Group Foliation of Finite Difference Equations Using Equivariant Moving Frames

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Symmetry of Differential Equations

Definition: Given a differential equation

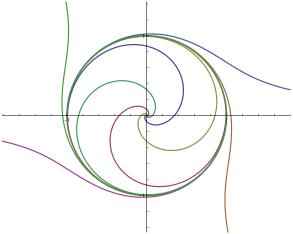
$$\Delta(x, y^{(k)}) = 0$$

a Lie group G is a symmetry group of the equation if it sends solutions to solutions:

$$\Delta(g \cdot (x, y^{(k)})) = 0 \qquad \text{whenever} \qquad \Delta(x, y^{(k)}) = 0$$

Example: $\dfrac{dy}{dx} = \dfrac{y^3 + x^2 y - x - y}{x^3 + xy^2 - x + y}$ is invariant under rotations

$$X = x\cos\theta - y\sin\theta$$
$$Y = x\sin\theta + y\cos\theta$$



Sophus Lie (1842–1899)

Using symmetries, Lie developed a theory for solving differential equations.

Differentialgleichungen (1891):

The older examinations on ordinary differential equations as found in standard books are not systematic. The writers developed special integration theories for homogeneous differential equations, for linear differential equations, and other special integrable forms of differential equations. However, the mathematicians did not realize that these special theories are all contained in the term infinitesimal transformations, which is closely connected with the term of a one parametric group.

Group Foliation: Historical Overview

Group foliation of differential equations:

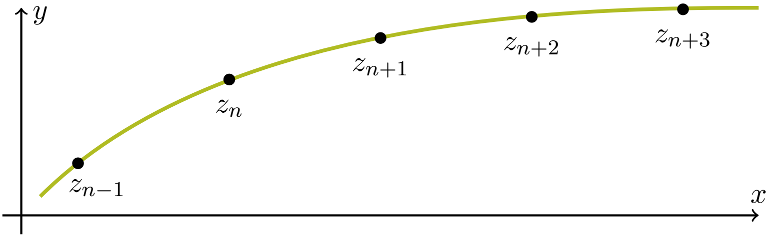
- 1895: Lie laid out the basic ideas in 2 examples
- 1904: Vessiot formalized Lie's ideas
- 1969 –: Fluid dynamics (Ovsiannikov and Soviet mathematicians)
- 2001: Heavenly and complex Monge–Ampère equations (Martina, Nutku, Sheftel, and Winternitz)
- 2005/08: EDS formulation (Anderson, Fels, and Pohjanpelto)
- 2015: Moving frame formulation (Thompson – V)

Group foliation of finite difference equations

- Today: (With Thompson, R.) Group foliation of finite difference equations, Commun. Nonlinear Sci. Numer. Simul. **59** (2018), 235–254.

Geometric setting

Consider $\{z_n = (x_n, y_n) \,|\, n \in \mathbb{Z}\}$:



Definition: The k^{th} order forward discrete jet at n is

$$z_n^{[k]} = (z_n, z_{n+1}, \ldots, z_{n+k})$$
$$= \text{ minimum \# of points to approximate } x, y, \frac{dy}{dx}, \ldots, \frac{d^k y}{dx^k}$$

The k^{th} order forward discrete jet space is

$$\mathrm{J}^{[k]} = \bigcup_{n \in \mathbb{Z}} z_n^{[k]}$$

Finite difference equations

Definition: A finite difference equation of order k is

$$E(n, z_n^{[k]}) = E_n(z_n, \dots, z_{n+k}) = 0$$

In many applications finite difference equations are used to approximate differential equations.

Example: $\dfrac{dy}{dx} = (k + x^a)y^b$ can be approximated by

$$\left(\frac{x_{n+1}^{a+1}}{a+1} + \frac{y_{n+1}^{1-b}}{b-1}\right) - \left(\frac{x_n^{a+1}}{a+1} + \frac{y_n^{1-b}}{b-1}\right) + k(x_{n+1} - x_n) = 0, \qquad x_{n+1} - x_n = h$$

Applications:

- Numerical modeling
- Discrete Quantum/General Relativity theory
- Discrete time economics
- Chaos

Symmetry of Finite Difference Equations

Definition: A Lie group G is a symmetry group of $E_n(z_n^{[k]}) = 0$ if it sends solutions to solutions:

$$E_n(g \cdot z_n^{[k]}) = 0 \qquad \text{whenever} \qquad E_n(z_n^{[k]}) = 0$$

Note: G acts on $z_n^{[k]}$ by the product action:

$$g \cdot (z_n, z_{n+1}, \ldots, z_{n+k}) = (g \cdot z_n, g \cdot z_{n+1}, \ldots, g \cdot z_{n+k})$$

Example: The equations

$$\left(\frac{x_{n+1}^{a+1}}{a+1}+\frac{y_{n+1}^{1-b}}{b-1}\right)-\left(\frac{x_n^{a+1}}{a+1}+\frac{y_n^{1-b}}{b-1}\right)+k(x_{n+1}-x_n)=0 \qquad x_{n+1}-x_n = h$$

are invariant under $G = (\mathbb{R}, +)$:

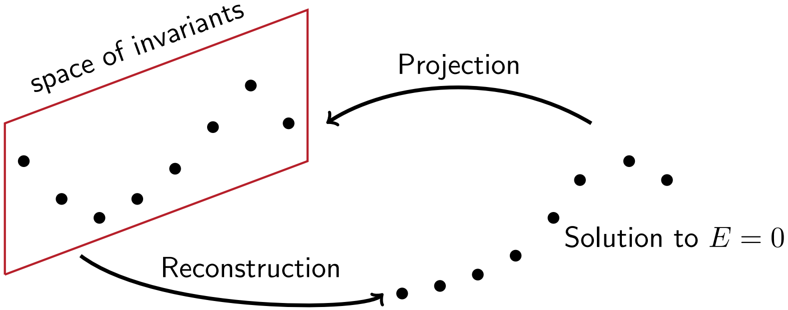
$$X_n = x_n + \epsilon \qquad \frac{Y_n^{1-b}}{b-1} = \frac{y_n^{1-b}}{b-1} + \frac{x_n^{a+1}}{a+1} - \frac{(x_n+\epsilon)^{a+1}}{a+1}$$

Group Foliation – Outline

Goal: Solve finite difference equations that admit a group of symmetry

Outline of the solution:

1. Project the (unknown) solutions into the space of invariants
2. Solve the problem in the space of invariants
   - Typically easier to solve than the original equation
3. Reconstruct the solution to the original equation



EQUIVARIANT MOVING FRAMES!

Equivariant Moving Frames

Let G act on $\mathrm{J}^{[k]}$

Defintion: A moving frame is a G-equivariant map

$$\rho\colon \mathrm{J}^{[k]} \to G$$

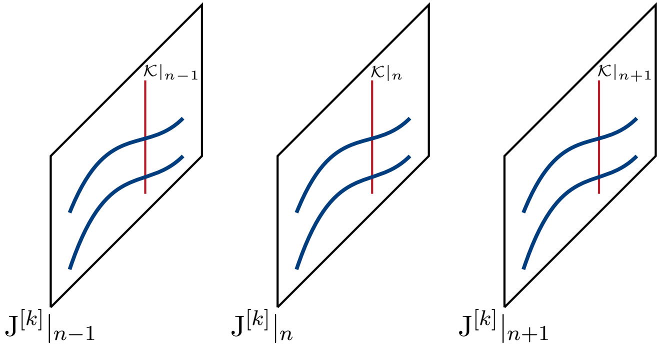
G-equivariance means

$$\rho_n(g \cdot z_n^{[k]}) = \rho_n(z_n^{[k]})\, g^{-1}$$

A moving frame is constructed by choosing a (discrete) cross-section.

Discrete Cross-Section

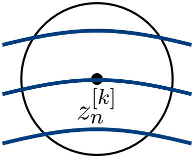
Definition: A subset $\mathcal{K} \subset \mathrm{J}^{[k]}$ is a cross-section if the restriction $\mathcal{K}|_n$ is a submanifold of $\mathrm{J}^{[k]}|_n$, which is transverse and of complementary dimension to the group orbits.



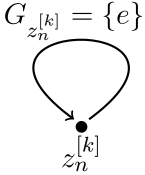
Moving Frame Construction

Provided the action is

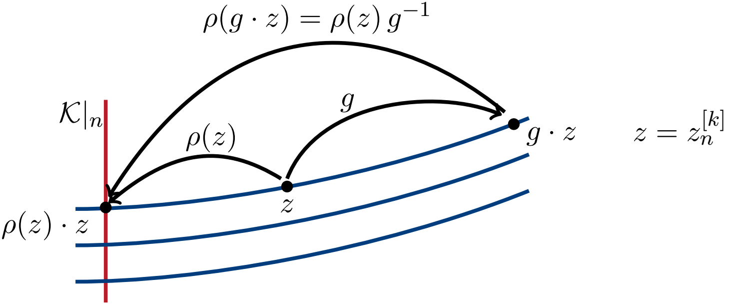
1) regular



2) free



the moving frame at $z_n^{[k]}$ is the unique group element $\rho_n(z_n^{[k]})$ sending $z_n^{[k]}$ onto the cross-section $\mathcal{K}$



Moving Frame: Example

- Product action

$$X_n = x_n + \epsilon, \qquad \frac{Y_n^{1-b}}{b-1} = \frac{y_n^{1-b}}{b-1} + \frac{x_n^{a+1}}{a+1} - \frac{(x_n+\epsilon)^{a+1}}{a+1}$$

- Choose a cross-section

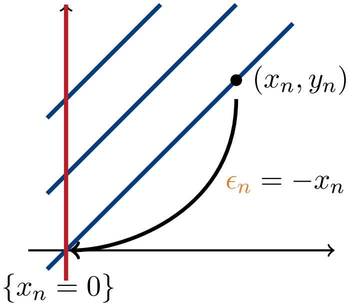
$$\mathcal{K} = \{x_n = 0\}$$

- Solve the normalization equation(s)

$$0 = X_n = x_n + \epsilon \qquad \Rightarrow \qquad \rho_n : \quad \epsilon_n = -x_n$$

$a = b = 0$:

$$X_n = x_n + \epsilon$$
$$Y_n = y_n + \epsilon$$



Invariantization

Definition: The invariantization of z_m w.r.t. $\rho_n = \rho_n(z_n^{[k]})$ is the invariant

$$\iota_n(z_m) = \rho_n \cdot z_m$$

Proof: $g \cdot \iota_n(z_m) = \rho_n(g \cdot z_n^{[k]}) \cdot g \cdot z_m = \rho_n(z_n^{[k]}) \cdot g^{-1} \cdot g \cdot z_m = \iota_n(z_m)$

Example: If

$$X_n = x_n + \epsilon, \qquad \frac{Y_n^{1-b}}{b-1} = \frac{y_n^{1-b}}{b-1} + \frac{x_n^{a+1}}{a+1} - \frac{(x_n + \epsilon)^{a+1}}{a+1}$$

then

$$\iota_n(x_{n+1}) = x_{n+1} + \epsilon_n \Big|_{\epsilon_n = -x_n} = x_{n+1} - x_n$$

$$J_n = \iota_n\left(\frac{y_n^{1-b}}{b-1}\right) = \frac{y_n^{1-b}}{b-1} + \frac{x_n^{a+1}}{a+1}$$

Notation: We introduce the normalized (joint) invariants

$$\mathbf{I}_n = \iota_n(z_n) \qquad \text{and} \qquad \mathbf{I}_n^{[k]} = \iota_n(z_n^{[k]})$$

Generating Invariants

Let

$$\mathbf{S}\colon \mathbb{Z} \to \mathbb{Z} \qquad\qquad \mathbf{S}(n) = n + 1$$

denote the forward shift operator

Definition: A set of invariants $\mathbf{I}_{\text{gen}}$ generates the algebra of joint invariants if any invariant I can be expressed as a function of the invariants in $\mathbf{I}_{\text{gen}}$ and their shifts.

Definition: Let ρ_n be moving frame. The Maurer–Cartan invariant(s) is (are)

$$\mathfrak{m}_n = \rho_n \, \rho_{n+1}^{-1} \;\in\; G$$

The invariance of $\mathfrak{m}_n$ follows from the equivariance of ρ_n:

$$\begin{aligned} \mathfrak{m}_n(g \cdot z_n^{[k]}) &= \rho_n(g \cdot z_n^{[k]}) \, \rho_{n+1}^{-1}(g \cdot z_{n+1}^{[k]}) \\ &= \rho_n(z_n^{[k]}) \, g^{-1} \, g \, \rho_{n+1}^{-1}(z_{n+1}^{[k]}) = \mathfrak{m}_n(z_n^{[k]}) \end{aligned}$$

Generating Invariants

Proposition: The order zero normalized invariants

$$\mathbf{I}_n = \iota_n(z_n)$$

together with the Maurer–Cartan invariants

$$\mathfrak{m}_n = \rho_n \, \rho_{n+1}^{-1}$$

generate the algebra of joint invariants.

To prove this statement, we introduce the recurrence relations that relate normalized invariants and their shifts.

Recurrence Relations

Proposition: The invariants

$$\iota_n(z_m) \qquad \iota_{n+1}(z_m)$$

are related by the recurrence relation

$$\iota_n(z_m) = \mathfrak{m}_n \cdot \iota_{n+1}(z_m),$$

Proof: $\iota_n(z_m) = \rho_n \cdot z_m = \rho_n \cdot \rho_{n+1}^{-1} \cdot \rho_{n+1} \cdot z_m = \mathfrak{m}_n \cdot \iota_{n+1}(z_m)$

In general,

$$\iota_n(z_m) = \mathfrak{m}_n \cdot \mathfrak{m}_{n+1} \cdots \mathfrak{m}_{n+k-1} \cdot \iota_{n+k}(z_m)$$

Letting $m = n + k$ yields

$$\iota_n(z_{n+k}) = \mathfrak{m}_n \cdot \mathfrak{m}_{n+1} \cdots \mathfrak{m}_{n+k-1} \cdot \mathbf{S}^k(\mathbf{I}_n) \qquad \mathbf{I}_n = \iota_n(z_n)$$

Generating Invariants

From

$$\iota_n(z_{n+k}) = \mathfrak{m}_n \cdot \mathfrak{m}_{n+1} \cdots \mathfrak{m}_{n+k-1} \cdot \mathbf{S}^k(\mathbf{I}_n)$$

⇒ The normalized invariants $\iota_n(z_{n+k})$ are expressible in terms of

$$\mathbf{I}_n = \iota_n(z_n), \qquad \mathfrak{m}_n, \tag{1}$$

and their shifts.

Let $\mathcal{I}_n(z_n^{[k]})$ be an invariant function. Since

$$\mathcal{I}_n = \iota_n(\mathcal{I}_n)$$

we have that

$$\mathcal{I}_n(z_n^{[k]}) = \mathcal{I}_n(\iota_n(z_n^{[k]})) = \mathcal{I}_n(\mathbf{I}_n^{[k]})$$

⇒ Any invariant can be expressed in terms of (1) and there shifts.

Example

For

$$\rho_n = \epsilon_n = -x_n$$

where have

$$\mathfrak{m}_n = \rho_n \, \rho_{n+1}^{-1} = \epsilon_n - \epsilon_{n+1} = -x_n + x_{n+1}$$

and

$$\iota_n(x_n) = 0 \qquad J_n = \iota_n\left(\frac{y_n^{1-b}}{b-1}\right) = \frac{y_n^{1-b}}{b-1} + \frac{x_n^{a+1}}{a+1}$$

$\Rightarrow$ $\mathfrak{m}_n$ and J_n generate the algebra of joint invariants:

$$\iota_n(x_{n+1}) = \mathfrak{m}_n \cdot \mathbf{S}[\iota_n(x_n)] = \mathfrak{m}_n \cdot 0 = \mathfrak{m}_n$$

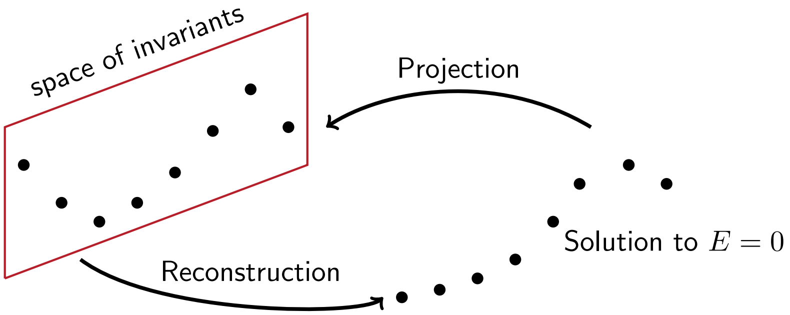
$$\iota_n\left(\frac{y_{n+1}^{1-b}}{b-1}\right) = \mathfrak{m}_n \cdot \mathbf{S}\left[\iota_n\left(\frac{y_n^{1-b}}{b-1}\right)\right] = \mathfrak{m}_n \cdot J_{n+1} = J_{n+1} - \frac{\mathfrak{m}_n^{a+1}}{a+1}$$

and so on.

Group Foliation

Solution steps:

1. Project the (unknown) solutions into the space of invariants
2. Solve the problem in the space of invariants
3. Reconstruct the solution to the original equation



Step 1: Projection

Let $E_n(z_n^{[k]}) = 0$ be a system of finite difference equations with symmetry group G.

- Construct a moving frame
- Invariantize the equations

$$E_n(\iota_n(z_n^{[k]})) = E_n(\mathbf{I}_n^{[k]}) = 0$$

- Use the recurrence relations to express $\mathbf{I}_n^{[k]}$ in terms of $\mathbf{I}_n$, $\mathfrak{m}_n$ and their shifts

$$\widetilde{E}_n(\mathbf{I}_n, \mathfrak{m}_n, \ldots \mathbf{I}_{n+k}, \mathfrak{m}_{n+k}) = 0 \qquad \text{(resolving system)}$$

Example:. In terms of $\mathfrak{m}_n$ and $J_n = \iota_n(y_n)$ the equations

$$\left(\frac{y_{n+1}^{1-b}}{b-1}+\frac{x_{n+1}^{a+1}}{a+1}\right)-\left(\frac{y_n^{1-b}}{b-1}+\frac{x_n^{a+1}}{a+1}\right)+k(x_{n+1}-x_n)=0 \qquad x_{n+1}-x_n=h$$

are

$$J_{n+1} - J_n + k\,\mathfrak{m}_n = 0 \qquad \mathfrak{m}_n = h$$

Step 2: Solve the Resolving System

Solving

$$\widetilde{E}_n(\mathbf{I}_n, \mathfrak{m}_n, \ldots \mathbf{I}_{n+k}, \mathfrak{m}_{n+k}) = 0$$

we obtain

$$\mathbf{I}_n = \mathbf{I}(n) \qquad \mathfrak{m}_n = \mathfrak{m}(n)$$

Example:. The solution to

$$J_{n+1} - J_n + k\,H_n = 0 \qquad\qquad \mathfrak{m}_n = h$$

is

$$J_n = J_0 - (k\,h)n \qquad\qquad \mathfrak{m}_n = h$$

Step 3: Reconstruction

Solution in the space of invariants $\rightsquigarrow$ original solution

Definition: Let

$$\overline{\rho}_n = \rho_n^{-1}$$

The reconstruction equation is

$$\overline{\rho}_{n+1} = \overline{\rho}_n \, \mathfrak{m}_n$$

Since

$$\mathbf{I}_n = \iota_n(z_n) = \rho_n \cdot z_n$$

the solution to $E_n(z_n^{[k]}) = 0$ is

$$z_n = \rho_n^{-1} \cdot \mathbf{I}_n = \overline{\rho}_n \cdot \mathbf{I}_n$$

Example

Let $\overline{\rho}_n = \overline{\epsilon}_n$. Since $\mathfrak{m}_n = h$, the reconstruction equation is

$$\overline{\rho}_{n+1} = \overline{\rho}_n \, \mathfrak{m}_n \qquad \Rightarrow \qquad \overline{\epsilon}_{n+1} = \overline{\epsilon}_n + \mathfrak{m}_n = \overline{\epsilon}_n + h$$

so that $\overline{\epsilon}_n = h\,n + \overline{\epsilon}_0$. Since

$$\iota_n(x_n) = 0 \qquad \iota_n(y_n) = [(b-1)J_n]^{1/(1-b)}$$

we have

$$x_n = \overline{\rho}_n \cdot 0 = h\,n + \overline{\epsilon}_0$$
$$y_n = \overline{\rho}_n \cdot [(b-1)J_n]^{1/(1-b)} = (1-b)^{1/(1-b)} \left[k\,x_n + \frac{x_n^{1+a}}{1+a} + C \right]^{1/(1-b)}$$

where $C = -J_0 - k\,\epsilon_0$.

Concluding Remarks

- Computations can be done symbolically:
  - Does not require the coordinate expressions for

$$\rho_n \qquad \mathbf{I}_n \qquad \mathfrak{m}_n$$

  - Requires expressions for the group action, the choice of a cross-section, and the recurrence relations
- Ideas developed in this talk can be adapted to differential equations
- Results appear in

(With Thompson, R.). Group foliation of finite difference equations, *Commun. Nonlinear Sci. Numer. Simul.* **59** (2018), 235–254.

Thank you!