Rota-Baxter Type Operators, Rewriting Systems, and Gröbner-Shirshov Bases, Part I

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Part I Outline

- Introduction: Rota's Classification Problem
- Free Operated Algebras
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Rota's Classification Problem for Linear Operators

In a series of papers, I have tried to show that other linear operators satisfying algebraic identities may be of equal importance in studying certain algebraic phenomena, and I have posed the problem of finding all possible algebraic identities that can be satisfied by a linear operator on an algebra. Simple computations show that the possibilities are very few, and the problem of classifying all such identities is very probably completely solvable.

Baxter operators, an introduction (*1995*)
Gian-Carlo Rota

Guiding Principle

I will lay my cards on the table: a revision of the notion of a sample space is my ultimate concern. I hasten to add that I am not about to put forth concrete proposals for carrying out such a revision. We will, however, be guided by a belief that has been a guiding principle of the mathematics of this century. Analysis will play second fiddle to algebra.The algebraic structure sooner or later comes to dominate, whether or not it is recognized when a subject is born. Algebra dictates the analysis.

Twelve problems in probability no one likes to bring up
—Gian-Carlo Rota

More on Rota's guiding principle

"Beyond specific problems, ideas from universal algebra lie at the heart of Rota's work. For example, a motivation for Rota's change of focus was his recognition of the central role which identities play in analysis and probability theory."

"The key result here is Garrett Birkhoff's theorem on free algebras. Philosophically, Birkhoff's theorem says that there are free structures associated with every set of identities and the best way to study identities is to describe the free structures."

"In a *tour de force* of universal algebra, the theory of symmetric functions, and combinatorics, Rota constructed the free Baxter algebra. The free Baxter algebra provides a common structure underlying identites in probability theory, integration by parts, symmetric functions, q-integration and q-series."

Gian-Carlo Rota on Combinatorics (*1998*)
—Kung and Yan

Examples of Operator Algebraic Identities

- Rota was most interested in the following operators arising from analysis, probability and combinatorics:

$$
\begin{aligned}
\text{Endomorphism operator} &: \quad d(xy) = d(x)d(y), \\
\text{Differential operator} &: \quad d(xy) = d(x)y + xd(y), \\
\text{Average operator} &: \quad P(x)P(y) = P(xP(y)), \\
\text{Inverse average operator} &: \quad P(x)P(y) = P(P(x)y), \\
\text{(Rota-)Baxter operator} & \quad\;\; P(x)P(y) = P(xP(y) + P(x)y + \lambda xy), \\
\text{of weight } \lambda &: \qquad \text{where } \lambda \text{ is a fixed constant}, \\
\text{Reynolds operator} &: \quad P(x)P(y) = P(xP(y) + P(x)y - P(x)P(y))
\end{aligned}
$$

Some Applications

- Endomorphism operator: difference algebras, Galois theory.
- Differential operator: differential algebra, quantum differential operators.
- Rota-Baxter operator: probability, classical Yang-Baxter equation, operads, combinatorics, Hopf algebra, renormalization of quantum field theory.
- Average operator and Reynolds operator: probability theory, fluid dynamics, functional analysis, invariant theory, lattice theory, and logic.
- Inverse average operator: multivariate interpolation

Construction of Free Algebras

- Polynomial rings
- Differential polynomial rings
- Free differential algebra of weight λ

$$d(xy) = d(x)y + xd(y) + \lambda d(x)d(y)$$

- Free Rota-Baxter algebras (of weight λ)
- Free Nijenhuis algebras

$$P(x)P(y) = P(xP(y) + P(x)y - P(xy))$$

- Free TD algebras

$$P(x)P(y) = P(xP(y) + P(x)y - xP(1)y)$$

Symbolic Computation in Algebras

- Let **k** be a commutative unitary ring. An **algebra** R in this talk means an associative **k**-algebra that is **free as a k-module**, but R need not be commutative or unitary.
- Given a set Z, an algebra R is **free over** Z if there is a set map $Z \to R$ which is an initial object in the category whose objects are set maps $Z \to A$, where A is an algebra.
- Symbolic computation is based on (1) rewriting systems in these free algebras, (2) the notion of irreducibility, (3) a monomial order, and (4) a Gröbner-Shirshov basis.
- To study Rota's Classification Problem for algebras with a linear operator, a starting point is the construction of (a) free operated algebras, (b) a rewriting system (c) a monomial order, and (d) a Gröbner-Shirshov basis. The last three are related to a given operated identity.

Free Operated Algebras

- **Ω-algebras**, first introduced by A.G. Kurosh, are algebras with (one or more) linear operators. An algebra with one linear operator is called an **operated algebra**.
- An **operated monoid** is a monoid U with a **set** map $P : U \to U$.
- A morphism from an operated monoid U to an operated monoid V is a monoid homomorphism $f : U \to V$ such that $f \circ P = P \circ f$.
- The notions of an **operated k-module** and **operated k-algebra** are defined similarly, with a **k-linear** map $P : U \to U$.
- The construction of a **free operated monoid** $\mathfrak{M}(Z)$ over a set Z is done as a direct limit. The **free operated algebra** $\mathbf{k}\|Z\|$ is the free $\mathbf{k}$-module with basis $\mathfrak{M}(Z)$ and multiplication induced via linearity from the monoid structure.

Bracketed (or Operated) Polynomials

- Elements of Z are called **operated variables** or **indeterminates**.
- An element $\varphi \in \mathbf{k}\lfloor Z \rfloor$ will be called an **operated**, or a **bracketed**, **polynomial in Z with coefficients in k**, and **we will implicitly assume that $\varphi \notin \mathbf{k}$, unless otherwise noted**. For free operated algebras $\mathbf{k}\lfloor Z \rfloor$, we often write $P(\varphi)$ as $\lfloor \varphi \rfloor$, and so $\lfloor\ \rfloor$ is the operator.
- Elements of $\mathfrak{M}(Z) \subset \mathbf{k}\lfloor Z \rfloor$ are called **bracketed words** or **bracketed monomials, or simply just monomials, in Z**.
- When there is no danger of confusion, we often omit the adjectives "bracketed" and "operated" as well as "in Z". Example: if $x, y \in Z$, then $\lfloor x \rfloor^2 \lfloor y \rfloor x y^3 \lfloor \lfloor xy \rfloor \rfloor$ is a monomial.

Operated Polynomial Identities (OPI)

- Let $X = \{ x_1, \ldots, x_k \}$ and $\varphi \in \mathbf{k}\|x_1, \cdots, x_k\|$. We say that an operated algebra R with operator P is a **φ-algebra** and that P is a **φ-operator**, if $\varphi(r_1, \ldots, r_k) = 0$ for all $r_1, \ldots, r_k \in R$.

- An **operated polynomial identity algebra** is any algebra which is also a φ-algebra for some φ.

- If R is a φ-algebra, we will say loosely that $\varphi = 0$ (or by abuse, φ, or even $\varphi_1 = \varphi_2$ if $\varphi = \varphi_1 - \varphi_2$) is an **operated polynomial identity (OPI) satisfied by R**.

Free Operated Polynomial Identity Algebras

- Let $X = \{ x_1, \ldots, x_k \}$ and $\varphi \in \mathbf{k}\|x_1, \cdots, x_k\|$.
- Given any set Z, let $I_\varphi(Z)$ be the operated ideal of $\mathbf{k}\|Z\|$ generated by the set

$$S_\varphi(Z) := \{ \varphi(u_1, \cdots, u_k) \mid u_1, \cdots, u_k \in \mathbf{k}\|Z\| \}. \qquad (1)$$

  Then the quotient algebra $\mathbf{k}_\varphi\|Z\| := \mathbf{k}\|Z\|/I_\varphi(Z)$ has a natural structure of a φ-algebra.
- **Theorem 3.5.6 (F. Baader and T. Nipkow, 1998)**: The quotient operated algebra $\mathbf{k}_\varphi\|Z\|$ is the free φ-algebra on Z.

Brief discussion on Rota's Classification Problem

- Every OPI can be satisfied by some operated algebra, but $\mathbf{k}_{\varphi}\|Z\|$ may be trivial!
- Can we decide for what φ would $I_{\varphi}(Z)$ be the unit operated ideal? Is there a Nullstellensatz type theorem for free operated algebras? If so, is there an effective version?
- To be effective, we need a well-ordering on the operated monomials compatible with the operator and a convergent rewriting system.
- To construct the free operated φ-algebra and perform computation in it, it would help to know a $\mathbf{k}$-module basis consisting of irreducible elements.
- Algorithms and complexity analyses, and bounds.
- Isomorphism problems.

What is a Rewriting System?

- An (abstract) **rewriting system** (ARS) is simply a set V together with a binary relation, traditionally denoted by $\rightarrow$. A **relation** is just a subset of $V \times V$.

- An ARS is also known as a **reduction system**, or a **state transition system**.

- The binary relation is understood as performing some action that changes one element to another; the action may be called "rewriting", "reducing", or "transforming" an element a to b if $a \rightarrow b$ (or a state a being transited to another state b).

- A **rule** is just a pair (a, b) in the relation: $a \rightarrow b$.

Basic Notions for Rewriting Systems

- The **transitive reflexive closure** of $\to$ is denoted by $\xrightarrow{*}$ or $\twoheadrightarrow$. So $a \xrightarrow{*} b$ means there is a finite chain of reductions: $a = a_0 \to a_1 \to \cdots \to a_n = b$ with $n \geqslant 0$.
- An element $a \in V$ is **reducible** if for some $b \in V$, $a \to b$ with $b \neq a$. Otherwise, it is called **irreducible**.
- **A normal form** of a is some b such that b is irreducible, and $a \xrightarrow{*} b$. A normal form for a need not exist nor be unique.
- If every element $a \in V$ has a normal form, we say $\to$ is **normalizing**.
- **terminating** or **noetherian** if there is no infinite chain of reductions $a_0 \to a_1 \to a_2 \cdots$. Terminating implies normalizing.

Good ARS for Symbolic Computation

- Two elements a and b are **joinable** if there is a c such that $a \xrightarrow{*} c$ and $b \xrightarrow{*} c$. This is denoted by $a \downarrow b$.
- A pair of distinct reductions $(a \xrightarrow{*} b_1, a \xrightarrow{*} b_2)$ (resp. $(a \to b_1, a \to b_2)$) is called a **fork** (resp. **local fork**) at a. The fork is **joinable** if $b_1 \downarrow b_2$.
- An ARS is **confluent** (resp. **locally, or weakly, confluent**) if every fork (resp. local fork) is joinable, and
- **convergent** if it is both terminating and confluent. In a normalising and confluent system, every element has a unique normal form.
- **Theorem (Newman's Lemma)**: **A terminating ARS is confluent if and only if it is locally confluent.**

Free Modules with Basis

- Let V be a free **k**-module with a given **k**-basis W.
- For $f \in V$, the **support** or **W-support** $\mathrm{Supp}(f) = \mathrm{Supp}_W(f)$ of f is the set consisting of $w \in W$ appearing in f (with non-zero coefficients), **when f is expressed as a unique linear combination of $w \in W$ with coefficients in k**.
- Let $f, g \in V$. We use $f \dotplus g$ to indicate the relation that $\mathrm{Supp}(f) \cap \mathrm{Supp}(g) = \emptyset$. If this is the case, we say $f + g$ is a **direct sum** of f and g, and by abuse, we use $f \dotplus g$ also for the sum $f + g$.
- Note $\mathrm{Supp}(0) = \emptyset$ and hence $f \dotplus 0$ for any $f \in V$.

Simple Term Rewriting Systems on Free Modules

- A **base-rewriting system $\to$ on V** is a binary relation (as a subset Π) of $W \times V$.
- The image $\pi_1(\Pi)$ of Π under the first projection map $\pi_1 : W \times V \to W$ will be denoted by T, which is the set of reducibles in the basis.
- We say $\to$ is **simple** if $t \dot{+} v$ for all $t \to v$.
- We extend and expand $\to$ to a rewriting system $\to_\Pi$ on V by linearity. A rewriting system obtained this way is a **term rewriting system (TRS)**, where "term" refers to $w \in W$. A TRS is **simple** if the base-rewriting system $\to$ is simple.
- Example: Let $W = \{x, y\}$. Then $\Pi = \{x \to y, y \to x\}$ is simple. Every element is reducible, none has a normal form, and Π is neither normalizing nor terminating, but is confluent.

The Hierarchy Lemma for Simple TRS

Lemma: Let V be a free $\mathbf{k}$-module with a $\mathbf{k}$-basis W and let Π be a term-rewriting system on V relative to W. For any $f, g \in V$, consider the following properties:

(1). $f \to_\Pi g$;

(2). $(f - g) \to_\Pi 0$;

(3). $(f - g) \xrightarrow{*}_\Pi 0$; (equivalently, $(f - g) \downarrow_\Pi 0$);

(4). $f \downarrow_\Pi g$;

Then

- (A). (1) $\implies$ (4), and (2) $\implies$ (3) $\implies$ (4).
- (B). If Π is simple, then (1) $\implies$ (2) and for any $f, g, h \in V$, $f \to_\Pi g \implies (f + h) \downarrow_\Pi (g + h)$.
- (C). None of the reverse implications holds in general (best-possible).

Main Theorem on Confluence

Let V be a free $\mathbf{k}$-module with a $\mathbf{k}$-basis W and let Π be a simple term-rewriting system on V relative to W. **The following properties on Π are equivalent.**

(a). $\to_\Pi$ is confluent, that is, for any $f, g, h \in V$, **confluence**

$$(f \xrightarrow{*}_\Pi g,\ f \xrightarrow{*}_\Pi h) \implies g \downarrow_\Pi h.$$

(b). For all $f, g, h \in V$, **transitivity of $\downarrow_\Pi$**

$$f \downarrow_\Pi g,\ g \downarrow_\Pi h \implies f \downarrow_\Pi h.$$

(c). For all $f, g, f', g' \in V$, **2-additivity of $\downarrow_\Pi$**

$$f \downarrow_\Pi g,\ f' \downarrow_\Pi g' \implies (f + f') \downarrow_\Pi (g + g').$$

Theorem Continued

(d). For all $r \geqslant 1$ and $f_1, \ldots, f_r, g_1, \ldots, g_r \in V$,

***n*-additivity of $\downarrow_\Pi$**

$$f_i \downarrow_\Pi g_i \quad (1 \leqslant i \leqslant r) \implies \left(\sum_{i=1}^{r} f_i\right) \downarrow_\Pi \left(\sum_{i=1}^{r} g_i\right).$$

(e). For all $f, g, h' \in V$,

1-additivity of $\downarrow_\Pi$

$$f \downarrow_\Pi g \implies (f + h') \downarrow_\Pi (g + h').$$

(f). For all $f, g \in V$,

1-transposition of $\downarrow_\Pi$

$$f \downarrow_\Pi g \implies (f - g) \xrightarrow{*}_\Pi 0 \quad (\text{that is}, (f - g) \downarrow_\Pi 0).$$

(g). For all $r \geqslant 1$ and $f_1, \ldots, f_r, g_1, \ldots, g_r \in V$,

***n*-transposition of $\downarrow_\Pi$**

$$f_i \downarrow_\Pi g_i \quad (1 \leqslant i \leqslant r) \implies \left(\sum_{i=1}^{r} f_i\right) - \left(\sum_{i=1}^{r} g_i\right) \xrightarrow{*}_\Pi 0.$$

Main Theorem Continued

(h). For all $r \geqslant 1$ and $f_1, \ldots, f_r, g_1, \ldots, g_r \in V$,

zero-sum of $\downarrow_\Pi$

$$f_i \downarrow_\Pi g_i \quad (1 \leqslant i \leqslant r) \text{ and } \sum_{i=1}^{r} g_i = 0 \implies \left(\sum_{i=1}^{r} f_i\right) \xrightarrow{*}_\Pi 0.$$

(i). For all $r \geqslant 1$ and $f_1, \ldots, f_r, g_1, \ldots, g_r \in V$,

n-transpose of $\xrightarrow{*}_\Pi$

$$f_i \xrightarrow{*}_\Pi g_i \quad (1 \leqslant i \leqslant r) \implies \left(\sum_{i=1}^{r} f_i\right) - \left(\sum_{i=1}^{r} g_i\right) \xrightarrow{*}_\Pi 0.$$

(j). For all $r \geqslant 1$ and $f_1, \ldots, f_r, g_1, \ldots, g_r \in V$,

zero-sum of $\xrightarrow{*}_\Pi$

$$f_i \xrightarrow{*}_\Pi g_i \quad (1 \leqslant i \leqslant r), \text{ and } \sum_{i=1}^{r} g_i = 0 \implies \left(\sum_{i=1}^{r} f_i\right) \xrightarrow{*}_\Pi 0.$$

Main Theorem Continued

(k). For all $r \geqslant 1$ and $f_1, \ldots, f_r \in V$, **zero-additivity of $\xrightarrow{*}_\Pi$**

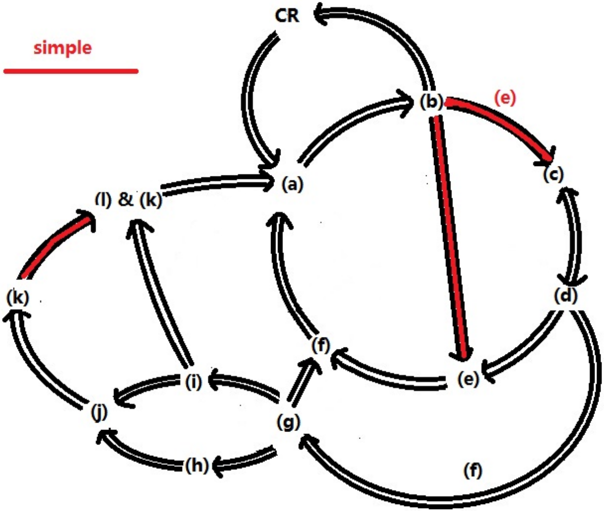
$$f_i \xrightarrow{*}_\Pi 0 \quad (1 \leqslant i \leqslant r) \implies \left(\sum_{i=1}^{r} f_i\right) \xrightarrow{*}_\Pi 0.$$

When any of the above holds, we also have
(ℓ). For all $f, g \in V$, **transposition of $\xrightarrow{*}_\Pi$**

$$f \xrightarrow{*}_\Pi g \implies f - g \xrightarrow{*}_\Pi 0.$$

.

Proof Steps



Remarks on (B) in Lemma and Main Theorem

- Conclusion (B) of Lemma (let alone the stronger property (ℓ)) need not hold if Π is not simple, even when $\to_\Pi$ is confluent.
- The failure of Property (ℓ) may be used to show $\to_\Pi$ is not confluent.
- The proof of the Theorem shows that none of $(c), (d), (g), (i)$ need hold when Π is confluent but not simple, or when Π is simple but not confluent, since these imply (ℓ) even when Π is neither simple nor confluent. These properties (including $(e), (f)$) are each strictly stronger than confluence.
- For a simple term-rewriting system Π with $\to_\Pi$ confluent, the relation $f \downarrow_\Pi g$ is transitive, additive, terms on either side are freely transposable, and $f \downarrow_\Pi g$ is interchangeable with $f - g \xrightarrow{*}_\Pi 0$.
- The property $f \xrightarrow{*}_\Pi 0$ for $f \in V$ is additive, yet for the relation $\xrightarrow{*}_\Pi$, only the *entire* right-hand side may be transposed.

Local Base-Forks and Local Base-Confluence

- A **local base-fork** is a fork $(ct \to_\Pi cv_1, ct \to_\Pi cv_2)$ where $t \to v_1, t \to v_2$ and $c \in \mathbf{k}$, $c \neq 0$. The rewriting system Π is **locally base-confluent** if for every local base-fork $(ct \to_\Pi cv_1, ct \to_\Pi cv_2)$, we have $c(v_1 - v_2) \xrightarrow{*}_\Pi 0$.
- Example. Let V be the free $\mathbf{k}$-submodule of the polynomial ring $\mathbf{k}[x, y, z, u, v]$ with a $\mathbf{k}$-basis $W = \{xyz, x, y, z, u, v\}$. Let Π consist of 6 rules:

$$\begin{aligned} xyz \to x + v, &\qquad x \to u, \\ xyz \to y, &\qquad y \to u + v, \\ xyz \to z + u, &\qquad z \to v. \end{aligned}$$

  Then $T = \{xyz, x, y, z\}$ and Π is a simple term-rewriting system. The only local base-forks start at xyz; and Π is locally base-confluent (in particular, the local base-forks are all joinable to $u + v$).

Linear Order on Reducible Terms

- Let T be the set of reducible terms under Π. A partial order $\preccurlyeq$ (or its corresponding strict partial order $\prec$) on T is **compatible with Π** if for all $(t, v) \in \Pi$, we have $t' \prec t$ for any $t' \in \mathrm{Supp}(v) \cap T$ (we shall abbreviate this property of (t, v) by $v \prec t$ or $t \succ v$). A necessary condition that such an order exists is that Π is simple.
- Let $\preccurlyeq$ be a linear order on T and let $f \in V$. The **reducible leader** of f, denoted by $L(f)$, or $L_{\preccurlyeq}(f)$ or $L_{\Pi,\preccurlyeq}(f)$ if necessary, is the unique maximum $t \in \mathrm{Supp}(f) \cap T$.
- **Lemma**: Let $\preccurlyeq$ be a linear order on $T = \pi_1(\Pi)$ that is compatible with a simple term-rewriting system Π on a free $\mathbf{k}$-module V with basis W. Let $f, g \in V$ and suppose $f \overset{(t,v)}{\to}_{\Pi} g$ for some $(t, v) \in \Pi$. Then $L(g) \preccurlyeq L(f)$, where equality holds if and only if $L(g) \neq t$.

Locally Base-Confluent implies Locally Confluent

- **Theorem**. Let V be a free $\mathbf{k}$-module with a $\mathbf{k}$-basis W and let Π be a simple term-rewriting system on V. Suppose we have a linear order $\preccurlyeq$ on T compatible with Π. If Π is locally base-confluent, it is locally confluent.

- **Corollary**. If $\rightarrow_\Pi$ is terminating, then it is locally base-confluent if and only if it is confluent, in which case, $\rightarrow_\Pi$ is converging.

- Proof of Corollary. If $\rightarrow_\Pi$ is confluent and $(ct \rightarrow_\Pi cv_1, ct \rightarrow_\Pi cv_2)$ is a local base-fork, then $cv_1 \downarrow_\Pi cv_2$. By Property (f), $cv_1 - cv_2 \stackrel{*}{\rightarrow}_\Pi 0$ and hence Π is locally base-confluent. The converse follows from Newman's Lemma.

Extension to Free Algebras

- Polynomial Algebra: basis consists of monomials
- Differential Polynomial Algebra: basis consists of differential monomials
- Free Rota-Baxter Algebra: basis consists of Rota-Baxter words
- Free Operated Polynomial Algebra: basis consists of operated monomials
- All these are free **k**-modules with a "monomial" basis, and while it is possible to explicitly define all rewrite rules in terms of all monomials that are reducible, a term-rewriting system is often given more concisely by omitting those that can be derived through the operations in the algebra as well as by substitutions. In our case, rewrite rules will be derived from a single operated polynomial identity (OPI) φ.

Term Rewriting Systems in Free Operated Algebras

- Let V be a free operated $\mathbf{k}$-algebra over a set Z with a $\mathbf{k}$-basis $W = \mathfrak{M}(Z)$.
- Intuitively, a **term-rewriting system** for V can be determined concisely by a set of **term-rewriting rules**, which are pairs (ℓ, r), written as $\ell \to r$, where $\ell \in W, r \in V$. The rule may be applied to a term $t \in W$ if some subterm of t matches ℓ at some position, in which case, t can be rewritten by substituting that occurrence of ℓ by r resulting in some $v \in V$. The rule $\ell \to r$ thus generates many more (abstract) rewriting rules $t \to v$ for V.
- If V is an operated φ-algebra, where $\varphi = \varphi_1 - \varphi_2$ with $\varphi_1 \in \mathfrak{M}(x_1, \ldots, x_k)$ and $\varphi_2 \in \mathbf{k}\|x_1, \ldots, x_k\|$, then for any $r_1, \ldots, r_k \in \mathfrak{M}(Z)$, we will have a term-rewrite rule $\ell := \varphi_1(r_1, \ldots, r_k) \to r := \varphi_2(r_1, \ldots, r_k)$.

Precise Definitions: Place-holder $\star$

- Let $Z^\star = Z \cup \{\star\}$. By a **$\star$-bracketed word** (respectively, **$\star$-bracketed polynomial**) on Z, we mean any word in $\|Z^\star\| = \mathfrak{M}(Z^\star)$ (respectively, polynomial in $\mathbf{k}\cdot\mathfrak{M}(Z^\star)$) **with exactly one** occurrence of $\star$. The set of all $\star$-bracketed words (respectively, $\star$-bracketed expressions) on Z is denoted by $\|Z\|^\star$ or $\mathfrak{M}^\star(Z)$ (respectively, $\mathbf{k}^\star\|Z\|$).
- Let $q \in \|Z\|^\star$ and $u \in \mathfrak{M}(Z)$. We will use $q|_u$ or $q|_{\star\mapsto u}$ to denote the bracketed word on Z obtained by replacing the symbol $\star$ in q by u.
- Next, we extend by linearity this notion to elements $s = \sum_i c_i u_i \in \mathbf{k}\mathfrak{M}(Z)$, where $c_i \in \mathbf{k}$ and $u_i \in \mathfrak{M}(Z)$.
- Finally, we extend again by linearity this notation to any $q \in \mathbf{k}^\star\|Z\|$.

Operated Ideals, Subwords, Placements, and RBNF

- The **operated ideal** $\mathrm{Id}(S)$ **generated by a subset** $S \subseteq \mathbf{k}\|Z\|$ is the set

$$\left\{ \sum_{i=1}^{k} c_i q_i|_{s_i} \;\middle|\; k \geqslant 1 \text{ and } c_i \in \mathbf{k}, q_i \in \mathfrak{M}^\star(Z), s_i \in S \text{ for } 1 \leqslant i \leqslant k \right\}.$$

- A bracketed word $u \in \mathfrak{M}(Z)$ is a **subword** of another bracketed word $w \in \mathfrak{M}(Z)$ if $w = q|_u$ for some $q \in \mathfrak{M}^\star(Z)$, where the specific occurrence of u in w defined by the $\star$ in q is denoted by the **placement** (u, q).
- We say $f \in \mathbf{k}\|Z\|$ is **Rota-Baxter irreducible** or is **in Rota-Baxter normal form** (RBNF) if no monomial $w \in \mathrm{Supp}(f)$ has the form $q|_{\lfloor u \rfloor \lfloor v \rfloor}$ for any $u, v \in \|Z\|$ and any $q \in \mathfrak{M}^\star(Z)$; otherwise, we say f is **Rota-Baxter reducible** and call (q, u, v) a **triple of f**. Let $\mathfrak{R}(Z)$ denote the set of *monomials* of $\|Z\|$ in RBNF.

Separated, Nested, and Intersecting Placements

Let $Z = \{x, y\}$. Consider placements in the monomials $w_1 = x\lfloor x \rfloor$ and $w_2 = \lfloor xyxy \rfloor$.

- The subword x appears at two locations in w_1. Their placements $(x, \star\lfloor x \rfloor)$ and $(x, x\lfloor \star \rfloor)$ are **separated**. In w_2, the two placements for the subword x are also separated, as are the two for xy.
- The placements $(x, \lfloor xy \star y \rfloor)$ and $(xy, \lfloor xy \star \rfloor)$ **nested** in w_2.
- Each of the two placements for xy overlaps partially the unique placement of yx in w_2. We say they are (properly) **intersecting**. The two placements of xx in xxx are also intersecting.

These notions can be formally defined.

Trichotomy Theorem on Placements

Theorem 4.11, Zheng and Guo, 2015. Let w be a bracketed word in $\mathfrak{M}(X)$. For any two placements (u_1, q_1) and (u_2, q_2) in w, exactly one of the following is true :

(a) : (u_1, q_1) and (u_2, q_2) are separated ;

(b) : (u_1, q_1) and (u_2, q_2) are nested ;

(c) : (u_1, q_1) and (u_2, q_2) are intersecting.

Rota-Baxter Rewriting Systems

- Let $\varphi(x,y) \in \mathbf{k}\|x,y\|$ be an OPI of the form $\lfloor x \rfloor \lfloor y \rfloor - \lfloor B(x,y) \rfloor$, where $B(x,y) \in \mathbf{k}\|x,y\|$.
- The rewriting system for $\mathbf{k}\|Z\|$ with basis $\mathfrak{M}(Z)$ associated to φ is the set $\Pi_\varphi(Z)$:

$$\{\, (q|_{\lfloor u \rfloor \lfloor v \rfloor}, q|_{\lfloor B(u,v) \rfloor}) \;\big|\; q \in \mathfrak{M}^\star(Z), u, v \in \mathfrak{M}(Z) \,\}$$

- We write $\rightarrow_\varphi$ instead of $\rightarrow_{\Pi_\varphi(Z)}$ when Z and $\Pi_\varphi(Z)$ are fixed.
- **Proposition**: $\Pi_\varphi(Z)$ is simple.
- $f \rightarrow_\varphi g$ if for some $u, v \in \mathfrak{M}(Z)$, g is obtained from f by replacing *exactly once* a subword $\lfloor u \rfloor \lfloor v \rfloor$ in *one* monomial $w \in \mathrm{Supp}(f)$ by $\lfloor B(u,v) \rfloor$. A bracketed polynomial $g \in \mathbf{k}\|Z\|$ is said to be **a normal φ-form** for f if g is in RBNF and $f \stackrel{*}{\rightarrow}_\varphi g$.

Totally Linear Expressions

- An expression $B \in \mathbf{k}\lfloor\!\lfloor X \rfloor\!\rfloor$ is **totally linear in X** if **every** variables $x \in X$, when counted with multiplicity in repeated multiplications, appears exactly once in **every** monomial $w \in \mathrm{Supp}(B)$.

- **Examples**: Let $X = \{x, y\}$. The expression $x\lfloor y \rfloor + \lfloor x \rfloor \lfloor y \rfloor + xy$ is totally linear in X, but the monomials $\lfloor x \rfloor$, $x^2 \lfloor y \rfloor$ and $x \lfloor y \rfloor^2$ are not.

Rota-Baxter Type OPI, Operators, and Algebras

An OPI $\varphi \in \mathbf{k}\|x, y\|$ is **of Rota-Baxter type** if φ has the form $\lfloor x \rfloor \lfloor y \rfloor - \lfloor B(x, y) \rfloor$ for some $B(x, y) \in \mathbf{k}\|x, y\|$ and if:

(*a*) : $B(x, y)$ is totally linear in x, y;

(*b*) : $B(x, y)$ is in RBNF;

(*c*) : for every well-ordered set Z, the rewriting system $\Pi_\varphi(Z)$ defined is terminating;

(*d*) : for every well-ordered set Z, the expression $B(B(u, v), w) - B(u, B(v, w))$ is φ-reducible to zero for all $u, v, w \in \mathfrak{M}(Z)$.

If $\varphi := \lfloor x \rfloor \lfloor y \rfloor - \lfloor B(x, y) \rfloor$ is of Rota-Baxter type, then we say the defining **operator** $P = \lfloor\ \rfloor$ **of a** φ**-algebra** R, and (by abuse) the **expression** $B(x, y)$, are **of Rota-Baxter type**.
By a **Rota-Baxter type algebra**, we mean **some** φ-algebra R where φ is **some** OPI in $\mathbf{k}\|x, y\|$ of Rota-Baxter type.

Examples

- Let $B(x,y) := x\lfloor y\rfloor$. Then $\varphi = 0$ is the OPI defining the **average operator** and it is of Rota-Baxter type. The identities defining a **Rota-Baxter operator** and that defining a **Nijenhuis operator** are OPIs of Rota-Baxter type.
- The expression $B(x,y) := y\lfloor x\rfloor$ is not of Rota-Baxter type. This is because in $\mathbf{k}\|u,v,w\|$, the operated polynomial $B(B(u,v),w) = w\lfloor B(u,v)\rfloor = w\lfloor v\lfloor u\rfloor\rfloor$ is in RBNF, while $B(u,B(v,w)) = B(v,w)\lfloor u\rfloor = w\lfloor v\rfloor\lfloor u\rfloor \to_\varphi w\lfloor u\lfloor v\rfloor\rfloor$ is also in RBNF and there being no other sequence of reduction for the expression $w\lfloor v\rfloor\lfloor u\rfloor$, the two operated polynomials are not φ-joinable. By the Hierarchy Lemma,

$$B(B(u,v),w) - B(u,B(v,w)) \not\stackrel{*}{\to}_\varphi 0.$$

Remarks on Definition of Rota-Baxter Type OPIs

- The **well-order on Z** need not be given if Z is denumerable or if Z is uncountable and we accept the Axiom of Choice.
- **Total linearity on $B(x, y)$** is imposed since we are considering linear operators.
- $B(x, y)$ **in RBNF** and $\Pi_\varphi(Z)$ **terminating** are necessary to avoid obvious infinite rewriting under $\Pi_\varphi(Z)$.
- φ**-reduction to zero of** $B(B(u, v), w) - B(u, B(v, w))$ is to ensure $\Pi_\varphi(Z)$ is confluent, but can be replaced with an apparently weaker condition for only one specific instance:
  - (e) For $Z' = \{ \mu, \nu, \omega \}$, the two operated polynomials $B(\mu, B(\nu, \omega))$ and $B(B(\mu, \nu), \omega)$ in $\mathbf{k}\|\mu, \nu, \omega\|$ are φ-joinable.
- φ**-joinability in** (e) is enough to ensure compatibility with the associative law for products of the form $\lfloor a \rfloor \lfloor b \rfloor \lfloor c \rfloor$ where $a, b, c \in R$ for any φ-algebra R.

List of Rota-Baxter Type Operators

Conjecture: For any $c, \lambda \in \mathbf{k}$, the operated polynomial $\varphi := \lfloor x \rfloor \lfloor y \rfloor - \lfloor B(x,y) \rfloor$, where $B(x,y)$ is taken from the list below, is of Rota-Baxter type. Moreover, any OPI φ of Rota-Baxter type is necessarily defined as above by a $B(x,y)$ from among this list.

- (a) : $x\lfloor y \rfloor$ (average operator),
- (b) : $\lfloor x \rfloor y$ (inverse average operator),
- (c) : $\underline{x\lfloor y \rfloor + y\lfloor x \rfloor}$, (symmetric average operator),
- (d) : $\underline{\lfloor x \rfloor y + \lfloor y \rfloor x}$, (symmetric inverse average operator),
- (e) : $x\lfloor y \rfloor + \lfloor x \rfloor y - \lfloor xy \rfloor$ (Nijenhuis operator),
- (f) : $x\lfloor y \rfloor + \lfloor x \rfloor y + \lambda xy$ (Rota-Baxter operator),
- (g) : $\underline{x\lfloor y \rfloor - x\lfloor 1 \rfloor y + \lambda xy}$, (average TD operator),
- (h) : $\underline{\lfloor x \rfloor y - x\lfloor 1 \rfloor y + \lambda xy}$, (inverse average TD operator),

List continued

(i) : $\underline{x\lfloor y\rfloor + \lfloor x\rfloor y - x\lfloor 1\rfloor y + \lambda xy}$ (TD operator),

(j) : $\underline{x\lfloor y\rfloor + \lfloor x\rfloor y - x\lfloor 1\rfloor y - xy\lfloor 1\rfloor + \lambda xy}$, (right TD operator),

(k) : $\underline{x\lfloor y\rfloor + \lfloor x\rfloor y - x\lfloor 1\rfloor y - \lfloor xy\rfloor + \lambda xy}$, (Nijenhuis TD operator),

(l) : $\underline{x\lfloor y\rfloor + \lfloor x\rfloor y - x\lfloor 1\rfloor y - \lfloor 1\rfloor xy + \lambda xy}$, (left TD operator),

(m) : $\underline{cx\lfloor 1\rfloor y + \lambda xy}$ (generalized endomorphism),

(n) : $\underline{cy\lfloor 1\rfloor x + \lambda yx}$ (generalized antimorphism).

- A new type is underlined followed by a proposed name. When λ is present, "of weight λ" should be added.