Differential schemes

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Abstract

An affine differential scheme, $X = \operatorname{diffspec} \mathcal{R}$, is similar to an affine scheme, except we start with a differential ring $\mathcal{R}$ and consider differential prime ideals. There is a canonical mapping of $\mathcal{R}$ into the ring of global sections of X. In scheme theory this mapping is an isomorphism, not so for differential schemes. We can easily determine the kernel. It is the differential ideal of "differential zeros". Surjectivity is missing because of the existence of "differential units" and the lack of "common denominators". We shall also discuss other "challenges" in the theory of differential schemes, such as the existence of products. For differential group schemes we have the problem that they need not be linear, $\mathcal{R}$ need not be a differential Hopf algebra. This is an elementary talk. We assume the audience knows the definition of spec but little else.

1 Introduction

Differential algebraic geometry has been developed Ritt, Raudenbush, Levi, Kolchin, Cassidy, Sit and others. For an excellent survey see Buium and Cassidy (1999) which has an extensive bibliography. The theory of differential algebraic groups is also well-established see the papers of Cassidy, e.g.(1972).

However all of this work is done in the classical "Weil" language. Kolchin (1985) broke from the Weil tradition by axiomatizing the notion of differential algebraic group. This approach is an elegant tour de force, but has not become widely accepted.

The language of schemes was introduced in the work of Keigher (1975, 1977, 1983, 1981, 1982) and was continued by Carrà Ferro (1978, 1985, 1990) and Buium (1982).

2 Differential rings

Rings are always commutative with identity. The 0 ring has $1 = 0$.

Definition 2.1. A Δ-ring $\mathcal{R}$ is called a *Keigher* ring if for any Δ-ideal $\mathfrak{a}$, $\sqrt{\mathfrak{a}}$ is also a Δ-ideal.

Keigher (1977, p. 242) calls such a ring *special.* Gorman (1973, p. 25) calls it a *d-MP* ring.

Every Δ-algebra over $\mathbb{Q}$ (a Ritt algebra) is a Keigher ring. Every ring with trivial derivations ($\delta a = 0$ for all a) is a Keigher ring.

Example 2.2. Let $\mathcal{R} = \mathbb{Z}[x]$ where $x' = 1$. This is *not* a Keigher ring. Then $\mathfrak{a} = (2, x^2)$ is a Δ-ideal. The radical of $\mathfrak{a}$ is

$$\sqrt{\mathfrak{a}} = (2, x)$$

which is not a Δ-ideal since it does not contain $x' = 1$.

Observe that $\mathfrak{a}$ is not contained in any prime Δ-ideal $\mathfrak{p}$. For x would be in $\mathfrak{p}$ and also $x' = 1 \in \mathfrak{p}$ which is a contradiction.

This implies that $\mathcal{S} = \mathcal{R}/\mathfrak{a}$ has *no* prime Δ-ideal. So $\operatorname{diffspec} \mathcal{S}$ is empty. This cannot happen for rings: $\operatorname{spec} R$ is empty if and only if R is the zero ring.

Lemma 2.3. *Let $\mathcal{R}$ be a Keigher Δ-ring and $\mathfrak{a}$ a Δ-ideal. Then $\mathfrak{a}$ is contained in a prime Δ-ideal if and only if $1 \notin \mathfrak{a}$.*

From now on every Δ-ring is assumed to be a Keigher ring. We use $\mathcal{R}$ to denote a (Keigher) ring.

3 Differential schemes

Definition 3.1. $X = \operatorname{diffspec} \mathcal{R}$ is the set of all prime Δ-ideals of $\mathcal{R}$. For a Δ-ideal $\mathfrak{a}$, $V(\mathfrak{a})$ is the set of $\mathfrak{p} \in \operatorname{diffspec} \mathcal{R}$ with $\mathfrak{p} \supset \mathfrak{a}$. For $f \in \mathcal{R}$, $D(f)$ is the set of $\mathfrak{p} \in \operatorname{diffspec} \mathcal{R}$ with $f \notin \mathfrak{p}$.

Definition 3.2. Define a topology on X, called the *Kolchin topology*, by taking the $V(\mathfrak{a})$ to be the closed sets of X.

X is a subset of $\operatorname{spec} \mathcal{R}$ and the Kolchin topology is the subspace topology of the Zariski topology on $\operatorname{spec} \mathcal{R}$. $D(f)$ form a basis of open sets.

Definition 3.3. For each open set U of X let $\mathcal{O}_X(U)$ be the set of functions

$$s : U \to \coprod_{\mathfrak{p} \in U} \mathcal{R}_{\mathfrak{p}}$$

satisfying:

1. $s(\mathfrak{p}) \in \mathcal{R}_{\mathfrak{p}}$, and
2. there is an open cover U_i of U and $a_i, b_i \in \mathcal{R}$, such that for each $\mathfrak{q} \in U_i$, $b_i \notin \mathfrak{q}$ and $s(\mathfrak{q}) = a_i/b_i \in \mathcal{R}_q$.

$\mathcal{O}_X(U)$ inherits the structure of a Δ-ring from that on $\mathcal{R}_\mathfrak{p}$, by

$$\delta(s)(\mathfrak{p}) = \delta(s(\mathfrak{p})) \in \mathcal{R}_\mathfrak{p}, \qquad \text{for } \delta \in \Delta.$$

$\mathcal{O}_X$ is a sheaf of Δ-rings.

Proposition 3.4. *For $\mathfrak{p} \in X$, the stalk $\mathcal{O}_{X,\mathfrak{p}}$ is Δ-isomorphic to $\mathcal{R}_\mathfrak{p}$.*

Proposition 3.5. *For every $f \in \mathcal{R}$ the open set $D(f) \subset X$ is canonically identified with* diffspec $\mathcal{R}_f$*.*

Definition 3.6. An LDR (Local Differential Ringed) space is a local ringed space whose sheaf is a sheaf of Δ-rings. A morphism of LDR spaces is a morphism of local ringed spaces such that the morphism of sheaves is Δ-.

Definition 3.7. An *affine Δ-scheme* is an LDR space which is isomorphic to $(X, \mathcal{O}_X)$ where $X = \operatorname{diffspec} \mathcal{R}$ for some Δ-ring $\mathcal{R}$. A *Δ-scheme* is an LDR space in which every point has an open neighborhood that is an affine Δ-scheme. A morphism of Δ-schemes is a morphism of LDR spaces.

Proposition 3.8. *If $\phi : \mathcal{S} \to \mathcal{R}$ is a Δ-homomorphism, then there is an induced morphism of schemes (the* adjoint*)*

$$\begin{aligned} {}^a\phi &: X = \operatorname{diffspec} \mathcal{R} \to Y = \operatorname{diffspec} \mathcal{S} \\ \phi^\# &: \mathcal{O}_Y \to ({}^a\phi)_* \mathcal{O}_X \end{aligned}$$

Up to this point things are going very well, the proofs for spec translate without difficulty to diffspec. Unfortunately this does not last.

4 Global sections

Proposition 4.1. *Denote the ring of global sections by*

$$\widehat{\mathcal{R}} = \mathcal{O}_X(X) = \Gamma(X, \mathcal{O}_X).$$

There is a canonical Δ-homomorphism $\iota \colon \mathcal{R} \to \widehat{\mathcal{R}}$ where

$$\iota(r)(\mathfrak{p}) = \frac{r}{1} \in \mathcal{R}_\mathfrak{p} \qquad (\mathfrak{p} \in X).$$

For spec, ι is an isomorphism. For diffspec it is neither injective nor surjective in general.

Example 4.2. $\mathcal{R} = \mathbb{Q}[x]\{\eta\} = \mathbb{Q}[x]\{y\}/[xy]$ where $x' = 1$. Thus

$$x\eta = 0, \qquad \text{so} \qquad \eta + x\eta' = 0$$

Multiply by η to get

$$0 = \eta(\eta + x\eta') = \eta^2 + \eta x\eta' = \eta^2 \ .$$

so any prime Δ-ideal of $\mathcal{R}$ contains $[\eta]$. But $\mathcal{R}/[\eta] \approx \mathbb{Q}[x]$ contains a single prime Δ-ideal, namely (0), i.e. it is Δ-simple. It follows that $[\eta]$ is the *unique* prime Δ-ideal of $\mathcal{R}$. It follows that $\widehat{\mathcal{R}} = \mathcal{R}_{[\eta]}$.

An aside: $\mathcal{R}$ has a unique maximal Δ-ideal $[\eta]$. It is not a maximal ideal so $\mathcal{R}$ is not a local ring - close but no cigar. As far as I am aware, such rings have not been studied.

Because $x\eta = 0$ and $x \notin [\eta]$ we have

$$\iota\eta = \frac{\eta}{1} = \frac{x\eta}{x} = 0 \in \mathcal{R}_{[\eta]} = \widehat{\mathcal{R}}$$

so ι is not injective.

It is not surjective either. Indeed, $x \notin [\eta]$ so

$$\frac{1}{x} \in \mathcal{R}_{[\eta]} = \widehat{\mathcal{R}}$$

but

$$\frac{1}{x} \notin \mathcal{R} = \mathbb{Q}[x] \ .$$

In fact, Kovacic (2002b, Example 5.5 p. 270) shows that

$$\widehat{\mathcal{R}} = \mathbb{Q}(x) \ .$$

Carrà Ferro (1990) addresses this challenge by using a different structure sheaf. See the paper for the exact definition.

Buium (1986) has yet another approach. He sets $X = \operatorname{spec} \mathcal{R}$ and takes the usual sheaf, which is then a sheaf of Δ-rings. This gives what some people, namely Umemura (1996, Definition 1.6, p. 8) and myself, call a "scheme with differentiation".

We stick with our definition.

Theorem 4.3. *The canonical mapping $\widehat{\mathcal{R}} \longrightarrow \widehat{\widehat{\mathcal{R}}}$ is an isomorphism and therefore*

$$\widehat{X} = \operatorname{diffspec} \widehat{\mathcal{R}} \approx \widehat{\widehat{X}} = \operatorname{diffspec} \widehat{\widehat{\mathcal{R}}}$$

This makes $\widehat{\mathcal{R}}$ into a "closure" of $\mathcal{R}$.

5 Differential zeros

In ring theory we have the theorem that if $a \in R$ goes to 0 in R_P for every prime ideal P, then $a = 0$. This is false in differential algebra.

Example 5.1. Let $\mathcal{R}[x]\{\eta\} = \mathbb{Q}[x]\{y\}/[xy]$. (Same as in Example 4.2.) There is only one prime Δ-ideal, $[\eta]$ and

$$\eta \longmapsto \frac{\eta}{1} = \frac{x\eta}{x} = 0 \in \mathcal{R}_{[\eta]} \ .$$

Suppose that $a \in \mathcal{R}$ goes to 0 in $\mathcal{R}_\mathfrak{p}$ for every prime Δ-ideal $\mathfrak{p}$. This means that there exists

$$a_\mathfrak{p} \in \mathcal{R}, \qquad a_\mathfrak{p} \notin \mathfrak{p}, \qquad a_\mathfrak{p} a = 0 \ .$$

I.e.

$$a_\mathfrak{p} \in \operatorname{ann}(a), \qquad a_\mathfrak{p} \notin \mathfrak{p}$$

And therefore $\operatorname{ann}(a)$ is not contained in any prime Δ-ideal, i.e.

$$1 \in [\operatorname{ann}(a)]$$

If we could conclude that $1 \in \operatorname{ann}(a)$ we would get $x = 0$. But we cannot, since $\operatorname{ann}(x)$ is not necessarily a Δ-ideal. Indeed, if $ax = 0$ then

$$0 = (ax)' = a'x + ax'$$

We can multiply by x to get

$$0 = a'x^2 + ax'x = a'x^2$$

so $a' \in \operatorname{ann}(x^2)$, but not necessarily in $\operatorname{ann}(x)$.

Definition 5.2. $r \in R$ is a Δ-*zero* if $1 \in [\operatorname{ann}(r)]$. The set of Δ-zeros of $\mathcal{R}$ is denoted by $\mathfrak{Z}(\mathcal{R})$.

Proposition 5.3. *$\mathfrak{Z}(\mathcal{R})$ is a Δ-ideal of $\mathcal{R}$.*

Proposition 5.4. *$R/\mathfrak{Z}(\mathcal{R})$ has no non-zero Δ-zero. $\mathfrak{Z}(\mathcal{R})$ is the smallest Δ-ideal with that property.*

Theorem 5.5. *The kernel of $\iota\colon \mathcal{R} \to \widehat{\mathcal{R}}$ is $\mathfrak{Z}(\mathcal{R})$.*

Theorem 5.6. $\operatorname{diffspec} \mathcal{R} \approx \operatorname{diffspec} \mathcal{R}/\mathfrak{Z}(\mathcal{R})$.

The only problem is that $\mathfrak{Z}$ behaves badly with respect to rings of quotients. In Kovacic (2002b, Example 5.3, p. 571) $\mathfrak{Z}(\mathcal{R}) = 0$ but, for some $a \in \mathcal{R}$, $\mathfrak{Z}(\mathcal{R}_a) \neq 0$.

Proposition 5.7. *$\mathfrak{Z}(\mathcal{R})$ is contained in the nil radical of $\mathcal{R}$. A reduced Δ-ring has no non-zero Δ-zero.*

Theorem 5.8. *If $\mathcal{R}$ is reduced then $\iota\colon \mathcal{R} \to \widehat{\mathcal{R}}$ is injective. In addition*

$$\operatorname{diffspec} \mathcal{R} \approx \operatorname{diffspec} \widehat{\mathcal{R}}$$

When $\mathcal{R}$ is reduced we identify $\mathcal{R}$ with a subring of $\widehat{\mathcal{R}}$.

Proposition 5.9. *If $\mathcal{R}$ is reduced then $\iota\colon \mathcal{R} \to \widehat{\mathcal{R}}$ is epi in the category of reduced Δ-rings.*

This means that if $f, g\colon \widehat{\mathcal{R}} \to \mathcal{S}$ with $\mathcal{S}$ reduced and

$$f \circ \iota = g \circ \iota\colon \mathcal{R} \to \mathcal{S}$$

then $f = g$. This theorem is true under weaker hypotheses, I don't know if it is true or false in general.

6 Differential units

The mapping $\iota : \mathcal{R} \to \widehat{\mathcal{R}}$ is rarely surjective, see Example 4.2. This was first discovered by Cassidy (1972, P. 901, paragraph preceding Section 6). Here is her example.

Example 6.1. Let $\mathcal{R} = \mathbb{Q}\{\eta\} = \mathbb{Q}\{y\}/[y' - y]$. This is reduced (in fact a domain) so $\iota\colon \mathcal{R} \to \widehat{\mathcal{R}}$ is injective. But it is not surjective. We claim that, for any $c \in \mathbb{Q}$, $c \neq 0$, $\eta - c$ is not contained in any prime Δ-ideal of $\mathcal{R}$. Indeed,

$$(\eta - c)' = \eta' = \eta \quad \Longrightarrow \quad c \in [\eta - c] \quad \Longrightarrow \quad 1 \in [\eta - c]\ .$$

We can define $s \in \widehat{\mathcal{R}}$ by the formula

$$s(\mathfrak{p}) = \frac{1}{\eta - c} \in \mathcal{R}_{[\mathfrak{p}]}$$

and $s \notin \mathcal{R} = \mathbb{Q}\{\eta\}$.

Definition 6.2. $r \in \mathcal{R}$ is a Δ-unit if $1 \in [r]$. The set of Δ-units of $\mathcal{R}$ is denoted by $\mathcal{U}(\mathcal{R})$.

Proposition 6.3. *$\mathcal{U}(\mathcal{R})$ is a multiplicative set of $\mathcal{R}$.*

Proposition 6.4. *Every Δ-unit of $\widehat{\mathcal{R}}$ is a unit, i.e. $\mathcal{U}(\widehat{\mathcal{R}}) = \mathcal{R}^*$.*

Theorem 6.5. *The mapping $\iota\colon \mathcal{R} \to \widehat{\mathcal{R}}$ extends to a mapping*

$$\iota'\colon \mathcal{R}[\mathcal{U}(\mathcal{R})^{-1}] \to \widehat{\mathcal{R}}\ .$$

But even this is not surjective.

7 Denominators

In this section we assume that $\mathcal{R}$ is reduced so that $\mathcal{R} \subset \widehat{\mathcal{R}}$.

If $s \in \widehat{\mathcal{R}}$ then for each $\mathfrak{p} \in X$ we have

$$s(\mathfrak{p}) = \frac{a}{b} \in \mathcal{R}_{\mathfrak{p}}\ .$$

It is easy to see that only a finite number of denominators is needed, so that we have a "case" statement

$$s(\mathfrak{p}) = \begin{cases} \dfrac{a_1}{b_1} & \mathfrak{p} \in D(b_1) \\ \vdots & \vdots \\ \dfrac{a_n}{b_n} & \mathfrak{p} \in D(b_n) \end{cases}$$

For spec one first proves that we can choose $n = 1$ and then that the denominator is a unit. For diffspec we do not always have a common denominator, see Kovacic (2002b, Section 10, p. 276) for a counterexample. And even if we could, the denominator would only be a Δ-unit, not a unit.

From the case statement we know that

$$(a_i s - b_i)(\mathfrak{p}) = 0 \qquad \text{if } \mathfrak{p} \in D(b_i)$$

But we know nothing about $(a_i s - b_i)(\mathfrak{q})$ when $\mathfrak{q} \notin D(b_i)$.

Theorem 7.1. *Suppose that $\mathcal{R}$ is reduced and let $s \in \widehat{\mathcal{R}}$. Then there exists $n \in \mathbb{N}$, and $a_1, b_1, \ldots, a_n, b_n \in \mathcal{R}$ such that $1 \in [b_1, \ldots, b_n]$ and for every i*

$$b_i s = a_i \in \widehat{\mathcal{R}}\,.$$

This theorem is a very good replacement for surjectivity. Kovacic (2002a), (2002b), and (2003) use it extensively. The assumption that $\mathcal{R}$ be reduced can be weakened. But I do not know of an example of a Δ-ring for which the theorem fails.

8 Ring of quotients for a Gabriel topology

We put a linear topology on $\mathcal{R}$ called a Gabriel topology. The closed sets are

$$a + I$$

where $a \in \mathcal{R}$ and I is an ideal (not Δ-ideal) of $\mathcal{R}$ satisfying

$$1 \in [I]$$

With respect to this topology one can define a "ring of quotients" which we denote by $\mathfrak{Q}(\mathcal{R})$. For details see Kovacic (2002b).

Theorem 8.1. *There is a canonical isomomorphism*

$$\mathfrak{Q}(\mathcal{R}) \approx \widehat{\mathcal{R}}$$

So $\widehat{\mathcal{R}}$ turns out to be a certain type of ring of quotients of $\mathcal{R}$. See Kovacic (2002b, Section 12, p. 279) for more details.

9 Rittian Δ-rings

A Δ-ring is rarely Noetherian, even the ring $\mathcal{F}\{y\}$ of Δ-polynomials is not.

Example 9.1. This example is due to Ritt (1934, p. 12). Let $\mathcal{R} = \mathcal{F}\{y\}$. Then

$$\mathfrak{a} = [y'y'',\, y''y^{(3)},\, \ldots,\, y^{(n)}y^{(n+1)},\, \ldots] \, .$$

is not finitely Δ-generated.

Following Kolchin (1961, p. 7-14) we make the following definition.

Definition 9.2. A Δ-ring is said to be *Rittian* if every radical Δ-ideal has a finite set of generators.

This is equivalent the the ascending chain condition for radical Δ-ideals. This condition is weaker than Δ-Noetherian as it applies only to *radical* Δ-ideals.

Theorem 9.3. *If $\mathcal{R}$ is finitely Δ-generated over a Δ-field, then $\mathcal{R}$ is Rittian.*

Proposition 9.4. *If $\mathcal{R}$ is Rittian then so is $\widehat{\mathcal{R}}$.*

Theorem 9.5. *$X = \operatorname{diffspec} \mathcal{R}$ is Noetherian if and only if $\mathcal{R}$ is Rittian.*

It is not true that $\operatorname{spec} R$ being Noetherian implies that R is Noetherian. What makes it work here is that the condition Rittian refers only to radical ideals.

Theorem 9.6. *If $\mathcal{R}$ is reduced and Rittian, there is a canonical injection*

$$\widehat{\mathcal{R}} \hookrightarrow Q(\mathcal{R}) \ .$$

So, for reduced Rittian rings we have

$$\mathcal{R} \subset \widehat{\mathcal{R}} \subset Q(\mathcal{R})$$

This says that for $s \in \widehat{\mathcal{R}}$ we can write

$$s = \frac{a}{b} \in Q(\mathcal{R}) \ .$$

Unfortunately it does not say that

$$s(\mathfrak{p}) = \frac{a}{b} \in \mathcal{R}_{\mathfrak{p}}$$

because it may happen that $b \in \mathfrak{p}$. The example in Kovacic (2002b, Section 10, p. 276) illustrates this.

In classical language $\mathcal{R}$ is the ring of coordinate functions, $\widehat{\mathcal{R}}$ the ring of everywhere defined functions and $Q(\mathcal{R})$ the ring of all rational functions.

10 Reduced Δ-schemes

Recall that a scheme X is reduced if $\mathcal{O}_X$ is a sheaf of reduced rings. Suppose that $X = \operatorname{diffspec} \mathcal{R}$ is reduced. Must $\mathcal{R}$ be reduced? No.

Example 10.1. (Same as Example 4.2) $\mathcal{R} = \mathbb{Q}[x]\{\eta\} = \mathbb{Q}[x]\{y\}/[xy]$. X has a single point and

$$\mathcal{O}_X(U) = \begin{cases} 0 & \text{if } U = \emptyset \\ \mathbb{Q}(x) & \text{if } U = X \end{cases}$$

This is reduced but $\mathcal{R}$ is not, since $\eta^2 = 0$.

Theorem 10.2. *If X is a reduced affine Δ-scheme then there is a reduced Δ-ring $\mathcal{R}$ with $X \approx \operatorname{diffspec} \mathcal{R}$.*

More precisely, if $X = \operatorname{diffspec} \mathcal{R}$ is reduced then the nil radical of $\mathcal{R}$ is the Δ-ideal of Δ-zeros $\mathfrak{Z}(\mathcal{R})$ and

$$X \approx \operatorname{diffspec}(\mathcal{R}/\mathfrak{Z}(\mathcal{R}))\ .$$

11 Constrained points

Let $\mathcal{F}$ be a Δ-field and $\mathcal{R}$ a finitely Δ-generated Δ-$\mathcal{F}$-algebra and $X = \operatorname{diffspec} \mathcal{R}$.

Definition 11.1. $\mathfrak{p} \in X$ is *algebraic* if $\mathcal{R}/\mathfrak{p}$ is an algebraic extension of $\mathcal{F}$.

This is so if and only if $\mathfrak{p}$ is a maximal ideal. An analogy for Δ-rings is the following, which, I believe, has not been studied at all.

Definition 11.2. A point $\mathfrak{p} \in X$ is *Δ-simple* if $\mathcal{R}/\mathfrak{p}$ is a Δ-simple ring, i.e. has no proper non-zero Δ-ideal.

Proposition 11.3. *Let $\mathfrak{p} \in X$. Then the following are equivalent.*

1. *$\mathfrak{p}$ is Δ-simple,*
2. *$\mathfrak{p}$ is a maximal Δ-ideal,*
3. *$\mathfrak{p}$ is a closed point, i.e. the unit set $\{\mathfrak{p}\}$ is closed.*

Definition 11.4. A point $\mathfrak{p} \in X$ is *constrained* if $\mathcal{R}/\mathfrak{p}$ is a constained extension of $\mathcal{F}$ (which I won't define).

Constrained extensions have been widely studied Kolchin (1974). A related notion, Δ-closure, is important in model theory, for a survey, see Scanlon (2002).

Proposition 11.5. *Let $\mathfrak{p} \in X$. Then the following are equivalent.*

1. $\mathfrak{p}$ *is constrained,*
2. *there exists* $c \in \mathcal{R}$ *(the constraint) such that* $\mathfrak{p}$ *is a* Δ*-ideal maximal with respect to the condition that it does not contain* c*,*
3. $\mathfrak{p}$ *is a locally closed point, i.e. there is an open neighborhood* U *of* $\mathfrak{p}$ *such that* $\overline{\{\mathfrak{p}\}} \cap U = \{\mathfrak{p}\}$*.*

In $\operatorname{spec} R$ a point is locally closed if and only if it is closed (provided R is finitely generated over a field). I don't think this is true in Δ-algebra but I don't have a counterexample.

12 Products

Suppose that $X = \operatorname{diffspec} \mathcal{R}$ is an affine Δ-scheme over $F = \operatorname{diffspec} \mathcal{F}$, where $\mathcal{F}$ is some field. This implies that O_X is a sheaf of Δ-$\mathcal{F}$-algebras. However it does not imply that $\mathcal{R}$ is an $\mathcal{F}$-algebra.

Example 12.1. Let $\mathcal{F} = \mathbb{Q}(x)$ and $\mathcal{R} = \mathbb{Q}[x]$. $\mathcal{R}$ is Δ-simple, i.e. there are no non-zero proper Δ-ideals. Therefore X has a unique point, the ideal (0). The sheaf is

$$\mathcal{O}_X(U) = \begin{cases} (0) & \text{if } U = \emptyset \\ \mathbb{Q}[x]_{(0)} = \mathbb{Q}(x) & \text{if } U = X \end{cases}$$

Thus X is an affine Δ-scheme over F but $\mathcal{R} = \mathbb{Q}[x]$ is not a Δ-algebra over $\mathcal{F} = \mathbb{Q}(x)$.

Now consider two affine Δ-schemes $X = \operatorname{diffspec} \mathcal{R}$ and $Y = \operatorname{diffspec} \mathcal{S}$ over F. One would like to define the product as

$$X \times_F Y \stackrel{?}{=} \operatorname{diffspec}(\mathcal{R} \otimes_{\mathcal{F}} \mathcal{S})$$

However we are in trouble unless both $\mathcal{R}$ and $\mathcal{S}$ are algebras over $\mathcal{F}$. There are two solutions.

Theorem 12.2. *If* $\mathcal{R}$ *and* $\mathcal{S}$ *are reduced and* $X = \operatorname{diffspec} \mathcal{R}$ *and* $Y = \operatorname{diffspec} \mathcal{S}$ *are* Δ*-schemes over* F*, then the product exists and*

$$X \times_F Y = \operatorname{diffspec}(\widehat{\mathcal{R}} \otimes_{\mathcal{F}} \widehat{\mathcal{S}})$$

But $\widehat{\mathcal{R}}$ and $\widehat{\mathcal{S}}$ are complicated so it is difficult to understand exactly what the product it. Or we can restrict the category.

Definition 12.3. An affine Δ-scheme X is called a Δ-$\mathcal{F}$-scheme if there is a Δ-$\mathcal{F}$-algebra $\mathcal{R}$ with $X \approx \operatorname{diffspec} \mathcal{R}$.

Theorem 12.4. *If $X = \operatorname{diffspec} \mathcal{R}$ and $Y = \operatorname{diffspec} \mathcal{S}$ where $\mathcal{R}$ and $\mathcal{S}$ are Δ-$\mathcal{F}$-algebras then*

$$X \times_F Y = \operatorname{diffspec}(\mathcal{R} \otimes_{\mathcal{F}} \mathcal{S}) .$$

13 Closed subschemes

Suppose that Y is a closed subscheme of $X = \operatorname{diffspec} \mathcal{R}$. Following the lead of algebraic geometry we would expect that $Y \approx \operatorname{diffspec}(\mathcal{R}/\mathfrak{a})$ for some Δ-ideal $\mathfrak{a}$. I don't know if this is true or false.

Theorem 13.1. *If X is reduced and Y is a reduced subscheme of X then there is a radical Δ-ideal $\mathfrak{a} \subset \mathcal{R}$ such that $Y \approx \operatorname{diffspec}(\mathcal{R}/\mathfrak{a})$.*

14 Morphisms

Let $X = \operatorname{diffspec} \mathcal{R}$ and $Y = \operatorname{diffspec} \mathcal{S}$. Suppose that $f \colon Y \to X$. Then there is a sheaf mapping $f^{\#} \colon \mathcal{O}_X \to f_* \mathcal{O}_Y$ and therefore a mapping of global sections

$$\widehat{f} \colon \widehat{\mathcal{R}} \to \widehat{\mathcal{S}} \qquad \text{and} \qquad \widehat{f} \circ \iota \colon \mathcal{R} \to \widehat{\mathcal{S}} .$$

This gives an aduction proved in Keigher (1975).

Theorem 14.1. *There is an isomorphism*

$$\operatorname{Mor}(Y, X) \approx \operatorname{Hom}(\mathcal{R}, \widehat{\mathcal{S}})$$

But there is a weird lack of symmetry.

In the category of reduced Δ-rings $\iota \colon \mathcal{R} \to \widehat{\mathcal{R}}$ is epi. This implies that

$$\hom(\widehat{\mathcal{R}}, \widehat{\mathcal{S}}) \to \hom(\mathcal{R}, \widehat{\mathcal{S}}) \qquad f \mapsto f \circ \iota$$

is injective. Using the previous theorem we can prove that it is surjective.

Theorem 14.2. *If* $\mathcal{R}$ *and* $\mathcal{S}$ *are reduced then*

$$\operatorname{Mor}(Y, X) \approx \operatorname{Hom}(\widehat{\mathcal{R}}, \widehat{\mathcal{S}}) .$$

I don't know if this theorem is true or false if $\mathcal{R}$ and $\mathcal{S}$ are not reduced.

15 Δ-Group schemes

It is known that every group scheme over a field of characteristic 0 is reduced. The theorem is due to Cartier. It is very important to us to know that every Δ-group scheme is reduced, particularly because we only have information about reduced subschemes. But I don't know if it is true or false.

Suppose that $X = \operatorname{diffspec} \mathcal{R}$ is a Δ-group scheme. Thus there is a multiplication

$$m \colon G \times_F G \longrightarrow G$$

And therefore a mapping of global sections

$$\widehat{m} \colon \widehat{\mathcal{R}} \longrightarrow \widehat{\mathcal{R} \otimes \mathcal{R}}$$

In algebraic geometry, this becomes

$$R = \widehat{R} \longrightarrow \widehat{R \otimes R} = R \otimes R$$

and it turns out that R a Hopf algebra.

Here, however,

$$\widehat{\mathcal{R} \otimes \mathcal{R}} \neq \widehat{\mathcal{R}} \otimes \widehat{\mathcal{R}}$$

so we do not get a Hopf algebra (in general). An example is due to Cassidy: neither $\mathcal{R}$ nor $\widehat{\mathcal{R}}$ is a Hopf algebra. In fact, the largest Hopf algebra in $\widehat{\mathcal{R}}$ that contains $\mathcal{F}$ is $\mathcal{F}$ itself and it is the trivial Hopf algebra. More precisely, she showed that the elliptic curve over constants can be made into an affine Δ-group scheme.

16 Regular functions

Cassidy and I are jointly writing a paper (book?) on Δ-group schemes. The idea is to "modernize" Cassidy (1972) and (1975). In these papers $\mathcal{U}$ is a universal Δ-field and $\mathcal{R}$ is supposed to be Δ-$\mathcal{U}$-algebra. We also assume it to be reduced and finitely Δ-generated over $\mathcal{U}$. We do not yet know which results are true over an arbitrary Δ-field $\mathcal{F}$ or which need some modification.

Definition 16.1. Let $s \in \widehat{\mathcal{R}}$. Then s is a *representative* section if

$$\widehat{m}(s) \in \widehat{\mathcal{R}} \otimes \widehat{\mathcal{R}}\,.$$

The set of representative sections is denoted by $\mathcal{R}^\star$.

Proposition 16.2. $\widehat{m}\colon \mathcal{R}^\star \to \mathcal{R}^\star \otimes \mathcal{R}^\star$.

Proposition 16.3. *$\mathcal{R}^\star$ is a Hopf algebra.*

Proposition 16.4. *Let $X^\star = \operatorname{diffspec}\mathcal{R}^\star$. There is a closed immersion of $X^\star$ in $\mathrm{GL}(n)$ for some n, i.e. $X^\star$ is a linear Δ-group scheme.*

Note that, by definition, $\mathcal{R}^\star \subset \widehat{\mathcal{R}}$. Recall that

$$X = \operatorname{diffspec}\mathcal{R} \approx \widehat{X} = \operatorname{diffspec}\widehat{\mathcal{R}}$$

so if $\mathcal{R} \subset \mathcal{R}^\star$ then we would have

$$X \approx X^\star = \operatorname{diffspec}\mathcal{R}^\star\,.$$

and X would be a linear Δ-algebraic group scheme.

Unfortunately Cassidy has examples where where $\mathcal{R} \not\subset \mathcal{R}^\star$ yet $X^\star \approx X$. She also has examples where $\mathcal{R}^\star$ is trivial, i.e. equal to $\mathcal{F}$.

More work needs to be done!

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