An algorithm for solving second order linear homogeneous differential equations

Jerald J. Kovacic
Department of Mathematics
The City College of The City University of New York
New York, NY 10031
email: jkovacic@verizon.net
URL: mysite.verizon.net/jkovacic

September 23, 2005

Abstract

The Galois group tells us a lot about a linear homogeneous differential equation - specifically whether or not it has "closed-form" solutions. Using it, we have been able to develop an algorithm for finding "closed-form" solutions.

First we will compute the Galois group of some very simple equations. We then will solve a more complicated one, using the techniques of the algorithm. This example illustrates how the algorithm was discovered and the kinds of calculations used by it.

1 Introduction

Every student of calculus wants a formula to solve differential equations. Of course that is impossible, at least if we want "closed-form" solutions. The situation is analogous to that of polynomial equations. We'd like to have a formula for solutions in terms of radicals, and we know from the Galois theory that we can't.

In fact, the Galois group tells us a great deal about the kinds of solutions that an equation has. That is, after all, the point of the Galois theory. Therefore one wants to compute the Galois group of an equation.

That was my point of view when I started working on [6]. I wanted to find some criteria to determine the Galois group of a differential equation. I wanted them to be explicit, and easy - at that time pencil and paper was the accepted form of symbolic computation.

What came out, to my surprise, was an explicit algorithm to either find a "simple" solution or to prove that none exist.

There are no new ideas in the algorithm. It is simply brute-force calculation. And the hardest parts were worked out in the 19th century.

According to Felix Ulmer and Jacques-Arthur Weil [13], algorithms for finding rational solutions are in Liouville [7] (1833).

Ulmer and Weil also claim that algorithms for finding algebraic solutions are in Fuchs [2] (1878) and Pèpin [9] (1881). But Michael Singer [11] claims that these authors did not give a complete decision procedure, and that Baldassarri and Dwork [1] were the first to have done so.

An algorithm for finding Liouvillian solutions requires the Picard-Vessiot (Galois) theory, and so awaited the work of Kolchin in this century.

The original algorithm has some severe implementation difficulties. But recent work as eliminated most of the problems. Most of the recent work done to improve the algorithm is due to Abramov, Bronstein, Singer, Ulmer, and Weil.

There is a web version for 2^{nd} and 3^{rd} order equations:

`http://www-sop.inria.fr/cafe/Manuel.Bronstein/sumit/bernina_demo.html`

A very nice, and complete, description of the history of the algorithm and further developments is Michael Singer's survey article in [5, Direct and inverse problems in differential Galois theory, p. 527–554]. The bibliography has 168 items!

For expositions of Picard-Vessiot (differential Galois) theory see: Kaplansky [3], Kolchin [4], Magid [8], and van der Put-Singer [14].

2 The DE

We consider a second order linear homogeneous ordinary differential equation

$$z'' + az' + bz = 0$$

where $a, b \in \mathcal{F} = \mathbb{C}(x)$ (and $x' = 1$). There is a change of variables that "normalizes" the equation. Let

$$y = e^{\frac{1}{2}\int a} z$$

then

$$y'' = (b - \tfrac{1}{4}a^2 - \tfrac{1}{2}a')y\,.$$

If we can find y we can find z, at least up to the problem of integrating $\frac{1}{2}\int a$. But that is a "easier" problem and we consider it "solved". In fact, there is an algorithm, called the *Risch algorithm*, for integration.

Throughout the remainder of this talk, we consider only

$$y'' = ry$$

where $r \in \mathcal{F} = \mathbb{C}(x)$. We call this *the DE*.

There are several types of solutions that we are particularly interested in.

Definition 2.1. Let η be a solution of the DE.

1. η is *algebraic* if it the solution of a polynomial equation over $\mathcal{F}$
2. η is *primitive* if $\eta' \in \mathcal{F}$, that is $\eta = \int f$ for some $f \in \mathcal{F}$,

3. η is *exponential* if $\eta'/\eta \in \mathcal{F}$, that is $\eta = e^{\int f}$.

The third should really be called "exponential of a primitive".

Definition 2.2. A solution η of the differential equation is said to be *Liouvillian* if there is a tower of differential fields

$$\mathcal{F} = \mathcal{G}_0 \subset \mathcal{G}_1 \subset ... \subset \mathcal{G}_m = \mathcal{G},$$

with $\eta \in \mathcal{G}$ and for each $i = 1, ..., m$, $\mathcal{G}_i = \mathcal{G}_{i-1}(\eta_i)$ with η_i either algebraic, primitive, or exponential over $\mathcal{G}_{i-1}$.

A Liouvillian solution is built up by integration and exponentiation. So we get log's, the trig functions, but not things like the Bessel functions. These are "closed-form" solutions familiar to a first year calculus student.

This is a little more generous than "elementary" functions (logs and exp's only) in that we allow arbitrary indefinite integration.

Proposition 2.3. *If $y'' = ry$ has one Liouvillian solution, then every solution is Liouvillian.*

Use reduction of order: set y = ηz where η is a Liouvillian solution. One finds that $\eta z'' + 2\eta' z' = 0$. Therefore $z = \int \frac{1}{\eta^2}$ and $y = \eta \int \frac{1}{\eta^2}$ is another Liovillian solution.

3 Picard-Vessiot (differential Galois) theory

We learned in college that the DE has a "fundamental system of solutions" η,ζ. This means that η and ζ are functions that satisfy the equation and are linearly independent over constants ($\mathbb{C}$). In addition, every solution is a linear combination over $\mathbb{C}$ of η and ζ.

We also learned that linear independence is equivalent to the Wronskian

$$W = \begin{pmatrix} \eta & \zeta \\ \eta' & \zeta' \end{pmatrix}$$

having non-zero determinant

$$\det W \neq 0$$

Observe that

$$(\det W)' = (\eta\zeta' - \eta'\zeta)' = \eta'\zeta' + \eta\zeta'' - \eta''\zeta - \eta'\zeta' = \eta r\zeta - r\eta\zeta = 0\,.$$

Thus $\det W \in \mathbb{C}$.

Consider the differential field ($\mathcal{F} = \mathbb{C}(x)$)

$$\mathcal{G} = \mathcal{F}\langle\eta, \zeta\rangle = \mathcal{F}(\eta, \zeta, \eta', \zeta')\,.$$

Definition 3.1. The group of all differential automorphisms of $\mathcal{G}$ that leave $\mathcal{F}$ invariant (element-wise) is called the *differential Galois group* of the $\mathcal{G}$ over $\mathcal{F}$ and is denoted by

$$G(\mathcal{G}/\mathcal{F})\,.$$

If $\sigma \in G(\mathcal{G}/\mathcal{F})$, is a differential automorphism over $\mathcal{F}$ then $\sigma\eta$ and $\sigma\zeta$ are solutions of the DE. Therefore

$$\sigma \begin{pmatrix}\eta & \zeta\end{pmatrix} = \begin{pmatrix}\eta & \zeta\end{pmatrix} \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}\,,$$

where

$$c(\sigma) = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}$$

is an invertible (since σ is an automorphism) matrix of constants, i.e.

$$c(\sigma) \in \mathrm{GL}_{\mathbb{C}}(2) = \mathrm{GL}(2)\,.$$

But we can do better. Since $a_\sigma, b_\sigma, c_\sigma, d_\sigma \in \mathbb{C}$, we have

$$\sigma \begin{pmatrix} \eta & \zeta \\ \eta' & \zeta' \end{pmatrix} = \begin{pmatrix} \eta & \zeta \\ \eta' & \zeta' \end{pmatrix} \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}\,.$$

i.e.

$$\sigma W = W c(\sigma)$$

Taking determinants we have

$$\sigma(\det W) = \det W \det c(\sigma)$$

But

$$\sigma(\det W) = \det W$$

since $\det W \in \mathbb{C}$. Therefore $\det c(\sigma) = 1$, i.e.

$$c(\sigma) \in \mathrm{SL}(2)\,.$$

Theorem 3.2. *The mapping*

$$c\colon G(\mathcal{G}/\mathcal{F}) \longrightarrow \mathrm{SL}(2) = \mathrm{SL}_{\mathbb{C}}(2)$$

is an injective homomorphism whose image is an algebraic subgroup of $\mathrm{SL}(2)$*.*

There is a "fundamental theorem of Galois theory", i.e. a bijection between algebraic subgroups of $G(\mathcal{G}/\mathcal{F})$ and intermediate differential fields $\mathcal{F} \subset \mathcal{E} \subset \mathcal{G}$.

Theorem 3.3. *If η_1, ζ_1 is another fundamental set of solutions of the DE then the image of c_1 in* $\mathrm{SL}(2)$ *is conjugate to the image of c.*

I.e. There is an element $X \in \mathrm{SL}(2)$ such that

$$c_1(G(\mathcal{G}/\mathcal{F})) = X\, c(G(\mathcal{G}/\mathcal{F}))\, X^{-1}\,.$$

4 Example 1

Consider $y'' = y$. Then e^x, e^{-x} is a fundamental system of solutions and

$$\mathcal{G} = \mathcal{F}\langle e^x, e^{-x}\rangle = \mathcal{F}(e^x)\,.$$

Because $(e^x)' = e^x$, we must have

$$(\sigma e^x)' = \sigma e^x$$

for every $\sigma \in G(\mathcal{G}/\mathcal{F})$. This implies that

$$\sigma e^x = d_\sigma e^x \qquad \text{and} \qquad \sigma e^{-x} = d_\sigma^{-1} e^{-x}$$

for some constant $d_\sigma \in \mathbb{C}$ Therefore

$$\sigma \begin{pmatrix} \eta & \zeta \end{pmatrix} = \begin{pmatrix} \eta & \zeta \end{pmatrix} \begin{pmatrix} d_\sigma & 0 \\ 0 & d_\sigma^{-1} \end{pmatrix}$$

i.e.

$$G(\mathcal{G}/\mathcal{F}) \approx \left\{ \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix} \middle| d \in \mathbb{C} \right\} \subset \mathrm{SL}(2)\,.$$

5 Example 2

Consider

$$y'' = -\frac{1}{4x^2} y\,.$$

One solution is $\eta = \sqrt{x}$. We get the other one by reduction of order, so a fundamental system of solutions is

$$\eta = \sqrt{x}, \qquad \zeta = \sqrt{x} \log x\,.$$

Now we can compute the Galois group. Let $\sigma \in G(\mathcal{G}/\mathcal{F})$. Then

$$\sigma\eta = \pm\eta\,.$$

$\log x$ is a solution of $y' = 1/x$ and every solution of that equation is of the form $\log x + c$ for some constant c. Therefore

$$\sigma\zeta = \pm\sqrt{x}(\log x + c_\sigma) = c_\sigma \eta \pm \zeta$$

Thus

$$G(\mathcal{G}/\mathcal{F}) \approx \left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \middle| c \in \mathbb{C} \right\} \cup \left\{ \begin{pmatrix} -1 & c \\ 0 & -1 \end{pmatrix} \middle| c \in \mathbb{C} \right\}$$

6 The four cases

Theorem 6.1. *There are precisely four cases that can occur.*

Case 1. *G is triangulisable, i.e. G is conjugate to a subgroup of*

$$\left\{ \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \middle| \, c, d \in \mathbb{C}, c \neq 0 \right\}$$

Case 2. *G is conjugate to a subgroup of*

$$\left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \middle| \, c \in \mathbb{C}, c \neq 0 \right\} \cup \left\{ \begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix} \middle| \, c \in \mathbb{C}, c \neq 0 \right\}$$

Case 3. *G is a finite group: the tetrahedral group, the octahedral group or the icosahedral group.*

Case 4. $G = \mathrm{SL}(2)$.

7 Case 1

Suppose that

$$G \subset \left\{ \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \middle| \, c, d \in \mathbb{C}, c \neq 0 \right\}$$

and η, ζ is a fundamental system of solutions relative to G. For every $\sigma \in G(\mathcal{G}/\mathcal{F})$,

$$\sigma \begin{pmatrix} \eta & \zeta \end{pmatrix} = \begin{pmatrix} \eta & \zeta \end{pmatrix} \begin{pmatrix} c_\sigma & d_\sigma \\ 0 & c_\sigma^{-1} \end{pmatrix}$$

so

$$\sigma\eta = c_\sigma \eta \,.$$

We say that η is a *semi-invariant*.

Let $\theta = \eta'/\eta$ then

$$\sigma\theta = \frac{\sigma\eta'}{\sigma\eta} = \frac{c_\sigma \eta'}{c_\sigma \eta} = \theta \qquad \Longrightarrow \qquad \theta \in \mathcal{F} = \mathbb{C}(x) \,.$$

We say that θ is an *invariant.*

θ satisfies the *Riccati equation*

$$\theta' + \theta^2 = \frac{\eta\eta'' - \eta'\eta'}{\eta^2} + \left(\frac{\eta'}{\eta}\right)^2 = \frac{\eta\eta''}{\eta^2} = r\,.$$

i.e. the Riccati equation has a rational solution.

8 Case 2

Suppose that

$$G \subset \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \middle| c \in \mathbb{C}, c \neq 0 \right\} \cup \left\{ \begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix} \middle| c \in \mathbb{C}, c \neq 0 \right\}$$

Then either

$$\begin{array}{lll} \sigma\eta = c_\sigma \eta & \text{and} \quad \sigma\zeta = c_\sigma^{-1}\zeta, & \qquad \text{or} \\ \sigma\eta = -c_\sigma^{-1}\zeta & \text{and} \quad \sigma\zeta = c_\sigma \eta & \end{array}$$

Therefore

$$\sigma(\eta\zeta) = \pm\eta\zeta$$

so $(\eta\zeta)^2$ is an invariant and is in $\mathbb{C}(x)$. We write

$$(\eta\zeta)^2 = a\prod_i (x - c_i)^{e_i}$$

for some $e_i \in \mathbb{Z}$, $a, c_i \in \mathbb{C}$.

Let

$$\phi = \frac{(\eta\zeta)'}{\eta\zeta} = \frac{1}{2}\frac{((\eta\zeta)^2)'}{(\eta\zeta)^2} = \frac{1}{2}a\sum_i \frac{e_i}{x - c_i}$$

One computes that

$$\phi'' + 3\phi\phi' + \phi^3 = 4r\phi + 2r'\,.$$

This is the Riccati equation associated to the third order linear homogeneous differential equation satisfied by $\eta\zeta$. In this case it has a solution of a very special sort.

9 Case 3

For the tetrahedral group,

$$\xi = (\eta^4 + 8\eta\zeta^3)$$

then ξ^3 is an invariant (and therefore is in $\mathbb{C}(x)$) and

$$\phi = \frac{\xi'}{\xi} = \frac{1}{3}\frac{(\xi^3)'}{\xi^3}$$

satisfies a 4$^{\text{th}}$ order Riccati equation.

For the octahedral group,

$$\xi = \eta^5\zeta - \eta\zeta^5$$

and ξ^2 is an invariant.

$$\phi = \frac{\xi'}{\xi} = \frac{1}{2}\frac{(\xi^2)'}{\xi^2}$$

satisfies a 6$^{\text{th}}$ order Riccati equation.

For the icosahedral group

$$\phi = \eta^{11}\zeta - 11\eta^6\zeta^6 - \eta\zeta^{11}$$

is invariant and satisfies a 12$^{\text{th}}$ order Riccati equation.

10 Case 4

This case is the easiest. The DE does not have Liouvillian solution.

11 Another example

Consider

$$y'' = \left(x^2 - 2x + 3 + \frac{1}{x} + \frac{7}{4x^2} - \frac{5}{x^3} + \frac{1}{x^4}\right) y = ry$$

We are going to try for Case 1, so we look for a rational solution θ of the Riccati equation

$$\theta' + \theta^2 = r$$

Since $\theta \in \mathbb{C}(x)$, it has a partial fraction decomposition

$$\begin{aligned}
\theta = & \frac{a_n}{x^n} + \cdots + \frac{a_1}{x} \\
& + \frac{b_{1m_1}}{(x-c_1)^{m_1}} + \cdots + \frac{b_{11}}{x-c_1} \\
& + \qquad \cdots \\
& + \frac{b_{rm_d}}{(x-c_d)^{m_d}} + \cdots + \frac{b_{d1}}{x-c_d} \\
& + f_0 + \cdots + f_e x^e
\end{aligned}$$

I separate out the pole $x = 0$ because it is a pole of r. The others $(c_1, \ldots, c_d)$ are not. It turns out that the Riccati equation can have singularities that are not present in the original equation.

It's easier to use Laurent series. Look first at 0:

$$\theta = \frac{a}{x^n} + \cdots + \frac{b}{x} + \cdots .$$

From the Riccati equation we get

$$-\frac{na}{x^{n+1}} + \cdots + \frac{a^2}{x^{2n}} + \cdots = r = \frac{1}{x^4} + \cdots$$

It immediately follows that

$$n = 2 \qquad \text{and} \qquad a = \pm 1$$

Using the Riccati equation again, we get

$$-\frac{2a}{x^3} - \frac{b}{x^2} + \cdots + \frac{a^2}{x^4} + \frac{2ab}{x^3} + \cdots = \frac{1}{x^4} - \frac{5}{x^3} + \cdots$$

therefore

$$-2a + 2ab = -5$$

So we have the possibilities:

$$a = \quad 1 \qquad b = -\frac{3}{2} \qquad\qquad \theta = \quad \frac{1}{x^2} - \frac{3/2}{x} + \cdots$$

$$a = -1 \qquad b = \quad \frac{7}{2} \qquad\qquad \theta = -\frac{1}{x^2} + \frac{7/2}{x} + \cdots$$

Now let's try some other point:

$$\theta = \frac{a}{x - c^n} + \cdots$$

From the Riccati equation we get

$$-\frac{na}{(x-c)^{n+1}} + \cdots + \frac{a^2}{(x-c)^2} + \cdots = 0 + \cdots$$

so

$$n = 1 \qquad \text{and} \qquad a = 1$$

So far

$$\theta = \quad \frac{1}{x^2} - \frac{3/2}{x} + \sum_{i=1}^{d} \frac{1}{x - c_i} + f_0 + \cdots + f_e x^e$$

or

$$\theta = -\frac{1}{x^2} + \frac{7/2}{x} + \sum_{i=1}^{d} \frac{1}{x - c_i} + f_0 + \cdots + f_e x^e$$

Unfortunately we do do not know, yet, what d is or what the c_i are (not to mention the polynomial part).

Next we look at ∞. Write

$$\theta = ax^n + \cdots + bx + cx^{-1} + \cdots$$

Then

$$nax^{n-1} + \cdots + a^2x^{2n} + \cdots = r = x^2 + \cdots$$

Therefore

$$n = 1 \qquad \text{and} \qquad a = \pm 1$$

So

$$\theta = ax + b + \frac{c}{x} + \cdot$$

and, from the Riccati equation,

$$a + \cdots + a^2x^2 + 2abx + 2ac + b^2 + \cdots = x^2 - 2x + 3 + \cdots$$

Comparing coefficients we get

$$a = \quad 1 \qquad b = -1 \qquad c = \quad \frac{1}{2} \qquad \theta = x - 1 + \frac{1/2}{x} + \cdots$$

$$a = -1 \qquad b = \quad 1 \qquad c = -\frac{3}{2} \qquad \theta = -x + 1 - \frac{3/2}{x} + \cdots$$

From our analysis of the finite poles we had two possibilities for θ. The first was

$$\theta = \frac{1}{x^2} - \frac{3/2}{x} + \sum_{i=1}^{d} \frac{1}{x - c_d} + f_0 + \cdots + f_e x^e$$

$$= f_e x^e + \cdots + f_0 + \frac{d - 3/2}{x} + \frac{?}{x^2} + \cdots$$

Comparing with the first case above we have

$$e = 1, \quad f_e = 1, \quad f_0 = -1, \quad d - 3/2 = 1/2, \quad d = 2$$

Comparing with the second case we have

$$e = 1, \quad f_e = -1, \quad f_0 = 1, \quad d - 3/2 = -3/2, \quad d = 0$$

But we had a second possibility for theta:

$$\theta = -\frac{1}{x^2} + \frac{7/2}{x} + \sum_{i=1}^{d} \frac{1}{x - c_d} + f_0 + \cdots + f_e x^e$$

$$= f_e x^e + \cdots f_0 + \frac{d + 7/2}{x} + \frac{?}{x^2} + \cdots$$

Comparing with the equations we got at ∞ we have

$$e = 1, \quad f_e = -1, \quad f_0 = 1, \quad d + 7/2 = -3/2, \quad d = -3$$

which is impossible.

The last case is

$$e = 1, \quad f_e = -1, \quad f_0 = 1, \quad d + 7/2 = -3/2, \quad d = -5$$

which is also impossible.

Let's try for $d = 0$. In that case

$$\theta = \frac{1}{x^2} - \frac{3/2}{x} + 1 - x$$

We try this in the Riccati equation and get

$$\theta' + \theta^2 = x^2 - 2x + 3 - \frac{5}{x} + \frac{23/4}{x^2} - \frac{5}{x^3} + \frac{1}{x^4} \neq r$$

This is *not* the right answer! So θ does not give a solution.

On to the case

$$\theta = \frac{1}{x^2} - \frac{3/2}{x} + \frac{1}{x - c_1} + \frac{1}{x - c_2} - 1 + x$$

We do not know what c_1 and c_2 are.

Let

$$\omega = \frac{1}{x^2} - \frac{3/2}{x} - 1 + x$$

and

$$P = (x - c_1)(x - c_2)$$

Then

$$\eta = e^{\int \theta} = P e^{\int \omega}$$

is supposed to be a solution of the original DE ($y'' = ry$). This gives

$$P'' + 2\omega P' + (\omega' + \omega^2 - r)P = 0$$

or

$$P'' + \left(\frac{2}{x^2} - \frac{3}{x} - 2 + 2x\right) P' + (\frac{4}{x} - 4)P = 0$$

Substituting $P = x^2 + ax + b$ one easily finds that

$$P = x^2 - 1 = (x-1)(x+1)$$

So $c_1 = 1$, $c_2 = -1$.

The solution to the original DE

$$y'' = ry$$

is

$$\eta = Pe^{\int \omega} = (x^2 - 1)e^{\int \frac{1}{x^2} - \frac{3}{2x} - 1 + x} = x^{-3/2}(x^2 - 1)e^{-1/x + -x + x^2/2}$$

12 Higher order

First of all, there really are only two cases: either the equation has a Liouvillian solution or it doesn't. And if it does, the Lie-Kolchin theorem tells us that the DE will have a Liouvillian solution if and only if the connected component of the identity of G, denoted by G^o, is triangulizable:

$$G^o = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ 0 & c_{22} & \cdots & c_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & 0) \cdots & c_{nn} & \end{pmatrix}$$

For every $\sigma \in G^o$,

$$\sigma\eta = c_{11}\eta$$

so

$$\sigma\frac{\eta'}{\eta} = \frac{\eta'}{\eta}$$

Because G^o has finite index in G,

$$\frac{\eta'}{\eta}$$

is algebraic over $\mathbb{C}(x)$. The degree d is the index of G^o in G. Then the symmetric functions in η, ζ of degree d are invariants. These are solutions of a Riccati equation of order at most d.

Unfortunately, the index of G^o in G may be arbitrarily large. However we do have:

Theorem 12.1. *If $G \subset \mathrm{SL}(n)$ has a non-trivial triangularizable subgroup (not necessarily G^o, but always G^0), then the index is no greater than a computable number $I(n)$.*

$I(n)$ tends to be rather large, for example

$$I(2) = 384,064\,.$$

The following was proven by Michael Singer [10] and [11].

Theorem 12.2. *Given a linear homogeneous differential equation of order n*

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = 0\,,$$

there is an algorithm that either finds a Liouvillian solution or proves that it has none.

13 The Galois group

The algorithm actually tells us something about the Galois group of the differential equation. In case 1, for example, the group is reducible (triangularizable). We can break the cases into various subcases and refine the algorithm to determine which subcase the equation belongs to.

For example, in case 1 could have $d = 0$. In this case the group is diagonal

$$G = \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix},$$

and $\eta\zeta$ is an invariant. If c is an n-th root of unity, then η^n is an invariant.

Singer and Ulmer [12] actually calculate the Galois group.

References

[1] F. Baldassarri and B. Dwork, *On second order linear differential equations with algebraic solutions*, Amer. J. Math. **101** (1979), no. 1, 42–76. MR 527825 (81d:34002)

[2] L. Fuchs, *Über die linearen Differentialgleichungen zweiter Ordnung welche algebraische Integrale besitzen, zweiter Abhandlung*, J. für Math. **85** (1878).

[3] Irving Kaplansky, *An introduction to differential algebra*, Actualités Sci. Ind., No. 1251 = Publ. Inst. Math. Univ. Nancago, No. 5, Hermann, Paris, 1957. MR 0093654 (20 #177)

[4] E. R. Kolchin, *Differential algebra and algebraic groups*, Academic Press, New York, 1973. MR 0568864 (58 #27929)

[5] Ellis Kolchin, *Selected works of Ellis Kolchin with commentary*, American Mathematical Society, Providence, RI, 1999. MR 1677530 (2000g:01042)

[6] Jerald J. Kovacic, *An algorithm for solving second order linear homogeneous differential equations*, J. Symbolic Comput. **2** (1986), no. 1, 3–43. MR 839134 (88c:12011)

[7] J. Liouville, *Sur la détermination des intégrales dont la valeur est algébrique*, J. de l'École Polytechnique **22** (1833).

[8] Andy R. Magid, *Lectures on differential Galois theory*, University Lecture Series, vol. 7, American Mathematical Society, Providence, RI, 1994. MR 1301076 (95j:12008)

[9] P. Th. Pépin, *Méthode pour obtenir les intégrales altébriques des équations différentielles linéaires du second ordre*, Atti dell' Accad. Pont. de Nuovi Lincei **XXXIV** (1881), 243–388.

[10] Michael F. Singer, *Algebraic solutions of nth order linear differential equations*, Proceedings of the Queen's Number Theory Conference, 1979 (Kingston, Ont., 1979), 1980, pp. 379–420. MR 634699 (83b:12022)

[11] ———, *Liouvillian solutions of nth order homogeneous linear differential equations*, Amer. J. Math. **103** (1981), no. 4, 661–682. MR 623132 (82i:12028)

[12] Michael F. Singer and Felix Ulmer, *Galois groups of second and third order linear differential equations*, J. Symbolic Comput. **16** (1993), no. 1, 9–36. MR 1237348 (94i:34015)

[13] Felix Ulmer and Jacques-Arthur Weil, *Note on Kovacic's algorithm*, J. Symbolic Comput. **22** (1996), no. 2, 179–200. MR 1422145 (97j:12006)

[14] Marius van der Put and Michael F. Singer, *Galois theory of linear differential equations*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 328, Springer-Verlag, Berlin, 2003. MR 1960772 (2004c:12010)