Height Functions

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Definition 1. *Let $Y \subseteq \mathbb{P}^n$ be a quasi-projective variety. A function $f : Y \to k$ is regular at a point $P \in Y$ if there is an open neighborhood U with $P \in U \subseteq Y$, and homogeneous polynomials $g, h \in k[x_0, ..., x_n]$, of the same degree, such that h is nowhere zero on U, and $f = g/h$ on U.*

Definition 2. *If X is a variety, we define the function field $K(X)$ of X as follows, an element of $K(X)$ is an equivalence class of pairs $< U, f >$ where U is a nonempty open subset of X and f is a regular function on U, and where we identify two pairs $< U, f >$ and $< V, g >$ if $f = g$ on $U \cap V$. The elements of $K(X)$ are called rational functions on X.*

It's much easier if we just describe the function field of a projective variety V as the field of rational functions $F(X) = \frac{f(X)}{g(X)}$ such that:

(a) f and g are homogeneous polynomials of the same degree.

(b) $g \notin I(V)$.

(c) Two functions $\frac{f}{g}$ and $\frac{f'}{g'}$ are identified if $fg' - f'g \in I(V)$.

Consider the variety C, a smooth projective curve. The "valuation" on $k(C)$ is the function

$$ord_P : k(V) \to \mathbb{Z}$$

where $P \in C$. If $F \in k(C)$ then $ord_P(F)$ takes the value of the order of the zero of F at P if f has a zero at P, the negative of the order of the pole of f at P if f has a pole at P and takes the value zero otherwise.

We then define an absolute value on $k(C)$ for each $P \in K(V)$ by setting $|f|_P = e^{-ord_P(F)}$ This is a nonarchimedean absolute value because

(a) $ord_P(FG) = ord_P(F) + ord_P(G)$

(b) $ord_P(F + G) \geq \min\{ord_P(F), ord_P(G)\}$

Using this we have

$$|\cdot|_P : k(C) \to [0, \infty)$$

with the following three properties:

(1) $|x|_P = 0$ if and only if $x = 0$.

(2) $|xy|_P = |x|_P \cdot |y|_P$.

(3) $|x + y|_P \leq |x|_P + |y|_P$.

and more importantly we always have

(3') $|x + y|_P \leq \max\{|x|_P, |y|_P\}$.

If C is a smooth projective curve we have a product rule on $k(C)$ as well.

Theorem 3. *(Product Rule 3)*

$$\prod_{P \in C} |F|_P = 1$$

Proof.

$$\begin{aligned} \prod_{P \in C} |F|_P &= \prod_{P \in C} e^{-ord_P(F)} \\ &= e^{-\sum_{P \in C} ord_P(F)} \end{aligned}$$

Since we are considering $F = f/g$ where f and g are homogeneous polynomials of the same degree, F has the same number of poles and zeros counting multiplicity so

$$\sum_{P \in C} ord_P(F) = 0.$$

Therefore

$$\prod_{P \in C} |F|_P = e^0 = 1.$$

□

Because the whole theory of developing a well defined height on $\mathbb{P}^n$ over a number field depended on the "product rule." We can make the same exact construction of a function field minus the "boundedness" property.

The boundedness property fails because if we consider $K = k(C)$ over an infinite base field and consider constant functions each additive height will be equal to zero. But in this case there are an infinite amount of these.

We can define the height of a point $Q = (f_0, ..., f_n) \in \mathbb{P}^n$ over $K = k(C)$ as

$$h(Q) = \sum_{P \in C} \log \max\{|f_0|_P, ..., |f_n|_P\}$$

Now if we take a brief aside to try to at least sketch the proof of the basic theorems to create the Weil height machine. Which will work for a variety over a function field because the construction just depends on the construction of height for points in $\mathbb{P}^n$

Definition 4. *Let X be an algebraic variety, the group of* ***Weil Divisors*** *on X denoted $Div(X)$ is the free abelian group generated by the closed subvarieties of codimension one on X.*

Weil Divisors are of the form

$$D = \sum n_Y Y$$

where Y is a closed subvariety of codimension one of X and all but finitely many n_Y's are nonzero.

Example 5. *If X is a curve the group of Weil Divisors is the free group generated by the points of X.*

If X is a surface the group will be the free group generated by the irreducible curves in X.

Definition 6. *Let X be a variety, and let $f \in k(X)^*$ be a rational function on X. The divisor of f denoted $div(f)$ or just (f) is the divisor*

$$div(f) = \sum_Y ord_Y(f)Y \in Div(X).$$

A divisor is principal if it can be written as a divisor of a rational function.

A divisor is said to be effective or positive if for every $n_Y \geq 0$.

The support of a divisor $D = \sum n_Y Y$ is the union of Y's such that the associated n_Y are not zero.

Two divisors are said to be linearly equivalent, written $D \sim D'$, if $D - D'$ is a principal divisor.

A small description of where linear equivalence comes from suppose that $D \sim D'$ we say $D' = D + div(f)$. For each point in $(a, b) \in \mathbb{P}^1$, define the divisor $D_{(a,b)} = D + div(a + bf)$. Then the divisors $D_{(1,0)} = D$ and $D_{(0,1)} = D'$. So there is a family of divisors parameterized by points of a line, that deforms D to D'.

Definition 7. *For each divisor D on a variety X we have and associated vector space*

$$L(D) = \{f \in k(K)^* | D + div(f) \geq 0\} \cup \{0\}$$

The set of effective divisors linearly equivalent to D is a linear system called the complete linear system associated to D. It is denoted by $|D|$

If X is projective then $L(D)$ is finite dimensional with dimension $l(D)$.

Definition 8. *Let L be a linear system of dimension n parameterized by a projective space $\mathbb{P}(V) \subset \mathbb{P}(L(D))$. Select a basis $f_0, ..., f_n$ of $V \subset L(D)$. The rational map associated to L, denoted by ϕ_L is the map*

$$\begin{aligned} \phi : X &\rightarrow \mathbb{P}^n \\ x &\mapsto (f_0(x), ..., f_n(x)). \end{aligned}$$

A linear system L on a projective variety X, is very ample if the associated rational map $\phi_L : X \rightarrow \mathbb{P}^n$ is an embedding, that is ϕ_L is a morphism that maps X isomorphically onto its image ϕ_L.

The divisor D is said to be very ample if the complete linear system $|D|$ is very ample. A divisor D is said to be ample if some positive multiple of D is very ample.

The sum of two very ample divisors is very ample and every divisor can be written as the difference of two (very) ample divisors.

Proposition 9. *Let*

$$\begin{aligned} S_{n,m} : \mathbb{P}^n \times \mathbb{P}^m &\rightarrow \mathbb{P}^N \\ (x, y) &\mapsto (x_0y_0, x_0y_1, ..., x_iy_j, ..., x_ny_m). \end{aligned}$$

where $N = (n + 1)(m + 1) - 1$. Let H_n, H_m and H_N be hyperplanes in $\mathbb{P}^n$, $\mathbb{P}^m$ and $\mathbb{P}^N$ respectively.

(a) $S^*_{n,m}(H_N) \sim H_n \times \mathbb{P}^m + \mathbb{P}^n \times H_m \in Div(\mathbb{P}^n \times \mathbb{P}^m)$.

(b) $h(S_{n,m}(x, y)) = h(x) + h(y)$ *for all* $x \in \mathbb{P}^n(\overline{\mathbb{Q}})$ *and* $y \in \mathbb{P}^m(\overline{\mathbb{Q}})$.

(c) Let the map

$$\begin{aligned} \Phi_d : \mathbb{P}^n &\rightarrow \mathbb{P}^N \\ x &\mapsto (M_0(x), ..., M_N(x)) \end{aligned}$$

be the d-uple embedding. (i.e. $N = (\binom{n+d}{n}) - 1$ and the collection $M_0(x), ..., M_N(x)$ is the complete collection of monomials of degree d in the variables $x_0, ..., x_n$.) Then

$$h(\Phi_d(x)) = dh(x) \ \ for \ all \ x \in \mathbb{P}^n(\overline{\mathbb{Q}}).$$

Proof. (a) Let $(z_0, ..., z_N)$ be homogeneous coordinates on $\mathbb{P}^N$, and fix hyperplanes $H_N = z_0 = 0$, $H_n = x_0 = 0$ and $H_m = y_0 = 0$. Then

$$\begin{aligned} S^*_{n,m}(H_N) &= S^*_{n,m}\{z \in \mathbb{P}^N | z_0 = 0\} \\ &= \{(x, y) \in \mathbb{P}^n \times \mathbb{P}^m | x_0 y_0 = 0\} \\ &= H_n \times \mathbb{P}^m + \mathbb{P}^n \times H_m. \end{aligned}$$

(b) Let $x \in \mathbb{P}^n(k)$ and $y \in \mathbb{P}^m(k)$ for some number field k, and let $z \in S_{n,m}(x, y)$. Then for any absolute value $v \in M_k$ we have

$$\begin{aligned} \max_{0 \le l \le N} |z_l|_v &= \max_{\substack{0 \le i \le n \\ 0 \le j \le m}} |x_i y_j|_v \\ &= (\max_{0 \le i \le n} |x_i|_v)(\max_{0 \le j \le m} |y_j|_v). \end{aligned}$$

Now raise to the appropriate power and multiply over all v and take the log to get the appropriate result.

(c)It's pretty close to clear that

$$|M_j(x)|_v \le \max_{0 \le i \le n} |x_i|^d_v.$$

Since the particular monomials $x_0^d, ..., x_n^d$ appear in the list

$$\max_{0 \le j \le N} |M_j(x)|_v = \max_{0 \le i \le n} |x_i|^d_v.$$

Now raise take the appropriate power and multiply over all v and take the log to finish the proof.

□

Theorem 10. *Let $\phi : \mathbb{P}^n \to \mathbb{P}^m$ be a rational map of degree d definied over $\overline{\mathbb{Q}}$, so ϕ is given by a (m+1)-tuple $\phi = (f_0, ..., f_m)$ of homogeneous polynomials of degree d. Let $Z \subset \mathbb{P}^n$ be the subset of common zeros of the f_i's. Notice that $\mathbb{P}^n \backslash Z$.*

(a) We have

$$h(\phi(P)) \le dh(P) + O(1) \ for\ all\ P \in \mathbb{P}^n(\overline{\mathbb{Q}}) \backslash Z.$$

(b) Let X be a closed subvariety of $\mathbb{P}^n$ with the property that $X \cap Z = \emptyset$. (Thus ϕ defines a morphism $X \to \mathbb{P}^m$.) Then

$$h(\phi(P)) = dh(P) + O(1) \ for\ all\ P \in X(\overline{\mathbb{Q}})$$

Corollary 11. *Let $A : \mathbb{P}^n \to \mathbb{P}^m$ be a linear map defined over $\overline{\mathbb{Q}}$. In other words, A is given by $m+1$ linear forms $(L_0, ..., L_m)$. Let $Z \subset \mathbb{P}^n$ be the linear subspace where $L_0, ..., L_m$ simultaneously vanish, and let $X \subset \mathbb{P}^n$ be a closed subvariety with $X \cap Z = \emptyset$. Then*

$$h(A(P)) = h(P) + O(1) \text{ for all } P \in X(\overline{\mathbb{Q}}).$$

Proof. Fix the field of definition k for ϕ, so

$$\phi = (f_0, ..., f_m) \, with \, f_0, ..., f_m \in k[x_0, ..., x_n]_d.$$

(i.e., the f_i's are homogeneous polynomials of degree d.) Write the f_i's explicitly as

$$f_i(X) = \sum_{|e|=d} a_{i,e} X^e,$$

where $e = (e_0, ..., e_n)$ and $|e| = e_0, ..., e_n$, and $X^e = X_0^{e_0} ... X^{e_n}$. Notice the sum as $\binom{n+d}{n}$ terms, which is the number of monomials of degree d in $n+1$ variables.

For any point $P = (x_0, ..., x_n)$ with $x_j \in k$ and ny absolute value value $v \in M_k$, we will write $|P|_v = \max\{|x_j|_v\}$. Similarly, for any polynomial $f = \sum a_e X^e \in k[X]$ we will let $|f|_v = \max\{a_e\}$. We also set the notation

$$\epsilon_v(r) = \{ \begin{array}{c} r \; if \; v \; is \; archimedean, \\ 1 \; if \; v \; is \; nonarchimedean. \end{array}$$

With this notation, the triangle inequality can be written uniformly as

$$|a_1 + ... + a_r|_v \leq \epsilon(r) \max\{|a_1|_v, ..., |a_r|_v\}.$$

Now consider any point $P \in \mathbb{P}^n(\overline{\mathbb{Q}})$. Extending k if necessary, we may assume that $P \in \mathbb{P}^n(k)$ and write $P = (x_0, ..., x_n)$ with $x_i \in k$. Then for any $v \in M_k$ and any i we have

$$\begin{aligned} |f_i(P)|_v &= \left| \sum_{|e|=d} a_{i,e} x^e \right|_v \\ &\leq \epsilon_v \binom{n+d}{n} \left(\max_e |a_{i,e}|_v \right) \left(\max_e |x_0^{e_0} \cdots x_n^{e_n}|_v \right) \\ &\leq \epsilon_v \binom{n+d}{n} |f_i|_v \max_{e,j} |x_j|^{e_0 + \cdots + e_n} \\ &= \epsilon_v \binom{n+d}{n} |f_i|_v |P|_v^d. \end{aligned}$$

Now take the maximum over $0 \leq i \leq m$, raise to the appropriate power, and multiply over all $v \in M_k$. This gives

$$H(\phi(P)) \leq \binom{n+d}{n} H(\phi) H(P)^d,$$

where

$$H(\phi) = \prod_{v \in M_k} \max\{|f_0|_v, ..., |f_m|_v\}^{n_v/[k:\mathbb{Q}]}.$$

and making use of the identity

$$\prod_{v \in M_k} \epsilon_v(r)_v^n = \prod_{v \in M_k^\infty} r_v^n = r^{[k:\mathbb{Q}]},$$

which follows from the degree formula ($\sum n_v = [k : \mathbb{Q}]$).

Taking log's we get

$$h((\phi(P)) \leq dh(P) + h(\phi) + \log \binom{n+d}{n},$$

which proves (a).

(b)To get the other side of the inequality, we need to use the fact that we are choosing points in X and that ϕ is a morphism on X. Let $p_1, ..., p_r$ be homogeneous polynomials generating the ideal of X. Then we know that $p_1, ..., p_r, f_0, ..., f_m$ have no common zeros in $\mathbb{P}^n$. The Nullstellensatz tells us that the ideal they generated has a radical equal to the ideal generated by $X_0, ..., X_n$. This means that we can find polynomials g_{ij}, q_{ij} (which we may assume are homogeneous) and an exponent $t \geq d$ such that

$$g_{0j}f_0 + \cdots + g_{mj}f_m + q_{1j}p_1 + \cdots + q_{rj}p_r = X_j^t \quad for \ 0 \leq j \leq n.$$

Notice that the g_{ij}'s are homogeneous of degree $t - d$, since the f_i's are homogenous of degree d. Extending to k if necessary, we may also assume that the g_{ij}'s and the q_{ij}'s have coefficients in k. Now let $P = (x_0, ..., x_n) \in X(k)$. The assumption that $P \in X$ implies that $p_i(P) = 0$ for all i, so we can evaluate the above formula at P and obtain

$$g_{0j}(P)f_0(P) + \cdots + g_{mj}(P)f_m(P) = x_j^t, \ \ 0 \leq j \leq n.$$

Hence

$$\begin{aligned}
|P|_v^t &= \\
&= \max_j |g_{0j}(x)f_0(x) + g_{1j}(x)f_1(x) + \cdots + g_{mj}(x)f_m(x)|_v \\
&\leq \epsilon_v(m+1)\left(\max_{i,j} |g_{ij}(x)|_v\right)\left(\max_i |f_i(x)|_v\right) \\
&\leq \epsilon_v(m+1)\left[\epsilon\binom{t-d+n}{n}\left(\max_{i,j} |g_{ij}(x)|_v\right]|P|_v^{t-d}\right] \cdot \left(\max_i |f_i(x)|_v\right).
\end{aligned}$$

Now raise to the appropriate power and multiply over all $v \in M_k$ to get

$$H(P)^t \leq cH(P)^{t-d}H(\phi(P)),$$

where c is a certain constant depending on the f_i's, the g_{ij}'s and t, but independent of P. In other words, c depends on ϕ and X. Taking logarithms gives us the desired inequality

$$dh(P) \leq h(\phi(P)) + O(1).$$

This completes the proof and the corollary follows. □

Definition 12. *Let $\phi : V \to \mathbb{P}^n$ be a morphism. The (absolute logarithmic) height on V relative to ϕ is the function*

$$h_\phi : V(\overline{\mathbb{Q}}) \to [0, \infty), \quad h_\phi(P) = h(\phi(P)),$$

where $h : \mathbb{P}^n(\overline{\mathbb{Q}}) \to [0, \infty)$ is the height function on projective space defined earlier.

Theorem 13. *Let V be a projective variety defined over $\overline{\mathbb{Q}}$, let $\phi : V \to \mathbb{P}^n$ and $\psi : V \to \mathbb{P}^m$ be morphisms, and let H and H' be hyperplanes in $\mathbb{P}^n$ and $\mathbb{P}^m$ respectively. Suppose that $\phi^* H$ and $\psi^* H'$ are linearly equivalent. Then*

$$h_\phi(P) = h_\psi(P) + O(1) \, for \, all \, P \in V(\overline{\mathbb{Q}}).$$

Here the $O(1)$ constant will depend on V, ϕ and ψ, but independent of P.

Proof. Let $D \in Div(V)$ be any positive divisor in the linear equivalence class of $\phi^* H$ and $\psi^* H'$. The morphisms ϕ and ψ are determined by certain subspaces W and W' in the vector space $L(D)$ and the choice of bases for W and W'. In other words, if we chose $h_0, ..., h_N$ as a basis for $L(D)$, then there are linear combinations

$$f_i = \sum_{j=0}^{N} a_{ij} h_j, \ \ 0 \leq i \leq n,$$

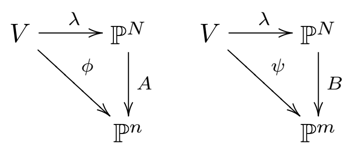
$$g_i = \sum_{j=0}^{N} b_{ij} h_j, \ \ 0 \leq i \leq m,$$

such that ϕ and ψ are given by

$$\phi = (f_0, ..., f_n) \; and \; \psi = (g_0, ..., g_m)$$

where the a_{ij}'s and b_{ij}'s are constants.

Let $\lambda = (h_0, ..., h_N) : V \to \mathbb{P}^N$ be the morphism corresponding to the complete linear system determined by D. Let A be the linear map $A : \mathbb{P}^N \to \mathbb{P}^n$ defined by the matrix (a_{ij}) and similarly let $B : \mathbb{P}^N \to \mathbb{P}^m$ be defined by (b_{ij}). Then we have commutative diagrams



The vertical maps A and B are not morphisms on all of $\mathbb{P}^N$, but the fact that ϕ and ψ are morphisms associated to the linear system $L(D)$ implies that A is defined at every point of the image $\lambda(V(\overline{\mathbb{Q}})$, and similarly for B. Hence we can apply the previous corollary so that

$$h(A(Q)) = h(Q) + O(1) \text{ and } h(B(Q)) = h(Q) + O(1) \text{ for all } Q \in \lambda(V(\overline{\mathbb{Q}}).$$

Letting $Q = \lambda(P)$ with $P \in V(\overline{\mathbb{Q}}$ and by using the commutative diagram

$$\begin{aligned} h(\phi(P)) &= h(A(\lambda(P))) \\ &= h(\lambda(P)) + O(1) \\ &= h(B(\lambda(P))) + O(1) \\ &= h(\psi(P)) + O(1). \end{aligned}$$

□

Theorem-Definition 14. *(Weil's Height Machine) Let k be a number field. For every smooth projective variety V/k there exists a map*

$$h_V : Div(V) \to \{functions\, V(\bar{k}) \to \mathbb{R}\}$$

with the following properties

(a) (Normalization)Let $H \subset \mathbb{P}^n$ be a hyperplane, and let $h(P)$ be the absolute logarithmic height on $\mathbb{P}^n$. Then

$$h_{\mathbb{P}^n,H}(P) = h(P) + O(1) \text{ for all } P \in \mathbb{P}^n(\bar{k}).$$

(b) (Functorality)Let $\phi : V \to W$ be a morphism and let $D \in Div(W)$. Then

$$h_{V,\phi^*D}(P) = h_{W,D}(\phi(P)) + O(1)\, for\, all\, P \in V(\bar{k}).$$

(c) (Additivity) Let $D, E \in Div(V)$. Then

$$h_{V,D+E}(P) = h_{V,D}(P) + h_{V,E}(P) + O(1) \text{ for all } P \in V(\bar{k}).$$

(d) (Uniqueness)The height functions $h_{V,D}$ are determined, up to $O(1)$, by normalization, functoriality just for embeddings $\phi : V \hookrightarrow \mathbb{P}^n$, and additivity.

(e) (Linear Equivalence)Let $D, E \in Div(V)$ with D linearly equivalent to E. Then

$$h_{V,D}(P) = h_{V,E}(P) + O(1) \text{ for all } P \in V(\bar{k}).$$

(f) (Positivity)Let $D \in Div(V)$ be an effective divisor, and let B be the base locus of the linear system $|D|$. Then

$$h_{V,D}(P) \geq O(1) \text{ for all } P \in (V \backslash B)(\bar{k}).$$

(g) *(Algebraic Equivalence) Let $D, E \in Div(V)$ with D ample and E algebraically equivalent to 0. Then*

$$\lim_{h_{V,D}(P)\to\infty} \frac{h_{V,E}(P)}{h_{V,D}(P)} = 0 \quad where\ P \in V(\overline{k})$$

(h) *(Finiteness)Let $D \in Div(V)$ be ample. Then for every finite extension k'/k and every constant B, the set*

$$\{P \in V(k')|h_{V,D}(P) \leq B\}$$

is finite.

Proof. We start with the construction: For every very ample every divisor $D \in Div(V)$ choose the morphism

$$\phi_D : V \to \mathbb{P}^n$$

associated to D and $L(D)$. Define

$$h_{v,D}(P) = h(\phi_D(P)) \ \ for\, all\, P \in V(\overline{k})$$

Next for every other divisor $D \in Div(V)$ we write $D = D_1 - D_2$ where D_1 and D_2 are divisors whose linear system has no base points. Then we define

$$h_{v,D}(P) = h_{v,D_1}(P) - h_{v,D_2}(P) \ \ for\, all\, P \in V(\overline{k})$$

This gives us a height function for every divisor D on every variety V.

Claim:The height function$h_{V,D}$ associated to a very ample divisor D is independent of of morphism ϕ_D, up to $O(1)$.

Let $\psi : V \to \mathbb{P}^m$ be another morphism associated to D. Then

$$\phi^* H \sim \psi^* H' \sim D$$

where H and H' are hyperplanes in $\mathbb{P}^n$ and $\mathbb{P}^m$ respectively. By our previous theorem

$$h(\phi_D(P)) = h(\psi_D(P)) + O(1) \ \ for\, all\, P \in V(\overline{k}).$$

Therefore through very ample divisors we can use the associated morphism to compute the height, proving our claim.

Next we check the additivity property (c) for very ample divisors.

Let D and E be very ample divisors with their associated morphisms $\phi_D : V \to \mathbb{P}^n$ and $\phi_E : V \to \mathbb{P}^m$. Composing the product

$$\phi_D \times \phi_E : V \to \mathbb{P}^n \times \mathbb{P}^m$$

with Segre embedding $S_{n,m}$ we get the morphism

$$\phi_D \otimes \phi_E : V \to \mathbb{P}^N \ \ N = (n+1)(n+m) - 1$$

$$\phi_D \otimes \phi_E(P) = S_{n,m}(\phi_D(P), \phi_E(P))$$

This morphism is associated to the divisor $D + E$. That is

$$(\phi_D \otimes \phi_E)^* H \sim D + E$$

(by a previous theorem.) From the above arguments we can compute the height for a very ample divisor by using the associated morphism. So

$$h_{V,D+E}(P) = h((\phi_D \otimes \phi_E)(P)) + O(1)$$

Now by the previous theorem about the Segre embedding

$$\begin{aligned} h_{V,D+E}(P) &= h((\phi_D \otimes \phi_E)(P)) + O(1) \\ &= h((S_{n,m}(\phi_D(P), \phi_E(P)))(P)) + O(1) \\ &= h(\phi_D(P)) + h(\phi_E(P)))(P)) + O(1) \\ &= h_{V,D}(P) + h_{V,E}(P) + O(1) \end{aligned}$$

This gives us (c) for very ample divisors.

We can write any divisor D as the difference of very ample divisors. Suppose we have two decompositions for a divisor D

$$D = D_1 - D_2 = E_1 - E_2$$

Then $D_1 + E_2 = E_1 + E_2$ and hence

$$\begin{aligned} h_{V,D_1} + h_{V,E_2} &= h_{V,D_1+E_2} + O(1) \\ &= h_{V,E_1+D_2} + O(1) \\ &= h_{V,E_1} + h_{V,D_2} + O(1) \end{aligned}$$

Therefore $h_{V,D_1} - h_{V,D_2} = h_{V,E_1} - h_{V,E_2} + O(1)$.

Now check (a) and (b).

If H is a hyperplane in $\mathbb{P}^n$, then the identity map $\mathbb{P}^n \to \mathbb{P}^n$, $P \mapsto P$, is associated to H giving (a).

To verify (b), we write $D \in Div(W)$ as the difference of very ample divisors $D = D_1 - D_2$. Let ϕ_{D_1} and ϕ_{D_2} be the corresponding morphisms into projective space. Then $\phi^* D_1$ and $\phi^* D_2$ are very ample, with associated morphisms $\phi_{D_1} \circ \phi$ and $\phi_{D_2} \circ \phi$ respectively. Hence

$$\begin{aligned} h_{V,\phi^* D} &= h_{V,\phi^* D_1} - h_{V,\phi^* D_2} + O(1) \\ &= h \circ \phi_{D_1} \circ \phi - h \circ \phi_{D_2} + O(1) \\ &= h_{W,D_1} \circ \phi - h_{W,D_2} \circ \phi + O(1) \\ &= h_{W,D} \circ \phi \end{aligned}$$

Now we check property (c) which we already know for very ample divisors. Let D and E be two divisors and write them as the difference of two very ample divisors, $D = D_1 - D_2$ and $E = E_1 - E_2$. Then $D_1 + E_1$ and $E_2 + D_2$ are very ample and we compute

$$\begin{aligned} h_{V,D+E} &= h_{V,D_1+E_1} - h_{V,D_2+E_2} + O(1) \\ &= h_{V,D_1} + h_{E_1} - h_{V,D_2} - h_{V,E_2} + O(1) \\ &= h_{V,D} + h_E + O(1) \end{aligned}$$

This completes the proof of (c)

By proving (a), (b) and (c) using the fact than any divisor can be written as the difference of two very ample divisors, we have proven (d).

Now to prove linear equivalence (e). Suppose D and E are linearly equivalent. Write $D = D_1 - D_2$ and $E = E_1 - E_2$ as the sum of very ample divisors. We have $D_1 + E_2 \sim E_1 + D_2$. This means the morphisms $\phi_{D_1+E_2}$ and $\phi_{E_1+D_2}$ are associated to the same linear system. By our previous theorem

$$h(\phi_{D_1+E_2}(P)) = h(\phi_{E_1+D_2}(P)) + O(1).$$

Using this equality and additivity we get

$$h_{V,D_1} + h_{V,E_2} = h_{V,D_1+E_2} + O(1) = h_{V,D_2+E_1} + O(1) = h_{V,D_2} + h_{V,E_1} + O(1)$$

Hence

$$h_{V,D} = h_{V,D_1} - h_{V,D_2} + O(1) = h_{V,E_1} - h_{V,E_2} + O(1) = h_{V,E} + O(1)$$

which proves (d).

□

Corollary 15. *Let V/k be a smooth variety defined over a number field, let $D \in Div(V)$, and let $\phi : V \to V$ be a morphism. Suppose that $\phi^* D \sim \alpha D$ for some $n \geq 1$. Then there exists a constant C such that*

$$|h_{V,D}(\phi(P)) - \alpha h_{V,D}(P)| \leq C \text{ for all } P \in V(\bar{k}).$$

Note 1. *The $O(1)$ here is dependent on the variety, divisor and morphism but not the points. It is possible to compute the $h_{V,D}$'s explicitly and to give bounds of $O(1)$ in terms of the defining equations the varieties, divisors and morphisms. However, it is difficult in practice to bound the $O(1)$'s.*

Theorem-Definition 16. *(Neron, Tate) Let V/k be a smooth variety defined over a number field, let $D \in Div(V)$, and let $\phi : V \to V$ be a morphism. Suppose that $\phi^* D \sim \alpha D$ for some $n \geq 1$. Then there exists a unique function, called the* ***canonical height*** *on V relative to ϕ and D,*

$$\widehat{h}_{V,\phi,D} : V(\bar{k}) \to \mathbb{R}$$

with the following two properties:

(i) $\widehat{h}_{V,\phi,D}(P) = h_{V,D}(P) + O(1)$ *for all* $P \in V(\overline{k})$.

(ii) $\widehat{h}_{V,\phi,D}(\phi(P)) = \alpha\widehat{h}_{V,\phi,D}(P)$ *for all* $P \in V(\overline{k})$.

The canonical height depends only on the linear equivalence class of D. Further, it can be computed as the limit

$$\widehat{h}_{V,\phi,D}(P) = \lim_{n\to\infty} \frac{h_{V,D}(\phi^n(P))}{\alpha^n},$$

where ϕ^n is the n-th iterate of ϕ.

Proof. By the previous corollary there exists a constant C such that

$$|h_{V,D}(\phi(Q)) - \alpha h_{V,D}(Q)| \leq C \;\; for \;\; all \;\; Q \in V(\overline{k}).$$

Now take a any point $P \in V(\overline{k})$. We prove the sequence $\{\alpha^{-n}h_{V,D}(\phi^n(P))\}$ converges by showing it is Cauchy. Take $n \geq m$ and

$$|\alpha^{-n}h_{V,D}(\phi^n(P)) - \alpha^{-m}h_{V,D}(\phi^m(P))|$$

$$= |\sum_{i=m+1}^{n} \alpha^{-i}h_{V,D}(\phi^i(P)) - \alpha h_{V,D}(\phi^{i-1}(P))|$$

by a telescoping sum. Then

$$\leq \sum_{i=m+1}^{n} |\alpha^{-i}|h_{V,D}(\phi^i(P)) - \alpha h_{V,D}(\phi^{i-1}(P))|$$

by the triangle inequality. Then

$$\leq \sum_{i=m+1}^{n} \alpha^{-i}C$$

from above and $Q = \phi^{i-1}P$. Then

$$\leq (\frac{\alpha^{-m} - \alpha^{-n}}{\alpha - 1})C.$$

This quantity goes to 0 as $n > m \to \infty$, which proves the sequence is Cauchy, hence converges. So we can define the $\widehat{h}_{V,\phi,D}(P)$ to be the limit

$$\widehat{h}_{V,\phi,D}(P) = \lim_{n\to\infty} \frac{h_{V,D}(\phi^n(P))}{\alpha^n}.$$

To verify property (i), take $m = 0$ and let $n \to \infty$ in the inequality above. This gives

$$|\widehat{h}_{V,Q,D}(P) - h_{V,D}(P)| \leq \frac{C}{\alpha - 1},$$

which gives us the desired inequality.

Property (ii) follows directly from the limit definition of canonical height.

$$\begin{aligned}\widehat{h}_{V,\phi,D}(\phi(P)) &= \lim_{n\to\infty}\frac{h_{V,D}(\phi^n(\phi(P)))}{\alpha^n}\\ &= \lim_{n\to\infty}\frac{\alpha h_{V,D}(\phi^{n+1}(P))}{\alpha^{n+1}}\\ &= \alpha\widehat{h}_{V,\phi,D}(P).\end{aligned}$$

What's left to prove is uniqueness. Let $\widehat{h}$ and $\widehat{h}'$ be two functions with properties (i) and (ii). Let $g = \widehat{h} - \widehat{h}'$. Then (i) implies that g is bounded, say $|g(P)| \leq C'$ for all $P \in V(\overline{k})$. While (ii) says that $g \circ \phi = \alpha^n g$ for all $n \leq 1$. Hence

$$|g(P)| = \frac{g(|\phi^n(P))|}{\alpha^n} \leq \frac{C'}{\alpha^n}$$

where $\frac{C'}{\alpha^n} \to 0$ as $n \to 0$. This says that $g(P) = 0$ for all P, so $\widehat{h} = \widehat{h}'$. □

Okay, now that that is done.

Another way to look at varieties over function fields is as follows.

Consider the smooth projective variety $\mathcal{V}/k$ over k, a number field, and a sooth projective curve C/k with a morphism $\pi : \mathcal{V} \to C$ over k whose generic fiber is smooth. We can think of the generic fiber V of $\mathcal{V}$ as a variety over a function field $K = \overline{k}(C)$.

Let $\phi : \mathcal{V} \to \mathcal{V}$ be a rational map over k that commutes with π and is a morphism on the generic fiber. Let C^0 be the subset of C having "good fibers"

$$C^0 = \{t \in C | \mathcal{V}_t = \pi^{-1}(t)\, is\, smooth\, and\, \phi_t : \mathcal{V}_t \to \mathcal{V}_t\, is\, a\, morphism\}$$

Example 17. *(Elliptic Surfaces) Let $\mathcal{E}$ be a smooth two dimensional projective variety with a morphism*

$$\pi : \mathcal{E} \to C$$

such that all but finitely many points $t \in C(\overline{k})$, the fiber

$$\mathcal{E}_t = \pi^{-1}(t)$$

is a smooth curve of genus 1 and we have a section to π

$$\sigma : C \to \mathcal{E}.$$

This surface an associated elliptic curve E/K where $K = k(C)$.

*In Silverman's book Advanced Topics in the Arithmetic of Elliptic Curves, he proves that we have an elliptic curve E/K with an associated elliptic surface $\mathcal{E} \to C$ that does not **split**, then if we fix some constant B the set*

$$\{P \in E(K) | h(P) \leq B\}$$

is finite.

Here an elliptic surface $\mathcal{E} \to C$ splits if there is a birational isomorphism

$$i : \mathcal{E} \to E_0 \times C$$

that commutes with the projections onto C.

Call and Silverman proved the following two theorems in their paper "Canonical heights on varieties with morphism."

Theorem 18. *Using the notation above the example, there exists constants c_1, c_2 depending on the family $\mathcal{V} \to C$, the map ϕ, the divisor class η and the choice of Weil height functions $h_{\mathcal{V},\eta}$ and h_C such that*

$$|\widehat{h}_{\mathcal{V}_t,\eta_t,\phi_t}(x) - h_{\mathcal{V},\eta}(x)| \geq c_1 h_C(t) + c_2 \;\; for\, all\, t \in C^0 \, and\, all\, x \in \mathcal{V}_t$$

If we consider a section $P : C \to \mathcal{V}$ to $\pi : \mathcal{V} \to C$ then P corresponds to a point $P_V \in V(K)$. Using the above theorem they prove

Theorem 19.

$$\lim_{\substack{t \in C^0(\overline{K}) \\ h_C(t) \to 0}} \frac{\widehat{h}_{\mathcal{V}_t,\eta_t,\phi_t}(P_t)}{h_C(t)} = \widehat{h}_{V,\eta_V,\phi_V}(P_V).$$

References 1. *Again, most of the theorems and proofs come from Marc Hindry and Joseph Silverman's "Diophantine Geometry, An Introduction." Other material comes from Robin Hartshorne's "Algebraic Geometry" and Joseph Silverman's "The Arithmetic of Elliptic Curves." The last page or so comes from Joseph Silverman's "Advanced Topics in the Arithmetic of Elliptic Curves" and Gregory Call and Joseph Silverman's paper "Canonical heights on varieties with morphism."*