mathbf{X}] = Q_i$.

Conclusions

- Approach is ideal theoretic, providing maximum flexibility in implementation
- Applies to all eventually quasi-linear systems without any transformation
- Applies to overdetermined as well as underdetermined systems
- Applies to non-linear systems either by a transformation or by dropping some non-linear equations
- Any system may be completed with no *a priori* conditions.
- Existence and Uniqueness theorem holds for computed initial conditions
- Provides equivalent explicit form ready for numerical methods and dynamical analysis

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