Differential Chow Form and Differential Resultant

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Motivations

- **Differential Chow Form**
  Algebraic chow form is a powerful tool in constructive algebraic geometry with fruitful applications in elimination theory, transcendental number theory, complexity analysis, etc.

  Differential Chow form is not studied before.

- **Differential Resultant**
  Resultant as a powerful tool in elimination theory is not fully explored in differential case.

- **Differential dimension conjecture**
  Proposed by Ritt (1950), still open now.

Outline of the Talk

- Intersection Theory for Generic Diff Polynomials
- Chow Form for an Irreducible Differential Variety
- Differential Chow Variety
- Generalized Differential Chow Form and Differential Resultant
- Summary

Main Tools:

- Differential Characteristic Set
- Differential Specialization
- Algebraic Chow Form

Intersection Theory for Generic Differential Polynomials

Notations

Ordinary differential field: $(\mathcal{F},\delta)$ with $\mathrm{char}(\mathcal{F})=0$. e.g. $(\mathbf{Q}(x),\frac{d}{dx})$

Universal differential field of $\mathcal{F}$: $(\mathcal{E},\delta)$.

Diff Indeterminates: $\mathbb{Y}=\{y_1,\ldots,y_n\}$. **Notation:** $y_i^{(k)}=\delta^k y_i$.

Differential polynomial ring: $\mathcal{F}\{\mathbb{Y}\}=\mathcal{F}[(y_i^{(k)})_{k\geq 0}]$

Ranking $\mathscr{R}$: a total order over $(y_i^{(k)})_{k\geq 0}$ satisfying
1) $\delta u > u$ and 2) $u > v \Longrightarrow \delta u > \delta v$.

Notations

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Ranking $\mathscr{R}$**:** a total order over $(y_i^{(k)})_{k\geq 0}$ satisfying
1) $\delta u > u$ and 2) $u > v \Longrightarrow \delta u > \delta v$.

Examples:

1) **Elimination ranking**:

$$y_i > y_j \Rightarrow \delta^k y_i > \delta^l y_j \text{ for any } k, l \geq 0$$

2) **Orderly ranking**:

$$k > l \Rightarrow \delta^k y_i > \delta^l y_j \text{ for any } 1 \leq i, j \leq n.$$

Characteristic set

Given $f \in \mathcal{F}\{\mathbb{Y}\}$ with fixed ranking $\mathscr{R}$:

Leader : u_f, the greatest derivative appearing in f.

$f = I_d u_f^d + I_{d-1} u_f^{d-1} + \ldots + I_0$.

- **Initial:** $\mathbf{I}_f = I_d$
- **Separant:** $\mathrm{S}_f = \frac{\partial f}{\partial u_f}$
- $\mathbf{ord}(f, y_i) = \max\{k : \mathbf{deg}(f, y_i^{(k)}) \geq 0\}$.
  Order of f: $\mathbf{ord}(f) = \max_i \mathbf{ord}(f, y_i)$

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Autoreduced set: $\mathcal{A} = A_1, \ldots, A_t$ with $\mathrm{ld}(A_i) = y_{c_i}^{(o_i)}$.

- **Parametric set of $\mathcal{A}$:** $\mathrm{Pm}(\mathcal{A}) = \mathbb{Y} \setminus \{y_{c_1}, \ldots, y_{c_p}\}$.
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$\mathbb{DP}$: A diff polynomial set

Characteristic set (CS): a smallest autoreduced set in $\mathbb{DP}$.

Dimension and order of a prime differential ideal

$\mathcal{I}$: a prime differential ideal in $\mathcal{F}\{\mathbb{Y}\}$.
$\mathcal{A}$: a CS of $\mathcal{I}$ w.r.t. any orderly ranking.
$\mathcal{B}$: a CS of $\mathcal{I}$ w.r.t. some elimination ranking and $U = \mathrm{Pm}(\mathcal{B})$.

Facts:

Dimension of $\mathcal{I}$: $\mathbf{dim}(\mathcal{I}) = n - |\mathcal{A}|$.
Order of $\mathcal{I}$: $\mathbf{ord}(\mathcal{I}) = \mathbf{ord}(\mathcal{A})$.
Relative order of $\mathcal{I}$ w.r.t. U: $\mathbf{ord}_U(\mathcal{I}) = \mathbf{ord}(\mathcal{B})$.

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Theorem (Relation between order and relative order)

$\mathbf{ord}(\mathcal{I})$ *is the maximum of all the relative orders of* $\mathcal{I}$, *that is,* $\mathbf{ord}(\mathcal{I}) = \max_{\mathbb{U}} \mathbf{ord}_{\mathbb{U}}(\mathcal{I})$, *where* $\mathbb{U}$ *is a parametric set of* $\mathcal{I}$.

A Property on Differential Specialization

Theorem (Differential Dependence under Diff Specialization)

$P_i(\mathbb{U}, \mathbb{Y}) \in \mathcal{F}\{\mathbb{U}, \mathbb{Y}\}$: diff polynomials in $\mathbb{U}$ and $\mathbb{Y} = (y_1, \ldots, y_n)$.

$\overline{\mathbb{Y}} = (\bar{y}_1, \ldots, \bar{y}_n)$: $\bar{y}_i \in \mathcal{E}$ free from $\mathbb{U}$.

If $P_i(\mathbb{U}, \overline{\mathbb{Y}})$ are diff dependent over $\mathcal{F}\langle\mathbb{U}\rangle$, then for any specialization $\mathbb{U}$ to $\overline{\mathbb{U}} \subset \mathcal{F}$ over $\mathcal{F}$, $P_i(\overline{\mathbb{U}}, \overline{\mathbb{Y}})$ are diff dependent over $\mathcal{F}$.

Remark. This is a key tool of our theory.

The algebraic analog is also a basic tool in algebraic elimination theory. But, the differential case needs an essentially new proof.

Dimension of the Intersection Variety

Generic diff poly: Complete diff polynomial with given order, degree, and coeffs diff independent over $\mathcal{F}$.

Quasi-generic diff poly: Sparse generic diff polynomial contains **degree zero term** and a term in $\mathcal{F}\{y_i\}\backslash\mathcal{F}$ for each i.

$\mathcal{I}$: a prime diff ideal in $\mathcal{F}\{\mathbb{Y}\}$ with dimension d.

In algebraic case, $\mathbf{dim}((\mathcal{I}, f)) \geq d - 1$ always holds.

In differential case, $\mathbf{dim}([\mathcal{I}, f]) < d - 1$ may happen (Ritt 1950).

Theorem

f: a (quasi-)generic diff polynomial with $\mathbf{deg}(f) > 0$ and coefficients $\mathbf{u}_f$.

If $d > 0$, then $\mathcal{I}_1 = [\mathcal{I}, f]$ is a prime diff ideal in $\mathcal{F}\langle\mathbf{u}_f\rangle\{\mathbb{Y}\}$ with dimension $d - 1$.

And if $d = 0$, then $\mathcal{I}_1$ is the unit ideal $\mathcal{F}\langle\mathbf{u}_f\rangle\{\mathbb{Y}\}$.

Order of the Intersection Variety

$\mathcal{I} \subset \mathcal{F}\{\mathbb{Y}\}$: a prime diff ideal with dimension $d > 0$ and order h.

Theorem

f: a generic diff polynomial of order s with $\mathbf{u}_f$ the set of its coefficients.

Then $\mathcal{I}_1 = [\mathcal{I}, f]$ is a prime diff ideal in $\mathcal{F}\langle \mathbf{u}_f \rangle\{\mathbb{Y}\}$ with dimension $d - 1$ and order $h + s$.

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Corollary (Basis for differential Chow form)

The intersection of V with a generic hyperplane is of dimension $\mathbf{dim}(V) - 1$ and order $\mathbf{ord}(V)$.

$\mathcal{I}_1 = [\mathcal{I}, u_0 + u_1 y_1 + \ldots + u_n y_n]$ is a prime diff ideal in $\mathcal{F}\langle u_0, u_1, \ldots, u_n \rangle\{\mathbb{Y}\}$ with dimension $d - 1$ and order h.

Dimension Conjecture in Generic Case

Dimension conjecture: $f_1, \ldots, f_r$ diff polynomial with $r \leq n$, is every component of $\mathbb{V}(f_1, \ldots, f_r)$ of dimension at least $n - r$?

Theorem (Generic Dimension Theorem)

$f_1, \ldots, f_r$ ($r \leq n$): quasi-generic diff polynomials in $\mathbb{Y}$ with $\mathbf{u}_{f_i}$ coefficient set of f_i.

Then $[f_1, \ldots, f_r]$ is a prime diff ideal over $\mathcal{F}\langle \mathbf{u}_{f_1}, \ldots, \mathbf{u}_{f_r} \rangle$ with dimension $n - r$.

Together with order considered, results above can be strengthened:

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Together with order considered, results above can be strengthened:

Theorem

$F_1, \ldots, F_r$ $(r \leq n)$: generic diff polynomials with $\mathbf{ord}(F_i) = s_i$.

Then $\mathbb{V}(F_1, \ldots, F_r)$ is an irreducible variety over $\mathcal{F}\langle \mathbf{u}_{F_1}, \ldots, \mathbf{u}_{F_r} \rangle$ with dimension $n - r$ and order $\sum_{i=1}^{r} s_i$.

Differential Chow Form

Definition of Differential Chow Form

$\mathcal{I} \subset \mathcal{F}\{\mathbb{Y}\}$: prime differential ideal of dimension d.

$d+1$ **Generic Differential Hyperplanes**:

$\mathbb{P}_i = u_{i0} + u_{i1}y_1 + \cdots + u_{in}y_n \, (i = 0, \ldots, d)$.

$\mathbf{u}_i = (u_{i0}, \ldots, u_{in})$: coefficient set of $\mathbb{P}_i$

Theorem

By intersecting $\mathcal{I}$ with the $d+1$ hyperplanes,

$$[\mathcal{I}, \mathbb{P}_0, \ldots, \mathbb{P}_d] \cap \mathcal{F}\{\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d\} = \mathbf{sat}(F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d))$$

is a prime ideal of **co-dimension one**.

Call $F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)$ **Differential Chow form** of $\mathcal{I}$ or $\mathbb{V}(\mathcal{I})$

Example

V: an irreducible diff variety of dimension $n-1$ and $\mathbf{I}(V)=\mathbf{sat}(p)\subset\mathcal{F}\{\mathbb{Y}\}$.

Then its differential Chow form is $F(\mathbf{u}_0,\mathbf{u}_1,\ldots,\mathbf{u}_{n-1})=D^m p(\frac{D_1}{D},\ldots,\frac{D_n}{D})$, where

$$D=\begin{vmatrix} u_{01} & u_{02} & \cdots & u_{0n} \\ u_{11} & u_{12} & \cdots & u_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ u_{n-1,1} & u_{n-1,2} & \cdots & u_{n-1,n} \end{vmatrix}$$

and $D_i(i=1,\ldots,n)$ is the determinant of the matrix formed by replacing the i-th column of D by the column vector $(-u_{00},-u_{10},\ldots,-u_{n-1,0})^T$, and m is the minimal integer such that $D^m p(\frac{D_1}{D},\ldots,\frac{D_n}{D})\in\mathcal{F}\{\mathbf{u}_0,\ldots,\mathbf{u}_d\}$.

Order of Differential Chow Form

Chow form of $\mathcal{I}$: $F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)$

Symmetric Property of Chow form:

- $F(\ldots, \mathbf{u}_\sigma, \ldots, \mathbf{u}_\rho, \ldots) = (-1)^{r_{\sigma\rho}} F(\ldots, \mathbf{u}_\rho, \ldots, \mathbf{u}_\sigma, \ldots)$.
- $\mathbf{ord}(F, u_{00}) \neq 0$, $\mathbf{ord}(F, u_{00}) = \mathbf{ord}(F, u_{ij})$ if u_{ij} occurs in F

Order of Chow form: $\mathbf{ord}(F) = \mathbf{ord}(F, u_{00})$.

Theorem (Order of Chow Form)

$\mathbf{ord}(F) = \mathbf{ord}(\mathcal{I})$.

Definition (An equivalent definition for the order of $\mathcal{I}$)

Order of a prime diff ideal is defined to be the order of its Chow form.

Degree of Differential Chow Form

Differentially homogenous diff polynomial of degree m:

$$p(ty_0, ty_1 \ldots, ty_n) = t^m p(y_0, y_1, \ldots, y_n)$$

Theorem

$F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)$ is differentially homogenous of the same degree r in each set $\mathbf{u}_i$.

Definition (Differential degree)

r as above is defined to be the **differential degree** of $\mathcal{I}$, which is an invariant of $\mathcal{I}$ under invertible linear transformations.

Factorization of Differential Chow Form

$\mathcal{I}$: a prime diff ideal over $\mathcal{F}$ of dimension d and order h.
$F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)$: the differential Chow form of $\mathcal{I}$.

Theorem (Possion Type Product Formula)

$$\begin{aligned} F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d) &= A(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d) \prod_{\tau=1}^{g} \Big(u_{00} + \sum_{\rho=1}^{\mathbf{n}} \mathbf{u}_{\mathbf{0}\rho} \xi_{\tau\rho}\Big)^{(h)} \\ &= A(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d) \prod_{\tau=1}^{g} \mathbb{P}_0(\xi_{\tau 1}, \ldots, \xi_{\tau n})^{(h)} \end{aligned}$$

where $g = \mathbf{deg}(F, u_{00}^{(h)})$ *and* $\xi_{\tau\rho}$ *are in an extension field free from* $u_{0i}^{(h)}$.
$(\xi_{\tau 1}, \ldots, \xi_{\tau n})$ $(\tau = 1, \ldots, g)$ *are* **generic points** *of* $\mathcal{I}$.

Leading Differential Degree

Differential hyperplanes:

$$\mathbb{P}_i := u_{i0} + u_{i1} y_1 + \cdots + u_{in} y_n \, (i = 1, \ldots, d),$$

Algebraic hyperplanes:

$${}^a\mathbb{P}_0 := u_{00} + u_{01} y_1 + \cdots + u_{0n} y_n,$$

$${}^a\mathbb{P}_0^{(s)} := u_{00}^{(s)} + \sum_{j=1}^{n} \sum_{k=0}^{s} \binom{s}{k} u_{0j}^{(k)} y_j^{(s-k)} \, (s = 1, 2, \ldots)$$

Theorem

$(\xi_{\tau 1}, \ldots, \xi_{\tau n})\,(\tau = 1, \ldots, g)$ are the **only zeros** *of $\mathcal{I}$ which lie on $\mathbb{P}_1, \ldots, \mathbb{P}_d$ as well as on ${}^a\mathbb{P}_0, {}^a\mathbb{P}_0', \ldots, {}^a\mathbb{P}_0^{(h-1)}$.*

Definition

Number g is defined to be the **leading diff degree** *of $\mathcal{I}$.*

Example (2)

V: the general component of $(y')^2 - 4y$ in $\mathbf{Q}(t)\{y\}$.

$\mathbf{dim}(V) = 0$ and the diff Chow form of V is
$F(\mathbf{u}_0) = u_1^2(u_0')^2 - 2u_1u_1'u_0u_0' + (u_1')^2u_0^2 + 4u_1^3u_0$, where $\mathbf{u}_0 = (u_0, u_1)$.

Then
$F(\mathbf{u}_0) = u_1^2(u_0' + \xi_{11}u_1' + 2\sqrt{-1}\sqrt{u_0u_1})(u_0' + \xi_{21}u_1' - 2\sqrt{-1}\sqrt{u_0u_1}) = u_1^2(u_0 + \xi_{11}u_1)'(u_0 + \xi_{21}u_1)'$ where
$\xi_{i1} = -u_0/u_1|_{u_0' = \frac{u_0}{u_1}u_1' \mp 2\sqrt{-1}\sqrt{u_0u_1}}$ $(i = 1, 2)$.

Note that both ξ_{i1} $(i = 1, 2)$ satisfy ${}^a\mathbb{P}_0 = u_0 + u_1\xi_{i1} = 0$, but ${}^a\mathbb{P}_{00}^{(1)} = (u_0 + u_1\xi_{i1})' \neq 0$. And from the Chow form, we have $\xi_{i1}'^2 + 4u_0/u_1 = 0$.

Recover the Variety from Chow Form

- $\mathcal{I} \in \mathcal{F}\{\mathbb{Y}\}$: prime diff ideal over $\mathcal{F}$ of dimension d.
- $F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)$: diff Chow form of $\mathcal{I}$.
- $\zeta = (\zeta_0, \ldots, \zeta_d)$: generic point of $\mathbf{sat}(F) \subset \mathcal{F}\langle \mathbf{u} \rangle\{u_{00}, \ldots, u_{d0}\}$, where $\mathbf{u} = \cup_i \mathbf{u}_i \backslash \{u_{i0}\}$.

Theorem (Recover Generic Point from Chow Form)

$$\xi_\rho = \frac{\partial F}{\partial u_{0\rho}^{(h)}} \Big/ \frac{\partial F}{\partial u_{00}^{(h)}} \Big|_{(u_{00}, \ldots, u_{d0}) = \zeta}, \ (\rho = 1, \ldots, n)$$

Then $(\xi_1, \ldots, \xi_n)$ is a generic point of $\mathcal{I}$.

Relations between a Variety and Diff Chow Form

$F(\mathbf{u}_0, \ldots, \mathbf{u}_d)$: differential Chow form of V and $S_F = \frac{\partial F}{\partial u_{00}^{(h)}}$.

When u_{ij} specialize to $v_{ij} \in \mathcal{E}(\mathcal{F})$ over $\mathcal{F}$, $\mathbb{P}_i$ specialize to $\overline{\mathbb{P}}_i$.

Theorem

If $\overline{\mathbb{P}}_i = 0\,(i = 0, \ldots, d)$ meet V, then $F(\mathbf{v}_0, \ldots, \mathbf{v}_d) = 0$.

Furthermore, if $F(\mathbf{v}_0, \ldots, \mathbf{v}_d) = 0$ and $S_F(\mathbf{v}_0, \ldots, \mathbf{v}_d) \neq 0$, then the $d+1$ primes $\overline{\mathbb{P}}_i = 0$ $(i = 0, \ldots, d)$ meet V.

Recover the Generators from the Chow Form

Let $S = (s_{ij})$ be an $(n+1) \times (n+1)$ skew-symmetric matrix with $s_{ij}(i < j) \in \mathcal{E}$ diff independent over $\mathcal{F}$.

A generic prime passing through a point x is of the form Sx.

Theorem

$\mathcal{I}$: a diff prime ideal of dimension d and $F(\mathbf{u}_0, \ldots, \mathbf{u}_d)$ its Chow form.

Then $V \setminus \mathbb{V}(\mathcal{D}) = \mathbb{V}(\mathcal{P}) \setminus \mathbb{V}(\mathcal{D}) \neq \emptyset$, where $\mathcal{P}, \mathcal{D}$ are the diff polynomial sets obtained from $F(\mathbf{u}_0, \ldots, \mathbf{u}_d)$ and $S_F(\mathbf{u}_0, \ldots, \mathbf{u}_d)$ by substituting $\mathbf{u}_l$ with $S^l y$ and collecting coeffs in s_{ij}^l.

Differential Chow Variety

Differential Chow variety

$\mathcal{I}$ is said to be **of index** (n, d, h, g, m) if V is in $\mathcal{E}^n$, with dim d, order h, leading diff degree g, and diff degree m.

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$F(\mathbf{u}_0, \ldots, \mathbf{u}_d)$: a diff polynomial with index (n, d, h, g, m) and coefficients a_i.

Definition

A quasi-variety $\mathbb{CV}$ in a_i is the **Chow Variety** with index (n, d, h, g, m) if the following holds:

$(\bar{a}_i) \in \mathbb{CV}$

$\Leftrightarrow$ $\bar{F}$: Chow form with index (n, d, h, g, m_1) with $m_1 \leq m$.

$\Leftrightarrow$ $\mathcal{I}$: order-unmixed var of index (n, d, h, g, m_1) with $m_1 \leq m$.

Differential Chow variety

$\mathcal{I}$ is said to be **of index** (n, d, h, g, m) if V is in $\mathcal{E}^n$, with dim d, order h, leading diff degree g, and diff degree m.

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$\Leftrightarrow \mathcal{I}$: order-unmixed var of index (n, d, h, g, m_1) with $m_1 \leq m$.

Theorem

In the case $g = 1$, the differential Chow variety exists.

Sufficient condition for a polynomial to be a Chow form

Properties of Diff Chow Form:

1. $F(\mathbf{u}_0, \ldots, \mathbf{u}_d)$ is differentially homogenous of the same degree in each $\mathbf{u}_i$ and $\mathbf{ord}(F, u_{ij}) = h$ for all u_{ij} occurring in F.

2. $F(\mathbf{u}_0, \ldots, \mathbf{u}_d)$ can be factored uniquely into the following form

$F(\mathbf{u}_0, \ldots, \mathbf{u}_d) = A \prod_{\tau=1}^{g} (u_{00} + \sum_{\rho=1}^{n} u_{0\rho}\eta_{\tau\rho})^{(h)}$.

3. $\eta_\tau = (\eta_{\tau 1}, \ldots, \eta_{\tau n})\,(\tau = 1, \ldots, g)$ are on the differential hyperplanes $\mathbb{P}_\sigma = 0\,(\sigma = 1, \ldots, d)$ as well as on the algebraic hyperplanes ${}^a\mathbb{P}_0^{(k)} = 0\,(k = 0, \ldots, h-1)$.

4. For each τ, if $v_{i0} + v_{i1}\eta_{\tau 1} + \cdots + v_{in}\eta_{\tau n} = 0\ (i = 0, \ldots, d)$, then $F(\mathbf{v}_0, \ldots, \mathbf{v}_d) = 0$, where $\mathbf{v}_i = (v_{i0}, \ldots, v_{in})$ and $v_{ij} \in \mathcal{E}(\mathcal{F})$.

If F satisfies the above conditions, then it is the Chow form for a diff ideal of dimension d and order h.

Generalized Differential Chow Form and Differential Resultant

Generalized differential Chow form

$\mathcal{I}$: a prime differential ideal over $\mathcal{F}$ of dimension d and order h.

Generic differential polynomials:

$$\mathbb{P}_i = \mathbf{u_{i0}} + \sum_{j=1}^{n}\sum_{k=0}^{s_i} u_{ijk} y_j^{(k)} + \sum_{\substack{\alpha \in \mathbb{Z}_{\geq 0}^{n(s_i+1)} \\ 1 < |\alpha| \leq m_i}} u_{i\alpha}(\mathbb{Y}^{(s_i)})^{\alpha},\ (i = 0, \ldots, d)$$

where $(\mathbb{Y}^{(s_i)})^{\alpha} = \prod_{j=1}^{n}\prod_{k=0}^{s_i}(y_j^{(k)})^{\alpha_{jk}}$ and $|\alpha| = \sum_{j=1}^{n}\sum_{k=0}^{s_i}\alpha_{jk}$

Clearly, $\mathbf{ord}(\mathbb{P}_i) = s_i$ and $\mathbf{deg}(\mathbb{P}_i) = m_i$.

Definition

As a consequence of our intersection theory,

$$[\mathcal{I}, \mathbb{P}_0, \ldots, \mathbb{P}_d] \cap \mathcal{F}\{\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d\} = \mathbf{sat}(G(\mathbf{u}_0, \ldots, \mathbf{u}_d))$$

is a prime ideal of co-dimension one, where

$$\mathbf{u}_i = (u_{i0}, \ldots, u_{ijk}, \ldots, u_{i\alpha}, \ldots)\,(i = 0, \ldots, d)$$

We call $G(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d) = g(\mathbf{u}; u_{00}, \ldots, u_{d0})$ the **generalized differential Chow form** of $\mathcal{I}$ or $\mathbb{V}(\mathcal{I})$.

Generalized diff Chow form generalizes the result of Philippon to differential case, which has similar properties.

Differential Resultant

Existence Work on Differential Resultant:

- Ritt (1932) studied differential resultant for $n = 1$.
- Ferro (1997) defined differential resultant as algebraic Macaulay resultant.
  Not complete: Resultant of two generic diff polynomials of degree two is always zero.
- Rueda-Sendra (2010) differential resultant of linear system.

$\mathbb{P}_i\,(i = 0, \ldots, n)$: generic diff polynomials with $\mathbf{u}_i$ its coefficients.

Definition

The **differential resultant** of $\mathbb{P}_i$ is defined to be the generalized Chow form of [0], denoted by $\delta\mathrm{Res}(\mathbb{P}_0, \ldots, \mathbb{P}_n) = R(\mathbf{u}_0, \ldots, \mathbf{u}_n)$.

Properties of Differential Resultant

The differential resultant has the following properties:

a) $R(\mathbf{u}_0, \ldots, \mathbf{u}_n)$ is differentially homogeneous in each $\mathbf{u}_i$ and is of order $h_i = s - s_i$ in $\mathbf{u}_i$ $(i = 0, \ldots, n)$ where $s = \sum_{l=0}^{n} s_l$.

b) Algebraic case: $\mathrm{Res}(A(x), B(x)) = c \prod_{\xi, B(\xi)=0} A(\xi)$.

Differential case:
$R(\mathbf{u}_0, \ldots, \mathbf{u}_n) = A(\mathbf{u}_0, \ldots, \mathbf{u}_n) \prod_{\tau=1}^{t_0} \mathbb{P}_{\mathbf{0}}(\xi_{\tau \mathbf{1}}, \ldots, \xi_{\tau \mathbf{n}})^{(\mathbf{h_0})}$.

And $(\xi_{\tau 1}, \ldots, \xi_{\tau n})$ $(\tau = 1, \ldots, t_0)$ are generic points of the prime ideal $[\mathbb{P}_1, \ldots, \mathbb{P}_n]$.

c) Algebraic case: $\mathrm{Res}(A(x), B(x)) = A(x)T(x) + B(x)W(x)$, where $\mathbf{deg}(T) < \mathbf{deg}(B), \mathbf{deg}(W) < \mathbf{deg}(A)$.

Differential case: $R(\mathbf{u}_0, \ldots, \mathbf{u}_n)$ can be written as a linear combination of $\mathbb{P}_i$ and the derivatives of $\mathbb{P}_i$ up to the order $s - s_i$. Precisely, we have

$$R(\mathbf{u}_0, \ldots, \mathbf{u}_n) = \sum_{i=0}^{n} \sum_{j=0}^{s-s_i} h_{ij} \mathbb{P}_i^{(j)}.$$

And $\mathbf{deg}(h_{ij}\mathbb{P}_i^{(j)}) \leq (m+1)\mathbf{deg}(R)$ with $m = \max\{m_i\}$.

Conditions for Existence of Solutions

d) Algebraic case: $\text{Res}(A(x), B(x)) = 0 \Leftrightarrow A(x), B(x)$ have a common solution or leading coefficients of A and B vanish.

Differential case: When $\mathbb{P}_i$ specialize to $\overline{\mathbb{P}}_i$ with coefficients $\mathbf{v}_i$.

If $\overline{\mathbb{P}}_i = 0$ have a common solution, then $R(\mathbf{v}_0, \ldots, \mathbf{v}_n) = 0$.

And if $R(\mathbf{v}_0, \ldots, \mathbf{v}_n) = 0$ and $S_R(\mathbf{v}_0, \ldots, \mathbf{v}_n) \neq 0$, then $\overline{\mathbb{P}}_i = 0 (i = 0, \ldots, n)$ have a common solution, where $S_R = \frac{\partial R}{\partial u_{00}^{(h_0)}}$.

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  As a consequence, the differential dimension conjecture for generic differential polynomials is proved.

- Differential Chow form is defined and most properties of the algebraic Chow form are extended to its differential counterpart.

- The generalized diff Chow form is defined and its properties are proved.
  As an application, the differential resultant is defined and properties similar to that of the Sylvester resultant of two univariate polynomials are proved.

Thanks !

Reference.

- X. S. Gao, W. Li, C. M. Yuan. Intersection Theory in Differential Algebraic Geometry: Generic Intersections and the Differential Chow Form. Accepted by *Trans. of Amer. Math. Soc.*, 1-58. Also in arXiv:1009.0148v2.