On Certain Towers of Extensions by Antiderivatives

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Differential Fields

A field $\mathbf{F}$ with a map $' : \mathbf{F} \to \mathbf{F}$ satisfying

- $(a+b)' = a' + b'$
- $(ab)' = a'b + ab'$

is called a *differential field* and the map $'$ is called a *derivation* on $\mathbf{F}$. A *differential field extension* $\mathbf{E}$ of $\mathbf{F}$ is a differential field such that $\mathbf{E} \supset \mathbf{F}$ and the restriction of the derivation of $\mathbf{E}$ to $\mathbf{F}$ coincides with the derivation of $\mathbf{F}$.

Field of Constants: Let $(\mathbf{F}, \ ')$ be a differential field. The differential field $\mathbf{C} := \{c \in \mathbf{F} \mid c' = 0\}$ is called the field of constants.

Antiderivative Extensions

For the rest of the talk we will assume that the constants $\mathbf{C}$ of the differential field $\mathbf{F}$ is algebraically closed and of characteristic 0.

Definition 1. A differential field extension $\mathbf{E} \supset \mathbf{F}$ is a *No New Constant* (NNC) extension if the constants of $\mathbf{E}$ are the same as the constants of $\mathbf{F}$.

Definition 2. Let $\mathbf{E} \supset \mathbf{F}$ be a NNC extension. An element $u \in \mathbf{E}$ is an *antiderivative* if $u' \in \mathbf{F}$. A differential field extension $\mathbf{E} \supset \mathbf{F}$ is an *antiderivative extension* of $\mathbf{F}$ if for $i = 1, 2, \cdots, n$ there exists $u_i \in \mathbf{E}$ such that $u_i' \in \mathbf{F}$ and $\mathbf{E} = \mathbf{F}(u_1, u_2, \cdots, u_n)$.

THEOREM 3. *Let $\mathbf{E} \supset \mathbf{F}$ be a differential field extension and let $u \in \mathbf{E}$ such that $u' \in \mathbf{F}$. Then u is transcendental over $\mathbf{F}$ or $u \in \mathbf{F}$.*

• I. Kaplansky, **An Introduction to Differential Algebra**, Hermann, Paris, (1957).

• A. Magid, **Lectures on Differential Galois Theory**, University Lecture Series. American Mathematical society 1994, 2nd edn.

Some Observations: Consider the differential field $(\mathbf{C}(x), {}')$, where $' = \frac{d}{dx}$

1. $(\mathbf{C}(x), {}')$ has no nontrivial differential subfields

2. $(\mathbf{C}(x), {}')$ contains no solutions of the differential operator $L(Y) = Y' - \frac{1}{x}$.

THEOREM 4. *Let $\mathbf{E} \supset \mathbf{F}$ be a NNC differential field extension and for $1 = 1, 2, \cdots, n$ let $u_i \in \mathbf{E}$ be antiderivatives. If u_i are algebraically dependent over $\mathbf{F}$ then there is a tuple $(c_1, \cdots, c_n) \in \mathbf{C}^n \setminus \{0\}$ such that $\sum_{i=1}^n c_i u_i \in \mathbf{F}$.*

- A. Ostrowski, *Sur Les Relations Algébriques Entre Les Intégrales Indéfines*, Acta Mathematica, **78**, (1946), 315-318.

- E.R. Kolchin, *Algebraic Groups and Algebraic Dependence*, Amer. J. of Math, **90**, No.4. (1968), 1151-1164.

- J. Ax, *On Schanuel's Conjectures*, Ann. of Math (2) **93** (1971), 252-268. MR **43**

- M. Rosenlicht, *On Liouville's Theory of Elementary Functions*, Pacific J. Math (2) **65** (1976), 485-492

- The above theorem also works for any family of derivations $\{\partial_i | i \in I\}$ on $\mathbf{E}$ that satisfies $\cap_{i \in I} ker \partial_i = \mathbf{C}$

Exponential of an Integral

THEOREM 5. *Let* $\mathbf{E} \supset \mathbf{F}$ *be a differential field extension and let* $u \in \mathbf{E}$ *such that* $\frac{u'}{u} \in \mathbf{F}$*. Then* u *is transcendental over* $\mathbf{F}$ *or* $u^n \in \mathbf{F}$ *for some* $n \in \mathbb{N}$*.*

THEOREM 6. *Let* $\mathbf{E} \supset \mathbf{F}$ *be a NNC differential field extension and let* $\mathfrak{e}_1, \cdots, \mathfrak{e}_m \in \mathbf{E}$ *be such that* $\frac{\mathfrak{e}_j'}{\mathfrak{e}_j} \in \mathbf{F}$*. If* $\mathfrak{e}_1, \cdots, \mathfrak{e}_m$ *are algebraically dependent over* $\mathbf{F}$ *then there exist* $(r_1, \cdots, r_m) \in \mathbb{Z}^n \setminus \{0\}$ *such that* $\prod_{j=1}^m \mathfrak{e}_j^{r_j} \in \mathbf{F}$*.*

THEOREM 7. *(Kolchin-Ostrowski) Let* $\mathbf{E} \supset \mathbf{F}$ *be a NNC differential field extension and let* $\mathfrak{l}_1, \cdots, \mathfrak{l}_n$*,* $\mathfrak{e}_1, \cdots, \mathfrak{e}_m \in \mathbf{E}$ *be such that* $\mathfrak{l}_i' \in \mathbf{F}$ *and* $\frac{\mathfrak{e}_j'}{\mathfrak{e}_j} \in \mathbf{F}$*. If* $\mathfrak{l}_1, \cdots, \mathfrak{l}_n, \mathfrak{e}_1, \cdots, \mathfrak{e}_m$ *are algebraically dependent over* $\mathbf{F}$ *then there exist* $(c_1, \cdots, c_n) \in C^n \setminus \{0\}$ *such that* $\sum_{i=1}^n c_i \mathfrak{l}_i \in \mathbf{F}$ *or there exists* $(r_1, \cdots, r_m) \in \mathbb{Z}^n \setminus \{0\}$ *such that* $\prod_{j=1}^m \mathfrak{e}_j^{r_j} \in \mathbf{F}$*.*

Tower of Extensions by Antiderivatives

Let $\mathbf{F}$ be a characteristic zero differential field with an algebraically closed field of Constants $\mathbf{C}$ and let $\mathbf{F}_\infty$ be a complete Picard-Vessiot closure of $\mathbf{F}$ (every homogeneous linear differential equation over $\mathbf{F}_\infty$ has a full set of solutions in $\mathbf{F}_\infty$ and it has $\mathbf{C}$ as its field of constants and $\mathbf{F}_\infty$ is minimal with respect to these properties). All the differential fields under consideration are subfields of $\mathbf{F}_\infty$

Definition 8. A differential field extension $\mathbf{E}$ of $\mathbf{F}$ is called a *tower of extension by antiderivatives* if there are differential fields $\mathbf{E}_i$, $0 \leq i \leq n$ such that

$$\mathbf{E} := \mathbf{E}_n \supseteq \mathbf{E}_{n-1} \supseteq \cdots \supseteq \mathbf{E}_1 \supseteq \mathbf{E}_0 := \mathbf{F}$$

and $\mathbf{E}_i$ is an extension by antiderivatives of $\mathbf{E}_{i-1}$ for each $1 \leq i \leq n$.

THEOREM 9. *Let* $\mathbf{M} \supseteq \mathbf{F}$ *be differential fields and let*

$$\mathbf{E} := \mathbf{E}_n \supset \mathbf{E}_{n-1} \supset \cdots \supset \mathbf{E}_1 \supset \mathbf{E}_0 := \mathbf{F}$$

be a tower of extensions by antiderivatives. Then $u \in \mathbf{E}$ *is algebraic over* $\mathbf{M}$ *only if* $u \in \mathbf{M}$.

Thus the above theorem shows that if $\mathbf{E} \supseteq \mathbf{K} \supsetneq \mathbf{M} \supseteq \mathbf{F}$ are differentials fields and $\mathbf{E}$ is an extension by antiderivatives of $\mathbf{F}$ then $\mathbf{K}$ is purely transcendental over $\mathbf{M}$.

Infiniteness of $\mathbf{F}_\infty$

THEOREM 10. *Let* $\mathbf{E} \supseteq \mathbf{F}$ *be a NNC extension. If there is an* $\mathfrak{x} \in \mathbf{E} \setminus \mathbf{F}$ *such that* $\mathfrak{x}' \in \mathbf{F}$ *then for any* $n \in \mathbb{N}$ *and distinct* $\alpha_1, \cdots, \alpha_n \in \mathbf{C}$, *the elements* $\mathfrak{y}_i \in \mathbf{F}_\infty$ *such that* $\mathfrak{y}'_{\alpha_i} = \frac{1}{\mathfrak{x}+\alpha_i}$ *are algebraically independent over* $\mathbf{F}(\mathfrak{x})$. *Moreover, the differential field* $\mathbf{F}(\mathfrak{y}_\alpha, \mathfrak{x})$, *where* $\mathfrak{y}'_\alpha = \frac{1}{\mathfrak{x}+\alpha}$ *and* $\alpha \in \mathbf{C}$ *is not imbeddable in any Picard-Vessiot extension of* $\mathbf{F}$.

Let $\mathbf{F}_0 := \mathbf{F}$ and let $\mathbf{F}_i$ be the Picard-Vessiot closure of $\mathbf{F}_{i-1}$.

Remark 11. Thus if $\mathbf{E} \supseteq \mathbf{F}$ are differential fields such that $\mathfrak{x} \in \mathbf{E} \setminus \mathbf{F}$ and $\mathfrak{x}' \in \mathbf{F}$ then the differential field $\mathbf{F}(\mathfrak{y}_\alpha, \mathfrak{x})$, $\mathfrak{y}'_\alpha = \frac{1}{\mathfrak{x}+\alpha}$ and $\alpha \in \mathbf{C}$ is not imbeddable in any Picard-Vessiot extension of $\mathbf{F}$ and thus $\mathfrak{y}_\alpha \notin \mathbf{F}_1$. We may apply the above theorem again for the element $\mathfrak{y}_\alpha$ with $\mathbf{F}_1$ as the ground field. Then for any $\mathfrak{z}_\beta \in \mathbf{F}_\infty$ such that $\mathfrak{z}'_\beta = \frac{1}{\mathfrak{y}_\alpha+\beta}$, $\beta \in \mathbf{C}$, we obtain that the differential field $\mathbf{F}_1(\mathfrak{z}_\beta, \mathfrak{y}_\alpha)$ is not imbeddable in any Picard-Vessiot extension of $\mathbf{F}_1$ and thus $\mathfrak{z}_\beta \notin \mathbf{F}_2$. A repeated application of the theorem proves the following: If $\mathbf{F}$ is a differential field that has a proper extension by antiderivatives then for given any n, $\mathbf{F}_n$ has proper extensions by antiderivatives.

Algebraic Independence of Certain class of antiderivatives

THEOREM 12. *Let* $\mathbf{E} \supseteq \mathbf{F}$ *be differential fields,* $\mathfrak{x}_1, \cdots, \mathfrak{x}_l \in \mathbf{E}$ *be antiderivatives of* $\mathbf{F}$ *and assume that* $\mathfrak{x}_1, \cdots, \mathfrak{x}_l$ *are algebraically independent over* $\mathbf{F}$*. For each* $i = 1, \cdots, m$ *let* $A_i, B_i, C_i \in \mathbf{F}[\mathfrak{x}_1, \cdots, \mathfrak{x}_l]$*,* $(A_i, B_i) = (A_i, C_i) = (B_i, C_i) = 1$ *be polynomials satisfying the following condition*

C1: C_i *is an irreducible polynomial,* $C_i \nmid C_j$ *if* $i \neq j$ *and* $C_i \nmid B_j$ *for any* $1 \leq i, j \leq m$*.*

Let $\mathfrak{y}_1, \cdots, \mathfrak{y}_m \in \mathbf{F}_\infty$ *be antiderivatives of* $\mathbf{F}(\mathfrak{x}_1, \cdots, \mathfrak{x}_l)$ *with* $\mathfrak{y}_i' = \frac{A_i}{C_i B_i}$*. Then* $\mathfrak{y}_1, \cdots, \mathfrak{y}_m$ *are algebraically independent over* $\mathbf{F}(\mathfrak{x}_1, \cdots, \mathfrak{x}_l)$*.*

Tower of Extensions by J-I-E Antiderivatives

Let $\mathfrak{y}_{11}, \cdots, \mathfrak{y}_{1n_1}$ be algebraically independent antiderivatives of $\mathbf{F}$ and for $i = 1, 2, \cdots, k$, let $\mathbf{E}_i := \mathbf{E}_{i-1}(\mathfrak{y}_{i1}, \mathfrak{y}_{i2}, \cdots, \mathfrak{y}_{in_i})$, where $\mathbf{E}_0 := \mathbf{F}$ and for $i \geq 2$, let $\mathfrak{y}_{i1}, \mathfrak{y}_{i2}, \cdots, \mathfrak{y}_{in_i}$ are J-I-E antiderivatives of $\mathbf{E}_{i-1}$, that is, $\mathfrak{y}'_{ij} = \frac{A_{ij}}{C_{ij}B_{ij}}$ and for each $2 \leq i \leq k$ and for all $1 \leq j \leq n_i$, $A_{ij}, B_{ij}, C_{ij} \in \mathbf{E}_{i-2}[\mathfrak{y}_{i-11}, \cdots, \mathfrak{y}_{i-1n_{i-1}}]$ are polynomials such that $(A_{ij}, B_{ij}) = (B_{ij}, C_{ij}) = (A_{ij}, C_{ij}) = 1$ and satisfying the following conditions

C1: C_{ij} is an irreducible polynomial for each i, j. For every i, $C_{is} \nmid C_{it}$ (that is, they are non associates)if $s \neq t$ and $C_{is} \nmid B_{it}$ for any $1 \leq s, t \leq n_i$.
C2: For each i and for every j, $1 \leq j \leq n_i$ there is an element $\mathfrak{y}_{C_{ij}} \in \{\mathfrak{y}_{i-11}, \cdots, \mathfrak{y}_{i-1n_{i-1}}\}$ such that the partial $\frac{\partial C_{ij}}{\partial \mathfrak{y}_{C_{ij}}} \neq 0$ and $\frac{\partial A_{ij}}{\partial \mathfrak{y}_{C_{ij}}} = \frac{\partial B_{ij}}{\partial \mathfrak{y}_{C_{ij}}} = 0$.

Definition 13. We call

$$\mathbf{E} := \mathbf{E}_k \supset \mathbf{E}_{k-1} \supset \cdots \supset \mathbf{E}_2 \supset \mathbf{E}_1 \supset \mathbf{E}_0 := \mathbf{F}$$

a tower of extensions by J-I-E antiderivatives. Note that $\mathbf{E}_1$ is an ordinary antiderivative extension of $\mathbf{F}$.

Let $I_i := \{\mathfrak{y}_{ij} | 1 \leq j \leq n_i\}$, $\Lambda_t := Span_{\mathbf{C}} \cup_{i=1}^{t} I_i$, $\Lambda_0 = \{0\}$ and $\mathbf{E} := \mathbf{E}_k$.

Generalized Kolchin-Ostrowski Theorem

THEOREM 14. *Let $\mathbf{E}_k \supset \mathbf{K} \supset \mathbf{F}$ be an intermediate differential field. If $\cup_{j=1}^{k} I_i$ is algebraically dependent over $\mathbf{K}$ then there is a nonzero $\mathfrak{s} \in \mathbf{K} \cap \Lambda_k$.*

Differential Subfields of J-I-E tower

THEOREM 15. *For every differential subfield* $\mathbf{K}$ *of* $\mathbf{E} := \mathbf{E}_k$*, the field generated by* $\mathbf{F}$ *and* $S_k := \mathbf{K} \cap \Lambda_k$ *equals the differential field* $\mathbf{K}$*. That is*

$$\mathbf{K} = \mathbf{F}(S_k).$$

Moreover $\mathbf{K}$ *itself is a tower of extensions by antiderivatives, namely*

$$\mathbf{K} = \mathbf{F}(S_k) \supset \mathbf{F}(S_{k-1}) \supset \mathbf{F}(S_{k-2}) \supset \cdots \supset \mathbf{F}(S_1) \supset \mathbf{F},$$

where $S_i := S_k \cap \Lambda_i$*.*

Example

Let $\mathbf{C} := \mathbb{C}$ denote the complex numbers, $\mathbf{C}_\infty$ the complete Picard-Vessiot closure of $\mathbf{C}$, $x \in \mathbf{C}_\infty$ be an element whose derivative is 1, $\tan^{-1}(x) \in \mathbf{C}_\infty$ be an element such that

$$(\tan^{-1}(x))' = \frac{1}{1+x^2}$$

and let $\tan^{-1}(\tan^{-1}(x)) \in \mathbf{C}_\infty$ be an element such that

$$\Big(\tan^{-1}(\tan^{-1}(x))\Big)' = \frac{1}{(1+(\tan^{-1}(x))^2)(1+x^2)}.$$

Then

$$\mathbf{C}\langle\tan^{-1}(\tan^{-1}(x))\rangle$$
$$= \mathbf{C}(\tan^{-1}(\tan^{-1}(x)), \tan^{-1}(x), x).$$

Remark 16. The J-I-E extensions may have non-elementary functions. For example; if $a_i \in \mathbf{C}$ are distinct constants for $i = 1, \cdots, n$ then the elements $\mathfrak{y}_i := \int \frac{\ln(x)}{x-a_i}$ are J-I-E antiderivatives of the differential field $\mathbf{C}(x, \ln(x))$ with $\mathfrak{y}_i' := \frac{A_i}{C_i B_i}$ where $A_i := \ln(x)$, $B_i := 1$ and $C_i := x - a_i$. These $\mathfrak{y}_i$'s are non-elementary functions*. From theorem 12 we see that these $\mathfrak{y}_i$'s are algebraically independent over $\mathbf{C}(x, \ln(x))$ and from therorem15 we see that any differential field $\mathbf{K}$, $\mathbf{C}(x, \ln(x), \mathfrak{y}_i | 1 \leq i \leq n) \supseteq \mathbf{K} \supseteq \mathbf{C}$ is of the form $\mathbf{C}(S)$, where $S \subset \text{span}_{\mathbf{C}}\{x, \ln(x), \mathfrak{y}_i | 1 \leq i \leq n\}$ is a finite set. Moreover $\mathbf{C}(S)$ itself is a tower of (Picard-Vessiot) extensions by antiderivatives.

*Elena Anne Marchisotto, Gholam-Ali Zakeri, *An Invitiation to Integration in Finite Terms*, Math.Assoc.Amer (4) **25** (Sep., 1994), 295-308.

A Normal Tower of J-I-E Antiderivatives

Iterated Logarithms

Let $\mathbf{C}$ be an algebraically closed characteristic zero differential field with a trivial derivation and let $\mathbf{C}_\infty$ be the complete Picard-Vessiot Closure of $\mathbf{C}$.

Let $\Lambda_1 := \{\ln(x+c) | c \in \mathbf{C}\}$, where $\ln(x+c) \in \mathbf{C}_\infty$ and $\ln(x+c)' = \frac{1}{x+c}$. We observe that $\mathbf{C}(x, \Lambda_1;\ ' = \frac{d}{dx})$ is a differential field.

- Any subset $S \subset \{x\} \cup \Lambda_1$ is algebraically independent over $\mathbf{C}$.

Differential subfields of $\mathbf{C}(x, \Lambda_1)$

THEOREM 17. *For $u \in \mathbf{C}(x, \Lambda_1) \setminus \mathbf{C}(x)$, there is a set $S \subset \Lambda_1$ such that the singly generated differential field*

$$\mathbf{C}\langle u \rangle = \mathbf{C}(x, L_1, \cdots, L_t),$$

where $L_i \in Span_C S$. Moreover, if $u = \frac{P}{Q}$, $P, Q \in \mathbf{C}[x, \Lambda_1]$, $(P, Q) = 1$ then the linear forms L_i's can be explicitly computed.

Let $\mathcal{L}_{0,0} := x$ be a solution of the differential equation $Y' = 1$. We recursively define $\mathcal{L}_{\vec{c},n}$ for $\vec{c} \in \mathbf{C}^n$, $n \in \mathbb{N}$ as the solution of the differential equation

$$Y' = \frac{\mathcal{L}'_{\pi(\vec{c}),n-1}}{\mathcal{L}_{\pi(\vec{c}),n-1} + \psi_n(\vec{c})}, \tag{1}$$

where $\psi_n : \mathbf{C}^n \to \mathbf{C}$ is the map $\psi_n(c_1, \cdots, c_n) = c_n$ and $\pi : \mathbf{C}^n \to \mathbf{C}^{n-1}$ is the map

$$\begin{cases} \pi(c_1, \cdots, c_n) = (c_1, \cdots, c_{n-1}), & \text{when } n > 1; \\ \pi(c) = 0, & \text{when } n = 1. \end{cases}$$

- $\mathcal{L}_{\vec{c},n}$ is called an $n-$th level iterated logarithm.

- One can think of $\mathcal{L}_{\vec{c},n}$ as $\ln(\ln \cdots (\ln(x+c_1)) \cdots + c_{n-1}) + c_n)$.

We denote $\mathcal{L}_{\pi(\vec{c}),n-1}$ by $\pi(\mathcal{L}_{\vec{c},n})$.

Notations: $\Lambda_n := \{\mathcal{L}_{\vec{c},n} | \vec{c} \in \mathbf{C}^n\}$, $\Lambda_0 := \{x\}$, $\mathfrak{L}_0 = \mathbf{C}(\Lambda_0)$, $\mathfrak{L}_n := \mathbf{C}(\cup_{i=0}^{n} \Lambda_i)$ for all $n \in \mathbb{N}$, $\Lambda_\infty = \cup_{i=0}^{\infty} \Lambda_i$ and $\mathfrak{L}_\infty = \mathbf{C}(\Lambda_\infty)$.

We observe that

- Given any finite set $S \subset \Lambda_\infty$ there is an $n \in \mathbb{N}$ such that $\pi(S) = \{x\}$.

- $\mathbf{C}(S, \pi(S), \pi^2(S), \cdots, \pi^n(S) = x) = \mathbf{C}\langle S \rangle$ and we call the LHS, the container differential field of S.

Algebraic Independence of Iterated logarithms

THEOREM 18. *Let $S_{n-1} \subset \Lambda_{n-1}$ be a finite set whose elements are antiderivatives of a differential field $\mathbf{F}$ and let $S_n \subset \Lambda_n$ be such that $\pi(S_n) \subseteq S_{n-1}$. Suppose that S_{n-1} is algebraically independent over $\mathbf{F}$ then S_n is algebraically independent over $\mathbf{F}(S_{n-1})$.*

- Under the assumption $\pi(S_n) \subseteq S_{n-1}$ the differential field $\mathbf{F}(S_{n-1}, S_n)$ becomes an antiderivative extension of $\mathbf{F}(S_{n-1})$.

Note that x is not algebraic over $\mathbf{C}$ and therefore from the above theorem Λ_1 is algebraically independent over $\mathbf{C}(x)$, Λ_2 is algebraically independent over $\mathbf{C}(x, \Lambda_1)$ and so on..

Thus Λ_n is algebraically independent over $\mathfrak{L}_{n-1}$.

Normality: For any differential automorphism $\sigma \in \mathcal{G}(\mathbf{C}_\infty|\mathbf{C})$, $\sigma(x) = x + c_\sigma$ where $c_\sigma \in \mathbf{C}$ and therefore $\sigma(\ln(x+a)) = \ln(x+a+c_\sigma) + d(\sigma,a)$, $d(\sigma,a) \in \mathbf{C}$. Thus $\sigma\Big(\mathbf{C}(x)\Big) \subseteq \mathbf{C}(x)$ and $\sigma\Big(\mathbf{C}(x,\Lambda_1)\Big) \subseteq \mathbf{C}(x,\Lambda_1)$.

A similar argument should show that

- $\mathfrak{L}_n$ is a normal extension for every n and therefore $\mathfrak{L}_\infty$ is also normal.

THEOREM 19. *Let $\mathbf{F}$ be a differential field finitely generated over its constants $\mathbf{C}$, $\mathbf{E}$ be a Picard-Vessiot extension of $\mathbf{F}$, and let $\mathbf{F} \subset \mathbf{E} \subset \mathfrak{L}_\infty$. If $\sum_{j=1}^{s} a_j y_j \in \mathbf{E}$ for some $a_j \in \mathbf{C}^*$, $y_j \in \cup_{i=0}^{\infty}\Lambda_i$ and $s \in \mathbb{N}$ then $\pi^i(y_j) \in \mathbf{F}$ for all $i \in \mathbb{N}$ and thus $y_j' \in \mathbf{F}$.*

Essential Iterated logarithms:

Let $u \in \mathfrak{L}_\infty$. Then, there are $P, Q \in \mathbf{C}[\Lambda_\infty]$ such that $u = \frac{P}{Q}$. Thus $u \in \mathbf{C}[T]$, where $T \subset \Lambda_\infty$ and T is finite.

The *e*ssential iterated logarithms of u is the set $\mathcal{E} := \{y \in T | \frac{\partial P}{\partial y} \neq 0$ or $\frac{\partial Q}{\partial y} \neq 0\}$

THEOREM 20. *Let $u \in \mathfrak{L}_n \setminus \mathfrak{L}_{n-1}$, $\mathcal{E}$ the essential set of logarithms of u and let $\mathcal{F}$ be the container differential field $\mathcal{E}$. Then the differential field*

$$\mathbf{C}\langle u \rangle = \mathbf{C}(\mathcal{S}, \pi(\mathcal{E}), \pi^2(\mathcal{E}), \cdots, x),$$

where $\mathcal{S}$ is a finite nonempty subset of $Span_{\mathbf{C}}\mathcal{E}$. Moreover, if $u = \frac{P}{Q}$, $P, Q \in \mathbf{C}[\Lambda_\infty]$, $(P, Q) = 1$ then the set of linear forms $\mathcal{S}$ can be explicitly computed.

Examples

Let $\mathbf{C}$ be the field of Complex numbers.

1) Let

$$u = \frac{5x^3 \ln(x+1) + \ln(x+e) + 27x^3 \ln(x+\sqrt{2})}{\ln(x) + x\Big(\ln(x+2) - 17\ln(x+3)\Big)^2}.$$

Then

$$\mathbf{C}\langle u\rangle = \mathbf{C}\Big(x, \ln(x+e), \ln(x), 5\ln(x+1) + 27\ln(x+\sqrt{2}), \ln(x+2) - 17\ln(x+3)\Big)$$

2) Let $y_1 := \ln(\ln(\ln(x-i)+2)+3)$, $y_2 := \ln(\ln(x+i)+\sqrt{3})$, $y_3 := \ln(x+\frac{5}{6})$, $y_4 := \ln(\ln(x+\frac{1}{2})+\frac{1}{2})$, $y_5 := \ln(x+\sqrt{5})$, $y_6 := \ln(x+5+i)$, $y_7 := \ln(\ln(\ln(x)+i))$ and let

$$P := \ln(x+i)^2 \ln(x-i)(y_1-y_3)^5 + x^3 \ln(x)(y_2-y_5)^2,$$

$$\begin{aligned} Q := & \ln(\ln(x)+i)^2(y_5-y_7)^7 \\ & + x\ln(x-i)^3 \ln(\ln(x-i)+2)^2(y_6-y_4)^{12} \end{aligned}$$

and

$$u = \frac{P}{Q}.$$

Then

$$\begin{aligned} \mathbf{C}\langle u\rangle = & \mathbf{C}(\ln(x-i), \ln(x+i), \ln(\ln(x)+i), \ln(x+\frac{1}{2}), \\ & \ln(x), x, \ln(\ln(x-i)+2), y_1-y_3, y_2-y_5, \\ & y_6-y_4, y_5-y_7) \end{aligned}$$