

Differential Algebraic Subgroups of $SL(2)$

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- ◆ Let \mathcal{F} be a (partial differential) field of characteristic zero, with a set $\Delta = \{ \delta_1, \dots, \delta_m \}$ of derivations and let \mathcal{C} be the field of constants of \mathcal{F} .
- ◆ Let \mathcal{U} be a universal differential extension of \mathcal{F} and let \mathcal{K} be the field of constants of \mathcal{U} .
- ◆ \mathcal{K} and \mathcal{F} are linearly disjoint over \mathcal{C} .
- ◆ Let \mathcal{G} be an extension of \mathcal{F} , over which \mathcal{U} is still universal and let \mathcal{D} be its field of constants.
- ◆ An **isomorphism** of \mathcal{G} is a differential field isomorphism from \mathcal{G} onto its image \mathcal{G}^σ in \mathcal{U} .
- ◆ The compositum of two Δ -subfields of \mathcal{U} is denoted by concatenation.

Specialization of Isomorphisms

- ◆ We say an isomorphism σ of \mathfrak{G} **specializes to** another isomorphism σ' of \mathfrak{G} if for every finite family $(\alpha_1, \dots, \alpha_k) \in \mathfrak{G}^k$, $(\sigma\alpha_1, \dots, \sigma\alpha_k)$ specializes (differentially) **over \mathfrak{G}** to $(\sigma'\alpha_1, \dots, \sigma'\alpha_k)$.
- ◆ Specialization $\sigma \rightarrow \sigma'$ is a reflexive and transitive relation.
- ◆ **Generic specialization:** $\sigma \leftrightarrow \sigma'$
- ◆ σ is **isolated over \mathfrak{F}** if every specialization from any isomorphism to σ is generic.

Algebraic-geometric Structures of Isomorphisms

- ◆ Let \mathcal{G} be a finitely generated extension of \mathcal{F} , and let $\mathcal{G} = \mathcal{F}\langle\eta_1, \dots, \eta_n\rangle$.
- ◆ σ specializes to σ' if and only if $(\sigma\eta_1, \dots, \sigma\eta_n)$ (differentially) specializes to $(\sigma'\eta_1, \dots, \sigma'\eta_n)$ over \mathcal{G} .
- ◆ There exist finitely many isolated isomorphisms $\sigma_1, \dots, \sigma_k$ of \mathcal{G} over \mathcal{F} such that every isomorphism of \mathcal{G} over \mathcal{F} is a specialization of one and only one of these. The number k is unique.
- ◆ The differential field of invariants of $\sigma_1, \dots, \sigma_k$ is \mathcal{F} .
- ◆ If \mathcal{F}° is the algebraic closure of \mathcal{F} in \mathcal{G} , then the differential field of invariants of the component of the identity (the specializations of say σ_1 , where σ_1 specializes to the identity) is \mathcal{F}° .

Strong Isomorphisms

- ◆ Let σ be an isomorphism of \mathcal{G} over \mathcal{D} , and let \mathcal{D}_σ be the field of constants of $\mathcal{G}\mathcal{G}^\sigma$. The following conditions are equivalent:
 - 1 $\mathcal{G}^\sigma \subset \mathcal{G}\mathcal{K}$ and $\mathcal{G} \subset \mathcal{G}^\sigma\mathcal{K}$
 - 2 $\mathcal{G}\mathcal{K} = \mathcal{G}^\sigma\mathcal{K}$
 - 3 $\mathcal{G}\mathcal{D}_\sigma = \mathcal{G}\mathcal{G}^\sigma = \mathcal{G}^\sigma\mathcal{D}_\sigma$
- ◆ An isomorphism σ satisfying the conditions is said to be **strong** (e.g. automorphisms). Let $\text{SI}(\mathcal{G}) = \text{set of strong isoms.}$
- ◆ σ strong implies $\text{tr deg } \mathcal{G}\mathcal{G}^\sigma/\mathcal{G} = \text{tr deg } \mathcal{D}_\sigma/\mathcal{D}$.
- ◆ $\text{SI}(\mathcal{G}) \leftrightarrow \text{Aut}(\mathcal{G}\mathcal{K}/\mathcal{K})$.
- ◆ Every specialization of a strong isomorphism is strong.

Strongly Normal Extensions

- ◆ A **strongly normal extension of \mathcal{F}** is a Δ -finitely generated extension \mathcal{G} for which every isomorphism of \mathcal{G} **over \mathcal{F}** is strong.
- ◆ If \mathcal{G} is finitely generated over \mathcal{F} with $\mathcal{D} = \mathcal{C}$, and if $\mathcal{G}^{\sigma_i} \subset \mathcal{G}\mathcal{K}$ for $1 \leq i \leq k$, then \mathcal{G} is strongly normal over \mathcal{F} .
- ◆ Let \mathcal{G} be a strongly normal extension of \mathcal{F} .
- ◆ $\mathcal{D} = \mathcal{C}$.
- ◆ \mathcal{G} is finitely generated as a field over \mathcal{F} , and $\mathcal{D}_\sigma = \mathcal{C}_\sigma$ is finitely generated over \mathcal{C} for every isomorphism σ over \mathcal{F} .
- ◆ The set $\text{SI}(\mathcal{G}/\mathcal{F})$ of strong isomorphisms of \mathcal{G} over \mathcal{F} , when identified as $\text{Aut}(\mathcal{G}\mathcal{K}/\mathcal{F}\mathcal{K})$ has the structure of a \mathcal{C} -group, now denoted by $G(\mathcal{G}/\mathcal{F})$.
- ◆ The field associated to a “point” σ is \mathcal{C}_σ .

Examples: Primitives and Exponentials

- ◆ (primitive) Let $\mathcal{G} = \mathcal{F}\langle\alpha\rangle$ where $\delta\alpha \in \mathcal{F}$ for all $\delta \in \Delta$. If $\mathcal{D} = \mathcal{C}$, then \mathcal{G} is strongly normal over \mathcal{F} . We have $\delta(\sigma\alpha) = \sigma(\delta\alpha) = \delta\alpha$ and hence $c(\sigma) = \sigma\alpha - \alpha \in \mathcal{K}$ and σ is strong, with $\mathcal{C}_\sigma = \mathcal{C}(c(\sigma))$ and $c : G(\mathcal{G}/\mathcal{F}) \rightarrow \mathcal{K}$ is an injective \mathcal{C} -homomorphism.
- ◆ (exponential) Let $\mathcal{G} = \mathcal{F}\langle\alpha\rangle$ where $\ell\Delta(\alpha) \in \mathcal{F}^m$. If $\mathcal{D} = \mathcal{C}$, then \mathcal{G} is a strongly normal extension of \mathcal{F} . We have

$$(\sigma\alpha)^{-1}\delta(\sigma\alpha) = \sigma(\alpha^{-1}\delta\alpha) = \alpha^{-1}\delta\alpha$$

and hence $c(\sigma) = \alpha^{-1}\sigma\alpha \in \mathcal{K}^*$ and σ is strong, with $\mathcal{C}_\sigma = \mathcal{C}(c(\sigma))$ and $c : G(\mathcal{G}/\mathcal{F}) \rightarrow \mathcal{K}^*$ is an injective \mathcal{C} -homomorphism.

Examples: Picard-Vessiot and Weierstraussians

- ◆ Let $\alpha \in GL(n)$ be a matrix such that $\ell\Delta\alpha = (\Delta\alpha)\alpha^{-1}$ has entries in \mathcal{F} . Then $\mathcal{G} = \mathcal{F}\langle\alpha\rangle$ is strongly normal over \mathcal{F} . The matrix $c(\sigma) = \alpha^{-1}\sigma\alpha \in GL_{\mathcal{X}}(n)$ and σ is strong, with $\mathcal{C}_\sigma = \mathcal{C}(c(\sigma))$ and $c : G(\mathcal{G}/\mathcal{F}) \rightarrow GL_{\mathcal{X}}(n)$ is an injective \mathcal{C} -homomorphism.
- ◆ Let $g_2, g_3 \in \mathcal{C}$ be such that $g_2^3 - 27g_3^2 \neq 0$. Let $\mathcal{G} = \mathcal{F}\langle\alpha\rangle$, where α satisfies, for some $a_1, \dots, a_m \in \mathcal{F}$, the system:

$$(\delta_i\alpha)^2 = a_i^2(4\alpha^3 - g_2\alpha - g_3), \quad 1 \leq i \leq m.$$

Then \mathcal{G} is strongly normal over \mathcal{F} .

- ◆ Both primitive and exponential extensions are Picard-Vessiot, but if α is transcendental over \mathcal{F} , and Weierstraussian, then \mathcal{G} is not a Picard-Vessiot extension of \mathcal{F} .

Strongly Normal Subfields of a Simple Extension

- ◆ Let $t \in \mathcal{U}$ be differentially transcendental over \mathcal{F} . Let $\mathcal{G} = \mathcal{F}\langle t \rangle$.
- ◆ Problem: Find all intermediate differential subfields \mathcal{E} between \mathcal{F} and \mathcal{G} such that \mathcal{G} is strongly normal over \mathcal{E} .
- ◆ Quick answer: \mathcal{E} is generated basically over \mathcal{F} by:
 - 1 Schwarzian derivatives of t , $S\delta(t) = \frac{\delta^3 t}{\delta t} - \frac{3(\delta^2 t)^2}{2(\delta t)^2}$
 - 2 logarithmic derivatives of linear homogeneous differential polynomials in t with coefficients in \mathcal{F}
 - 3 powers of linear homogeneous differential polynomials in t with coefficients in \mathcal{F}
 - 4 polyhedral functions of t
 - 5 exceptions involving certain non-zero constants

Automorphisms of Simple Extensions

- ◆ Let \mathcal{H} be a differential subfield of \mathcal{U} over which \mathcal{U} need not be universal but there is an $s \in \mathcal{U}$ differentially transcendental over \mathcal{H} . For example, $\mathcal{H} = \mathcal{F}\mathcal{K}$.
- ◆ Let $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(n)$. Say $x \in \mathbf{P}(\mathcal{H})$ if there exists $u \in \mathcal{U}^*$ such that $ux \in GL_{\mathcal{H}}(n)$.
- ◆ For any $p \in \mathcal{U}$ such that $-cp + a \neq 0$, define

$$p^x = \frac{dp - b}{-cp + a} \in \mathcal{U}.$$

- ◆ Define τ_x to be the automorphism in $\mathcal{H}\langle s \rangle / \mathcal{H}$ with $\tau_x(s) = s^x$. Then $\tau_{xx'} = \tau_x \circ \tau_{x'}$ and if we let $\mathbf{Z} = \{+I, -I\} \subset SL(2)$, we have an exact sequence of groups:

$$1 \rightarrow \mathbf{Z} \rightarrow \mathbf{P}(\mathcal{H}) \xrightarrow{\tau} \text{Aut}(\mathcal{H}\langle s \rangle / \mathcal{H}) \rightarrow 1.$$

Differential Algebraic Groups

- ◆ A **differential algebraic group** is a differentially closed subset G of \mathcal{U}^n such that G is a group and the group laws are everywhere defined differential rational maps.
- ◆ G is **defined over** \mathcal{F} if the differentially closed set G and the group laws are defined over \mathcal{F} (that is, given by differentially polynomial or rational functions with coefficients in \mathcal{F}).
- ◆ (Cassidy) The components of G have the same differential dimension, are mutually disjoint, and are cosets of the component G° containing the identity, which is a connected, normal differential algebraic subgroup of G .
- ◆ G is **linear** if $G \subset GL(n)$. Viewed as a differential algebraic subgroup of $SL(n+1)$, the group laws are **polynomials** maps.
- ◆ (Cassidy) Every differential algebraic group whose group laws are given by differential polynomial functions is isomorphic to a linear differential algebraic group.

Galois Groups and $SL(2)$

- ◆ $G(\mathcal{G}/\mathcal{E})$ is identified with a subgroup of $\text{Aut}(\mathcal{GK}/\mathcal{FK}) = \text{Aut}(\mathcal{FK}\langle t \rangle/\mathcal{FK})$.
- ◆ Any $\sigma \in \text{Aut}(\mathcal{FK}\langle t \rangle/\mathcal{FK})$ can be represented by a matrix in $\mathbf{P}(\mathcal{FK}) \subset SL(2)$, or a matrix in $GL_{\mathcal{FK}}(2)$.
- ◆ There is a unique differential algebraic subgroup $H(\mathcal{E})$ of $SL(2)$, $\mathbf{Z} \subset H(\mathcal{E}) \subset \mathbf{P}(\mathcal{FK})$, such that the sequence is exact:

$$1 \rightarrow \mathbf{Z} \rightarrow H(\mathcal{E}) \xrightarrow{\tau} G(\mathcal{G}/\mathcal{E}) \rightarrow 1.$$

Moreover, the fixed field of $H(\mathcal{E})$ is \mathcal{E} .

- ◆ (Sketch) $H(\mathcal{E})$ is the set of $x \in SL(2)$ satisfying the system obtained after clearing denominators of the following:

$$A_\eta(y^x)B_\eta(y) = A_\eta(y)B_\eta((y^x)), \quad \eta \in \mathcal{E}$$

where $\eta = A_\eta(t)/B_\eta(t)$ and y is a differential indeterminate over \mathcal{U} . This just says σ_x fixes every $\eta \in \mathcal{E}$ when $y \mapsto t$.

Strategy to Compute the Subfields \mathcal{E}

- ◆ Goal: Compute all subfields \mathcal{E} such that \mathcal{G} is strongly normal over \mathcal{E} .
- ◆ (Strategy):
 - 1 Compute all differential algebraic subgroups H of $SL(2)$.
 - 2 Identify those satisfying $\mathbf{Z} \subset H \subset \mathbf{P}(\mathcal{F}\mathcal{K})$.
 - 3 For each such subgroup H , compute its fixed field $\mathcal{E}(H)$.
 - 4 Prove that \mathcal{G} is strongly normal over each $\mathcal{E}(H)$.
 - 5 Compute its Galois group $G(\mathcal{G}/\mathcal{E}(H))$.
 - 6 Show that its Galois group is H .
- ◆ Suffices to do this up to conjugation over \mathcal{F} , which means we need to first classify all differential algebraic subgroups of $SL(2)$.
- ◆ The Zariski closure G of a differential algebraic subgroup H defined over \mathcal{F} is an algebraic subgroup, also defined over \mathcal{F} .

Infinite Algebraic Subgroups of $SL(2)$

- ◆ Up to conjugation, the infinite algebraic subgroups of $SL(2)$ defined over a field K are classified into five types.
 - 1 $\dim G = 3$; then $G = SL(2)$.
 - 2 $\dim G = 2$; then G is connected and conjugate over K to $ST(2)$, the special triangular subgroup.
 - 3 $\dim G = 1$ and G° is unipotent; then G is conjugate over K to $U^n(2)$ for some positive integer n .
 - 4 $\dim G = 1$, G is connected and diagonalizable; then G is conjugate over K to $SO^e(2)$ for some $e \in K^*$.
 - 5 $\dim G = 1$, $G \neq G^\circ$, and G° is diagonalizable; then G is conjugate over K to $SO^e(2)^\dagger$ for some $e \in K^*$.
- ◆ Here, H and H' , subsets of $SL(2)$, are **conjugate over K** if there exists $z \in \mathbf{P}(K)$ such that $zHz^{-1} = H'$.

- ◆
$$\left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}, \alpha^n = 1 \right\}; \quad \left\{ \begin{pmatrix} \alpha & \pm e\gamma \\ \gamma & \pm \alpha \end{pmatrix}, \alpha^2 - e\gamma^2 = \pm 1 \right\}.$$

Finite Subgroups of $SL(2)$

- ◆ Let G be a finite subgroup of $SL(2)$ of order n . Then there are five possibilities:
 - 1 G is cyclic.
 - 2 $G/\mathbf{Z} \subset G$ and G/\mathbf{Z} is dihedral.
 - 3 $G/\mathbf{Z} \subset G$ and G/\mathbf{Z} is isomorphic to A_4 ; tetrahedral.
 - 4 $G/\mathbf{Z} \subset G$ and G/\mathbf{Z} is isomorphic to S_4 ; octahedral.
 - 5 $G/\mathbf{Z} \subset G$ and G/\mathbf{Z} is isomorphic to A_5 ; icosahedral.

- ◆ If K contains appropriate roots of unity, and if $G \subset \mathbf{P}(K)$, then G is defined over K and is conjugate over K to one of the groups: $C(n)$, $D(4m, e)$, $D(8, e, f, g)$, T , OC , DI .

Lie Vector Spaces and Fields of Constants

- ◆ Let $\mathfrak{D} = \mathcal{U}\Delta$, the vector space over \mathcal{U} with basis Δ . Let \mathfrak{E} be a subspace of dimension k , and let $\mathcal{K}(\mathfrak{E}) = \bigcap_{E \in \mathfrak{E}} \text{Ker } E$ be its field of constants, which is clearly algebraically closed.
- ◆ \mathfrak{E} is **Lie subspace** if the following equivalent conditions hold:
 - 1 $D \in \mathfrak{D}$ and $D(\mathcal{K}(\mathfrak{E})) = 0$ implies $D \in \mathfrak{E}$.
 - 2 \mathfrak{E} has a commuting basis E_1, \dots, E_k , where $[E_i, E_j] = 0$ for all i, j .
- ◆ $E = \sum_{i=1}^k a_i E_i \in \mathfrak{E}$ is **rational over \mathcal{F} relative to a basis E_1, \dots, E_k** if $a_i \in \mathcal{F}$ for all i . \mathfrak{E} is **defined over \mathcal{F}** if it has a basis where E_i is rational over \mathcal{F} relative to Δ .
- ◆ If \mathfrak{E} is defined over \mathcal{F} , then rationality of $E \in \mathfrak{E}$ relative to a rational basis is rationality relative to Δ . The subset of \mathfrak{E} consisting of all rational operators over \mathcal{F} is denoted by $\mathfrak{E}_{\mathcal{F}}$.

(Cassidy) Zariski Dense Subgroups of $SL(n)$

- ◆ Let \mathfrak{E} be Lie subspace of \mathfrak{D} , defined over \mathcal{F} and let $s \in SL(n)$ be such that $\ell E(s) = (Es)s^{-1} \in \mathfrak{sl}_{\mathcal{F}}(n)$ for all $E \in \mathfrak{E}_{\mathcal{F}}$. Let $SL(\mathfrak{E}, s) = sSL_{\mathcal{K}(\mathfrak{E})}(n)s^{-1}$.
- ◆ (Cassidy) $SL(\mathfrak{E}, s)$ is a Zariski dense differential algebraic subgroup of $SL(n)$ and is defined over \mathcal{F} and these are all.
- ◆ $SL(\mathfrak{E}, s)$ is conjugate over \mathcal{F} to $SL(\mathfrak{E}', s')$ if and only if $\mathfrak{E} = \mathfrak{E}'$ and there exists $z \in \mathbf{P}(\mathcal{F})$ such that $s'^{-1}zs \in SL_{\mathcal{K}(\mathfrak{E})}(n)$.
- ◆ Example. Let $\mathcal{F} = \mathbb{Q}(i \sin x, i \cos x)$, $\delta = d/dx$, $i = \sqrt{-1}$. Then $\mathfrak{C} = \mathbb{Q}$. Fix elements $\alpha, \gamma \in \mathcal{U}$ such that

$$\alpha^2 - \gamma^2 = \sin x, \quad 2\alpha\gamma = -\cos x, \quad \alpha^2 + \gamma^2 = -1.$$

Let $s = \begin{pmatrix} \alpha & \gamma \\ \gamma & -\alpha \end{pmatrix}$. Then $\ell\delta s = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix}$ and

$H = SL(\mathfrak{D}, s)$ is a Zariski dense differential algebraic subgroup, contained in $\mathbf{P}(\mathcal{F}\mathcal{K})$, but not conjugate over \mathcal{F} to $SL_{\mathcal{K}}(2)$.

Differential Algebraic Subgroups of \mathbf{G}_a^n

- ◆ (Cassidy) Every linear differential ideal \mathfrak{p} of $\mathcal{U}\{y_1, \dots, y_n\}$ defines a differential algebraic subgroup B of \mathbf{G}_a^n , which is connected, is a vector subspace of \mathcal{U}^n over \mathcal{K} , and is defined over \mathcal{F} if and only if \mathfrak{p} is. Moreover, these are all.
- ◆ The set $\mathcal{K}(B)$ of all $a \in \mathcal{U}$ such that $aB \subseteq B$ is the field of constants $\mathcal{K}(\mathcal{E})$ of a unique Lie subspace \mathcal{E} of \mathcal{D} , which is defined over \mathcal{F} . Note that $\mathcal{K}(0) = \mathcal{U}$ when $B = 0$ or $\mathcal{E} = 0$.
- ◆ Relative to any orderly ranking, \mathfrak{p} has a **canonical, linear, homogeneous** characteristic set $B \subset \mathcal{F}\{y_1, \dots, y_n\}$ such that

$$L(ay_1, \dots, ay_n) = aL(y_1, \dots, y_n), \quad (L \in B, a \in \mathcal{K}(B)).$$

Indeed, for every $L \in \mathfrak{p} \cap \mathcal{F}\{y_1, \dots, y_n\}_1$.

- ◆ An element $\beta \in \mathcal{U}^n$ is said to be **rational over \mathcal{F} mod B** , that is, the coset $\beta + B \in \mathbf{G}_a^n/B$ is rational over \mathcal{F} , if $L(\beta) \in \mathcal{F}$ for every linear homogeneous differential polynomial $L \in \mathfrak{p}$; or equivalently, if $L(\beta) \in \mathcal{F}$ for every $L \in B$. If $B = 0$, $\beta \in \mathcal{F}$.

Differential Algebraic Subgroups of \mathbf{G}_m

- ◆ The **logarithmic derivative map** $\ell\Delta : \mathbf{G}_m \rightarrow \mathbf{G}_a^m$ given by $\ell\Delta x = \left(\frac{\delta_1 x}{x}, \dots, \frac{\delta_m x}{x}\right)$ is a differential rational homomorphism defined over \mathbb{Q} with kernel $\mathbf{G}_{m\mathcal{K}} = \mathcal{K}^*$ and image \mathbf{I} , which is a differential algebraic subgroup of \mathbf{G}_a^m with corresponding linear differential ideal generated by $\{\delta_i y_j - \delta_j y_i \mid 1 \leq i < j \leq m\}$.
- ◆ (Cassidy) Every infinite differential algebraic subgroup A of \mathbf{G}_m contains $\mathbf{G}_{m\mathcal{K}}$, is connected, corresponds to a differential algebraic subgroup of \mathbf{I} , and is defined over \mathcal{F} if and only if $\ell\Delta(A)$ is. These are all.
- ◆ The finite (differential) algebraic subgroups of \mathbf{G}_m are the cyclic groups \mathbf{P}_n consisting of n -th roots of unity.

Differential Algebraic Subgroups of $ST(2)$

- ◆ B is a differential algebraic subgroup of \mathbf{G}_a defined over \mathcal{F} ;
 A is a (finite or infinite) differential algebraic subgroup of $\mathbf{G}_{m\mathcal{K}(B)} = \mathcal{K}(B)^*$, also defined over \mathcal{F} ;
 $\beta \in \mathcal{U}$ is rational over \mathcal{F} mod B .

- ◆ To such a triple (B, A, β) , we define

$$ST(B, A, \beta) = \left\{ \begin{pmatrix} a & (a - a^{-1})\beta + b \\ 0 & a^{-1} \end{pmatrix} \mid a \in A, b \in B \right\}.$$

- ◆ If $A = 1$ or $A = \mathbf{P}_2$, we only define this when $\beta = 0$.

- ◆ $ST(B, A, \beta)$ is a differential algebraic subgroup of $ST(2)$ defined over \mathcal{F} , and these are all. Moreover,

$$ST(B, A, \beta) = ST(B, 1, 0) \cdot ST(0, A, \beta) \quad (\text{semidirect product})$$

- ◆ $ST(B, A, \beta)$ is conjugate over \mathcal{F} to $ST(B', A', \beta')$ if and only if $A = A'$, and there exists $f \in \mathcal{F}^*$ such that $fB = B'$, and $(f\beta - \beta' + B') \cap \mathcal{F} \neq \emptyset$.

Dense Cases for $ST(2)$, $U^n(2)$, $SD(2)$

- ◆ For all $H = ST(B, A, \beta)$, one and only one of the following holds:
- 1 $B \neq 0$ and A is infinite; dense in $ST(2)$.
 - 2 $B \neq 0$ and $A = \mathbf{P}_n$; dense in $U^n(2)$.
 - 3 $B = 0$ and A is infinite; then $\beta \in \mathcal{F}$, H is conjugate over \mathcal{F} to $ST(0, A, 0)$, which is dense in $SD(2)$.
 - 4 $B = 0$ and $A = \mathbf{P}_n$; then $\beta \in \mathcal{F}$, H is finite and conjugate over \mathcal{F} to $C(n)$.

A Modified Logarithmic Derivative on $SO^e(2)$

◆ Recall for $e \in \mathcal{F}^*$, $SO^e(2) = \left\{ \begin{pmatrix} \alpha & e\gamma \\ \gamma & \alpha \end{pmatrix} \mid \alpha^2 - e\gamma^2 = 1 \right\}$.

◆ Let $\epsilon^2 = e$, $s = \begin{pmatrix} 1 & \epsilon \\ -1 & \epsilon \end{pmatrix}$ and let $\tau_s(u) = sus^{-1}$, $\ell_\epsilon(x) = \epsilon x$.

Then the map $\pi_e : SO^e(2) \xrightarrow{\tau_s} SD(2) \xrightarrow{p} \mathbf{G}_m \xrightarrow{\ell_\Delta} \mathbf{G}_a^m \xrightarrow{\ell_\epsilon} \mathbf{G}_a^m$ is a differential rational homomorphism defined over \mathbb{Q} , given by

$$\pi_e \left(\begin{pmatrix} \alpha & e\gamma \\ \gamma & \alpha \end{pmatrix} \right) = \begin{cases} \left(\frac{\delta_1 \alpha}{\gamma}, \dots, \frac{\delta_m \alpha}{\gamma} \right) & \text{if } \gamma \neq 0; \\ (0, \dots, 0) & \text{if } \gamma = 0. \end{cases}$$

with

$$\text{Ker } \pi_e = \left\{ \begin{pmatrix} \alpha & e\gamma \\ \gamma & \alpha \end{pmatrix} \mid \alpha \in \mathcal{K}, \gamma \in \mathcal{U}, \alpha^2 - e\gamma^2 = 1 \right\}.$$

Image = differential algebraic \mathcal{F} -subgroup \mathbf{I}_e of \mathbf{G}_a^m defined by

$$\delta_i y_j - \delta_j y_i + \frac{1}{2}(\ell \delta_j e) y_i - \frac{1}{2}(\ell \delta_i e) y_j \quad (1 \leq i < j \leq m).$$

Dense Differential Algebraic Subgroups of $SO^e(2)$

- ◆ $p \circ \tau_s : SO^e(2) \rightarrow \mathbf{G}_m$ is a rational isomorphism;
$$\begin{pmatrix} \alpha & e\gamma \\ \gamma & \alpha \end{pmatrix} \mapsto \alpha + e\gamma.$$
- ◆ Exact sequence: $1 \rightarrow \text{Ker } \pi_e \rightarrow SO^e(2) \xrightarrow{\pi_e} \mathbf{I}_e \rightarrow 1.$
- ◆ There is a bijection between the set of differential algebraic subgroups B of \mathbf{I}_e and the set of differential algebraic subgroup of $SO^e(2)$ containing $\text{Ker } \pi_e$. This bijection is given by $B \mapsto SO^e(B) = \pi_e^{-1}(B)$, B is defined over \mathcal{F} if and only if $SO^e(B)$ is.
- ◆ A differential algebraic subgroup H is dense in $SO^e(2)$ if and only if $H \supset \text{Ker } \pi_e$, in which case $H = SO^e(B)$ for a unique differential algebraic subgroup B of \mathbf{I}_e , H is connected, and H is defined over \mathcal{F} if and only if B is. These are all.
- ◆ $SO^e(B)$ and $SO^e(B')$ are conjugate over \mathcal{F} if and only if there exists $f \in \mathcal{F}^*$ such that $e'^{-1}e = f^2$ and $fB' = B$.

A Modified Logarithmic Derivative on $SO^e(2)^*$

◆ Recall for $e \in \mathcal{F}^*$,

$$SO^e(2)^* = \left\{ \begin{pmatrix} \alpha & -e\gamma \\ \gamma & -\alpha \end{pmatrix} \mid \alpha^2 - e\gamma^2 = -1 \right\}.$$

◆ The map $\pi_e^* : SO^e(2)^* \xrightarrow{\tau_s^*} SD(2)^* \xrightarrow{p^*} \mathbf{G}_m \xrightarrow{\ell\Delta} \mathbf{G}_a^m \xrightarrow{\ell\epsilon} \mathbf{G}_a^m$ is a differential rational map defined over \mathbb{Q} given by

$$\pi_e^* \left(\begin{pmatrix} \alpha & -e\gamma \\ \gamma & -\alpha \end{pmatrix} \right) = \begin{cases} \left(\frac{\delta_1\alpha}{\gamma}, \dots, \frac{\delta_m\alpha}{\gamma} \right) & \text{if } \gamma \neq 0; \\ (0, \dots, 0) & \text{if } \gamma = 0. \end{cases}$$

with

$$\text{Ker } \pi_e^* = \left\{ \begin{pmatrix} \alpha & -e\gamma \\ \gamma & -\alpha \end{pmatrix} \mid \alpha \in \mathcal{K}, \gamma \in \mathcal{U}, \alpha^2 - e\gamma^2 = -1 \right\}.$$

Image is the differential algebraic subgroup \mathbf{I}_e of \mathbf{G}_a^m .

◆ Note that $p^* \circ \tau_s^*$ is a bijective birational map.

Dense Differential Algebraic Subgroups of $SO^e(2)^\dagger$

- ◆ Recall that $SO^e(2)^\dagger = SO^e(2) \cup SO^e(2)^*$.
- ◆ Let $e \in \mathcal{F}^*$, B a differential algebraic subgroup of \mathbf{I}_e defined over \mathcal{F} , and $\beta \in \mathbf{I}_e$ be such that $\beta + B \in \mathbf{G}_a^m/B$ is rational over \mathcal{F} .
- ◆ The set $SO^e(B, \beta)^\dagger = \pi_e^{-1}(B) \cup \pi_e^{*-1}(\beta + B)$ is a Zariski dense differential algebraic subgroup of $SO^e(2)^\dagger$ defined over \mathcal{F} with $\pi_e^{-1}(B) = SO^e(B)$ as its component of the identity, and these are all.
- ◆ $SO^e(B, \beta)^\dagger$ is conjugate over \mathcal{F} to $SO^{e'}(B', \beta')^\dagger$ if and only if there exists $f \in \mathcal{F}^*$ such that $e'^{-1}e = f^2$, $fB' = B$, and

$$SO_{\mathcal{F}}^e(2) \cap \pi_e^{-1}((f\beta' - \beta + B) \cup (f\beta' + \beta + B)) \neq \emptyset.$$

Example: Dense DA_G of $SO^e(2)^\dagger$

- ◆ Example. Let $\mathcal{F} = \mathbb{Q}(i \sin x, i \cos x)$, $\delta = d/dx$. Then $\mathcal{C} = \mathbb{Q}$ and the field of constants of $\mathcal{F}(i)$ is $\mathbb{Q}(i)$ by linear disjointness.
- ◆ Let $e = -1$, $B = 0$, $h = i \sin x$, $g = i \cos x$, and $\beta = \frac{1}{2}$. Then $G_5 = SO^e(0, \beta)^\dagger$ and $G'_5 = SO^e(0, 2\beta)^\dagger$ are both dense differential algebraic subgroups of $SO^e(2)^\dagger$ contained in $\mathbf{P}(\mathcal{F}\mathcal{K})$, but neither is conjugate over \mathcal{F} to $SO^e_{\mathcal{K}}(2)^\dagger = SO^e(0, 0)^\dagger$, which is the union of

$$\text{Ker } \pi_e = \left\{ \begin{pmatrix} \alpha & e\gamma \\ \gamma & \alpha \end{pmatrix} \mid \alpha \in \mathcal{K}, \gamma \in \mathbf{u}, \alpha^2 - e\gamma^2 = 1 \right\}$$

and

$$\text{Ker } \pi_e^* = \left\{ \begin{pmatrix} \alpha & -e\gamma \\ \gamma & -\alpha \end{pmatrix} \mid \alpha \in \mathcal{K}, \gamma \in \mathbf{u}, \alpha^2 - e\gamma^2 = -1 \right\}.$$

Infinite $DA_G H$ of $SL(2)$ contained in $\mathbf{P}(\mathcal{FK})$

◆ If H is infinite, then H is conjugate over \mathcal{F} to one of the following:

1 $G_1 = sSL_{\mathcal{K}}(2)s^{-1}$, where $s \in \mathbf{P}(\mathcal{FC}_a)$ and $\ell\Delta s \in sl_{\mathcal{F}}(2)^m$.

2 $G_2 = ST(B, \mathcal{K}^*, 0)$, where B is a nonzero differential algebraic subgroup of \mathbf{G}_a defined over \mathcal{F} , and B , as a vector space over \mathcal{K} , has a finite basis in \mathcal{F} .

3 $G_3 = ST(B, \mathbf{P}_{2n}, 0)$, where $n \in \mathbb{N}$, and B is as above.

4 $G_4 = SO_{\mathcal{K}}^e(2)$, where $e \in \mathcal{C}^*$.

5 $G_5 = SO^e(0, \beta)^\dagger$ where $e \in \mathcal{C}^*$, $\beta = \left(\frac{1}{2} \frac{\delta_1 h}{g}, \dots, \frac{1}{2} \frac{\delta_m h}{g}\right)$
for some $h, g \in \mathcal{F}$, $g \neq 0$, and $h^2 - eg^2 \in \mathcal{C}^*$.

Finite $DA_G H$ of $SL(2)$ contained in $\mathbf{P}(\mathfrak{K})$

◆ If H is finite and if \mathcal{C} is algebraically closed, then $H \subset \mathbf{P}(\mathfrak{F})$, and H is conjugate over \mathfrak{F} to one of the following:

- 1 $G_6 = C(2n)$, where $n \in \mathbb{N}$.
- 2 $G_7 = D(4n, e)$, where $n \in \mathbb{N}$, $e \in \mathfrak{F}^*$.
- 3 $G_8 = D(8, e, f, g)$, where $e, f, g \in \mathfrak{F}$, $eg \neq 0$, and $f^2 + eg^2 = 1$.
- 4 $G_9 = T$
- 5 $G_{10} = OC$
- 6 $G_{11} = DI$.

$$D(4n, e) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a^n = 1 \right\} \cup \left\{ \begin{pmatrix} 0 & b \\ -b^{-1} & 0 \end{pmatrix} \mid b^{2n} = e^n \right\}$$

$$D(8, e, f, g) = \left\{ \pm 1, \pm \begin{pmatrix} 0 & \epsilon \\ -\epsilon^{-1} & 0 \end{pmatrix}, \pm \begin{pmatrix} \alpha & e\gamma \\ \gamma & -\alpha \end{pmatrix}, \pm \begin{pmatrix} -\epsilon\gamma & \epsilon\alpha \\ \epsilon^{-1}\alpha & \epsilon\gamma \end{pmatrix} \right\}$$
$$\epsilon^2 = e, \quad \alpha^2 + e\gamma^2 = -1, \quad \alpha^2 - e\gamma^2 = f, \quad 2\alpha\gamma = g$$

Example of $D(8, e, f, g)$

- ◆ Let $\mathcal{F} = \mathbb{C}(\sin^2 x, \sin x \cos x), \delta/dx$.
- ◆ Let $e = \sec^2 x, f = i \tan x$, and $g = 1$.
- ◆ Let $\epsilon = \sec x, \alpha = \left(\frac{-1+i \tan x}{2}\right)^{1/2}, \gamma = \frac{1}{2}\alpha^{-1}$.
- ◆ Then if $H = D(8, e, f, g)$, H is not conjugate over \mathcal{F} to $D(8, e)$.

$$D(4n, e) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a^n = 1 \right\} \cup \left\{ \begin{pmatrix} 0 & b \\ -b^{-1} & 0 \end{pmatrix} \mid b^{2n} = e^n \right\}$$

$$D(8, e, f, g) = \left\{ \pm 1, \pm \begin{pmatrix} 0 & \epsilon \\ -\epsilon^{-1} & 0 \end{pmatrix}, \pm \begin{pmatrix} \alpha & e\gamma \\ \gamma & -\alpha \end{pmatrix}, \pm \begin{pmatrix} -\epsilon\gamma & \epsilon\alpha \\ \epsilon^{-1}\alpha & \epsilon\gamma \end{pmatrix} \right\}$$

$$\epsilon^2 = e, \quad \alpha^2 + e\gamma^2 = -1, \quad \alpha^2 - e\gamma^2 = f, \quad 2\alpha\gamma = g$$

Strongly Normal Subfields: Infinite Galois Group

◆ If \mathcal{E} is a differential subfield of $\mathcal{G} = \mathcal{F}\langle t \rangle$ such that \mathcal{G} is strongly normal over \mathcal{E} and $G(\mathcal{G}/\mathcal{E})$ is infinite, then \mathcal{E} is conjugate over \mathcal{F} to one of the following:

- 1 $\mathcal{E}_1 = \mathcal{F}\langle S\delta_1(t^s), \dots, S\delta_m(t^s) \rangle$, where $s \in \mathbf{P}(\mathcal{F}\mathcal{C}_a)$ and $\ell\Delta(s) \in s/\mathcal{F}(2)^m$.
- 2 $\mathcal{E}_2 = \mathcal{F}\langle (\ell\delta_i L(t))_{1 \leq i \leq m, L \in \Gamma(B)} \rangle$, where B is a nonzero differential algebraic subgroup of \mathbf{G}_a defined over \mathcal{F} and B (as a vector space over \mathcal{K}) has a finite basis in \mathcal{F} , and $\Gamma(B)$ is the set of all non-zero linear homogeneous differential polynomials vanishing on B .
- 3 $\mathcal{E}_3 = \mathcal{F}\langle (L(t)^n)_{L \in \Gamma(B)} \rangle$, where $n > 0 \in \mathbb{N}$, and $B, \Gamma(B)$ as above.
- 4 $\mathcal{E}_4 = \mathcal{F}\langle (\frac{\delta_i t}{e-t^2})_{1 \leq i \leq m} \rangle$, where $e \in \mathcal{C}^*$.
- 5 $\mathcal{E}_5 = \mathcal{F}\langle ((\frac{2\delta_i t}{e-t^2} - \frac{\delta_i h}{2eg})_{1 \leq i \leq m}) \rangle$, where $e \in \mathcal{C}^*$, and $h, g \in \mathcal{F}$, $g \neq 0$, and $h^2 - eg^2 \in \mathcal{C}^*$.

Strongly Normal Subfields: Finite Galois Group

◆ If $G(\mathcal{G}/\mathcal{E})$ is finite, and if C is algebraically closed, then \mathcal{E} is conjugate over \mathcal{F} to one of the following, and \mathcal{G} is Galois over \mathcal{E} .

1 $\mathcal{E}_6 = \mathcal{F}\langle t^n \rangle$, where $n > 0 \in \mathbb{N}$.

2 $\mathcal{E}_7 = \mathcal{F}\langle t^n + \frac{e^n}{t^n} \rangle$, where $n > 0 \in \mathbb{N}$, $e \in \mathcal{F}^*$.

3 $\mathcal{E}_8 = \mathcal{F}\langle \frac{g(t^2 + e)^2}{t(gt^2 - 2ft - eg)} \rangle$, where $e, f, g \in \mathcal{F}$, $eg \neq 0$,
and $f^2 + eg^2 = 1$.

4 $\mathcal{E}_9 = \mathcal{F}\langle \frac{t^{12} - 33t^8 - 33t^4 + 1}{(t^4 - 1)^2 t^2} \rangle$.

5 $\mathcal{E}_{10} = \mathcal{F}\langle \frac{(t^{12} - 33t^8 - 33t^4 + 1)^2}{(t^4 - 1)^4 t^4} \rangle$.

6 $\mathcal{E}_{11} = \mathcal{F}\langle \frac{(t^{30} - 228t^{15} + 494t^{10} + 228t^5 + 1)^3}{t^5(t^{10} + 11t^5 - 1)^5} \rangle$.

The Differential Galois Group for $ST(B, \mathcal{K}^*, 0)$

- ◆ $G_2 = ST(B, \mathcal{K}^*, 0) = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathcal{K}^*, b \in B \right\}$ where B is a nonzero differential algebraic subgroup of \mathbf{G}_a defined over \mathcal{F} , and B , as a vector space over \mathcal{K} , has a finite basis in \mathcal{F} .
- ◆ $\mathcal{E}_2 = \mathcal{F}\langle (\ell\delta_i L(t))_{1 \leq i \leq m, L \in \Gamma(B)} \rangle$, where $\Gamma(B)$ is the set of all non-zero linear homogeneous differential polynomials vanishing on B . ($\Gamma(B)$ can be replaced by a finite subset.)
- ◆ G_2 is \mathcal{C} -isomorphic to the group $G(\ell, 0)$ of $(\ell + 1) \times (\ell + 1)$ matrices of the form

$$g(b_1, \dots, b_\ell; a) = \begin{pmatrix} 1 & 0 & \cdots & 0 & b_1 \\ 0 & 1 & \cdots & 0 & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & b_\ell \\ 0 & 0 & \cdots & 0 & a \end{pmatrix}$$

where $\ell = \dim_{\mathcal{K}} B$, and $b_1, \dots, b_\ell, a \in \mathcal{K}$.

The Differential Galois Group for $ST(B, \mathbf{P}_{2n}, 0)$

- ◆ $G_3 = ST(B, \mathbf{P}_{2n}, 0) = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a^{2n} = 1, b \in B \right\}$ where B is a nonzero differential algebraic subgroup of \mathbf{G}_a defined over \mathcal{F} , and B , as a vector space over \mathcal{K} , has a finite basis in \mathcal{F} .
- ◆ $\mathcal{E}_3 = \mathcal{F}\langle (L(t)^n)_{L \in \Gamma(B)} \rangle$, where $n > 0 \in \mathbb{N}$, and $B, \Gamma(B)$ as before. ($\Gamma(B)$ can be replaced by a finite subset.)
- ◆ G_3 is \mathcal{C} -isomorphic to the group $G(\ell, 0)$ of $(\ell + 1) \times (\ell + 1)$ matrices of the form

$$g(b_1, \dots, b_\ell; a) = \begin{pmatrix} 1 & 0 & \cdots & 0 & b_1 \\ 0 & 1 & \cdots & 0 & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & b_\ell \\ 0 & 0 & \cdots & 0 & a \end{pmatrix}$$

where $\ell = \dim_{\mathcal{K}} B$, and $b_1, \dots, b_\ell, a^{2n} = 1, a \neq 0$.

Open Questions

- ◆ $G_1 = sSL_{\mathcal{K}}(2)s^{-1}$, where $s \in \mathbf{P}(\mathcal{FC}_a)$ and $\ell\Delta s \in sl_{\mathcal{F}}(2)^m$.
- ◆ $\mathcal{E}_1 = \mathcal{F}\langle S\delta_1(t^s), \dots, S\delta_m(t^s) \rangle$
- ◆ Is $\mathcal{G}/\mathcal{E}_1$ a Picard-Vessiot Extension? If so, what is a representation of G_4 as an \mathcal{C} -group?
- ◆ Ditto: $G_4 = SO_{\mathcal{K}}^e(2)$, $e \in \mathcal{C}^*$
- ◆ $\mathcal{E}_4 = \mathcal{F}\langle (\frac{\delta_i t}{e-t^2})_{1 \leq i \leq m} \rangle$, where $e \in \mathcal{C}^*$.
- ◆ Ditto: $G_5 = SO^e(0, \beta)^\dagger$, $e \in \mathcal{C}^*$, $\beta = \left(\frac{1}{2} \frac{\delta_1 h}{g}, \dots, \frac{1}{2} \frac{\delta_m h}{g} \right)$ for some $h, g \in \mathcal{F}$, $g \neq 0$, and $h^2 - eg^2 \in \mathcal{C}^*$.
- ◆ $\mathcal{E}_5 = \mathcal{F}\langle \left(\left(\frac{2\delta_i t}{e-t^2} - \frac{\delta_i h}{2eg} \right)^2 \right)_{1 \leq i \leq m} \rangle$, where $e \in \mathcal{C}^*$, and $h, g \in \mathcal{F}$, $g \neq 0$, and $h^2 - eg^2 \in \mathcal{C}^*$.