Generic Picard-Vessiot extensions for connected by finite groups

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Polynomial Galois Theory case

Noether first introduced generic equations in connection with the following question:

Given a field $\mathcal{F}$*, a finite group* G*, a set of* $\mathcal{F}$*-indeterminates* $\{x_g | g \in G\}$ *and a* G*-action on the rational field* $\mathcal{F}(x_g | g \in G)$ *by* $g(x_h) = x_{gh}$*, is the extension* $\mathcal{F}(x_g | g \in G)^G \supset \mathcal{F}$ *purely transcendental?*

Definition. Let $\mathbf{s} = (s_1, \ldots, s_m)$ be indeterminates over the field K, and let G be a finite group. A monic polynomial $P(\mathbf{s}, X) \in K(\mathbf{s})[X]$ is then called a *generic G-polynomial* over K if the following conditions are satisfied:

(i) The splitting field of $P(\mathbf{s}, X)$ over $K(\mathbf{s})$ is a G-extension.

(ii) Every G-extension of a field L containing K is the splitting field (over L) of a polynomial $P(\mathbf{a}, X)$ for some $\mathbf{a} = (a_1, \ldots, a_n) \in L^n$. $P(\mathbf{a}, X)$ is called specialization of $P(\mathbf{s}, X)$.

Definition. *(Saltman)* Let K be a field and G a finite group. A Galois extension S/R of commutative rings with group G is called a *generic G-extension* over K if the following conditions are satisfied:

(i) R is a *localized polynomial ring*, i.e., of the form $K[\mathbf{s}, 1/s]$, for indeterminates $\mathbf{s} = (s_1, \ldots, s_m)$ and an element $s \in K[\mathbf{s}] \backslash \{0\}$.

(ii) Whenever L is an extension field of K and T/L is a Galois algebra with group G, there is a K-algebra homomorphism $\varphi : R \to L$ such that $S \otimes_\varphi L/L$ and T/L are isomorphic as Galois extensions (i.e., by a G-equivariant L algebra homomorphism). The map φ is called a specialization.

Theorem 1. *(Ledet) Let K be an infinite field and G a finite group. Then there is a generic G-extension over K if and only if there is a generic G-polynomial over K.*

Picard-Vessiot Theory case

Notation.

$\mathcal{C}$ denotes an algebraically closed field of characteristic zero.

$\mathbf{t} = \mathbf{t}_1, \ldots, \mathbf{t}_n$ are differentially independent elements over $\mathcal{C}$.

$\mathcal{C}\{\mathbf{t}\}$ = Ring of differential polynomials in $\mathbf{t}_i$.

Definition. *(Goldman)* A LDE:

$L(\mathbf{t}, Y) = b_0(\mathbf{t})Y^{(n)} + b_1(\mathbf{t})Y^{(n-1)} + \cdots + b_n(\mathbf{t})Y = 0$

where $b_j(\mathbf{t}) \in \mathcal{C}\{\mathbf{t}\}$ is called a generic G-equation if

1. There exists a fundamental system $\mathbf{y} = (y_1, \ldots, y_n)$ of solutions of $L(\mathbf{t}, Y) = 0$ with $\mathcal{C}\langle\mathbf{t}\rangle \subset \mathcal{C}\langle\mathbf{y}\rangle$ such that $\mathcal{C}\langle\mathbf{y}\rangle$ is a Picard-Vessiot extension of $\mathcal{C}\langle\mathbf{t}\rangle$ with differential Galois group G.

2. If $\eta_1, \ldots, \eta_n$ is a fundamental system of zeros of $L(Y) = Y^{(n)} + a_1Y^{(n-1)} + \cdots + a_nY = 0$, $a_i \in \mathcal{F}$, where $\mathcal{F}$ is a differential field with field of constants $\mathcal{C}$, and $\mathcal{E} = \mathcal{F}\langle\eta_1, \ldots, \eta_n\rangle$ is a Picard-Vessiot extension of $\mathcal{F}$ then there exists a specialization $t_i \to \bar{t}_i$ over $\mathcal{F}$, with $t_i \in \mathcal{F}$, such that $b_0(\bar{t}_1, \ldots, \bar{t}_r) \neq 0$ and $a_i = b_i(\bar{t}_1, \ldots, \bar{t}_r)/b_0(\bar{t}_1, \ldots, \bar{t}_r)$ if and only if the differential Galois group of the extension $\mathcal{E} \supset \mathcal{F}$ is a subgrup $H \leq G$.

Constructed for $G = \mathrm{GL}_n$, $\mathrm{SL_n}$, the orthogonal and symplectic group and $R = \{[a_{ij}]\} \leq \mathrm{SL_n}$, $a_{r+k,m} = 0$, $k = 1, \ldots, s$, $m = 1, \ldots, r$, $r + s = n$.

Bhandari-Sankaran changed 2. in Goldman's def. to

2'. If $\mathcal{E} \supset \mathcal{F}$, where $\mathcal{F}$ is a differential field with field of constants $\mathcal{C}$, is a PVE then there is an equation $L(Y) = Y^{(n)} + a_1 Y^{(n-1)} + \cdots + a_n Y = 0$, $a_i \in \mathcal{F}$, and a specialization $t_i \to \bar{t}_i$ over $\mathcal{F}$, with $\bar{t}_i \in \mathcal{F}$, such that $b_0(\bar{t}_1, \ldots, \bar{t}_r) \neq 0$ and $a_j = b_j(\bar{t}_i)/b_0(\bar{t}_i)$ if and only if the differential Galois group of the extension $\mathcal{E} \supset \mathcal{F}$ is a subgrup $H \leq G$.

and constructed a generic G-equation for the special orthogonal groups.

Necessary conditions

If there is a generic equation with group G then

1. The number r of parameters equals the order n of the equation.

2. $\mathcal{C}\langle \mathbf{y} \rangle^G \supset \mathcal{C}$ is purely transcendental.

The main difficulty in constructing an equation like Goldman's

$$L(\mathbf{t}, Y) = b_0(\mathbf{t})Y^{(n)} + b_1(\mathbf{t})Y^{(n-1)} + \cdots + b_n(\mathbf{t})Y = 0$$

is that the coefficient $b_0(\mathbf{t})$ must not vanish under specialization.

We can circumvent this problem by taking a *generic extension* approach which exploits the Lie algebra of the group to a great extent.

Torsors

Let G be a linear algebraic group defined over a field k. A *k-homogeneous space* for G is a k-affine variety together with a morphism $G \times V \mapsto V$ of k-varieties inducing a transitive action of $G(\overline{k})$ on $V(\overline{k})$, where $\overline{k}$ denotes the algebraic closure of k. If moreover the action is faithful, V is called a *principal k-homogeneous space* for G or a *G-torsor*. The group G itself is called the trivial G-torsor.

Theorem 2. *The set of G-torsors (up to G-isomorphism) maps bijectively to the first Galois Cohomology set $H^1(k, G)$.*

Structure

Theorem 3. *Let k be a differential field with field of constants $\mathcal{C}$ and let $E \supset k$ be a Picard-Vessiot extension with group G. Then E is the function field $k(V)$ of some k-irreducible G-torsor where the action of the Galois group on E is the same as the action resulting from $G(\mathcal{C})$ acting on V. Moreover, $E = k(v)$, for some E-point $v \in V$.*

Previous work

Generic Picard-Vessiot extensions with connected Galois group G which are isomorphic to the function field of the trivial G-torsor G.

Let $G = H \ltimes G^0$ where H is finite, G^0 is connected and H acts faithfully on the Lie algebra of G^0.

We use the construction for connected groups to produce an extension $\mathcal{F}(G^0) \supset \mathcal{F}^H$ with group $G = H \ltimes G^0$ which is the function field of a G-torsor of the form $\mathcal{W} \times G^0$, where $\mathcal{W}$ is an $\mathcal{F}^H$-irreducible H-torsor.

The generic extension specialize to G-extensions which are the function field of G-torsors of the same form.

The connected case

Assume that G is a connected linear algebraic group over $\mathcal{C}$ of dimension n.

Let $\mathcal{F} = \mathcal{C}\langle Y_1, \ldots, Y_n \rangle$, where $Y_1, \ldots, Y_n$ are differentially independent over $\mathcal{C}$.

Write $\mathcal{R}$ for $\mathcal{F}[G] = \mathcal{F} \otimes_{\mathcal{C}} \mathcal{C}[G]$, the coordinate ring of $G_{\mathcal{F}}$.

Le D_i denote a $\mathcal{C}$-basis of $\mathcal{G} = \mathrm{Lie}(G)$. Then

$$\mathcal{D} = D_{\mathcal{F}} \otimes 1 + \sum_{i=1}^{n} Y_i \otimes D_i$$

where $D_{\mathcal{F}}$ denotes the derivation of $\mathcal{F}$, is a G-equivariant derivation on $\mathcal{R}$ which extends as such to its quotient field $\mathcal{F}(G)$.

Let $X_1, \ldots, X_n$ be the coordinate functions on G so that $\mathcal{F}(G) = \mathcal{F}(X_1, \ldots, X_n)$. Under the above derivation on $\mathcal{F}(G)$, the X_i become differentially independent over $\mathcal{C}$ and

$$\mathcal{F}(G) = \mathcal{C}\langle X_1, \ldots, X_n \rangle.$$

Therefore the purely differentially transcendental extension $\mathcal{F}(G) = \mathcal{C}\langle X_1, \ldots, X_n \rangle \supset \mathcal{C}$ has no new constants and we have,

Theorem 4. *$\mathcal{F}(G) \supset \mathcal{F}$ is a Picard-Vessiot extension with group G.*

With $\mathcal{E} = \mathcal{F}(G)$,

Theorem 5. *The extension $\mathcal{E} \supset \mathcal{F}$ is a generic Picard-Vessiot G-extension in the class of trivial G-torsors. That is,*

(i) There are elements $Y_1, \ldots, Y_n \in \mathcal{F}$, which are differentially independent over $\mathcal{C}$, such that for every faithful representation of G in a $\mathrm{GL}_m(\mathcal{C})$, the differential equation giving rise to the extension $\mathcal{E} \supset \mathcal{F}$ is of the form $X' = A(Y_i)X$, where $A(Y_i) \in M_m(\mathcal{C}\langle Y_i \rangle)$.

(ii) Let F be a differential field with field of constants $\mathcal{C}$. Given a faithful representation of G in a $\mathrm{GL}_m(\mathcal{C})$, a matrix $A \in M_m(F)$ and a Picard-Vessiot extension $E \supset$

F for the equation $X' = AX$ with connected Galois group $G' \leq G$ and E isomorphic to the function field $F(G')$, there are $f_1, \ldots, f_n \in F$ such that the matrix $A(f_i)$ obtained from $A(Y_i)$ in part (i) via the differential specialization $Y_i \to f_i$ is gauge equivalent to A.

(iii) For every such well-defined specialization $Y_i \to f_i \in F$ and faithful representation of G in a $\mathrm{GL}_m(\mathcal{C})$, the differential equation $X' = A(f_i)X$ gives rise to a Picard-Vessiot extension $E \supset F$ with differential Galois group $G' \leq G$.

The connected-by-finite group case

Let $G = H \ltimes G^0$, where G^0 is connected, H is finite and the adjoint action of H on $\mathcal{G} = \mathrm{Lie}(G^0)$ is faithful.

Let $\mathcal{F} = \mathcal{C}\langle Y_1, \ldots, Y_n \rangle$ be as before. We will define a suitable H-action on $\mathcal{F}$ and produce a Picard-Vessiot extension $\mathcal{E} \supset \mathcal{F}^H$ with group $H \ltimes G^0$. The main ingredients are a Picard-Vessiot G^0-extension $\mathcal{E} = \mathcal{F}(G^0)$ as in Theorem 4, a condition for H-equivariance to be developed next and a criterion by Mitschi-Singer and Hartmann to obtain a Picard-Vessiot extension with group G.

We recall the following:

Let k be a differential field with field of constants $\mathcal{C}$ and E a Picard-Vessiot extension of k with group G. Let V be as in Theorem 3 and write $E = k(v)$ for some E-point $v \in V$. For $\sigma \in G$ and any E-point $r \in V$ denote by ${}^{\sigma}r$ the differential Galois action of σ on r and by $r \cdot \sigma$ the translation action of σ on r via the G-torsor V. We then have ${}^{\sigma}v = v \cdot \sigma$ for all $\sigma \in G(\mathcal{C})$.

Now suppose that $G = H \ltimes G^0$ and that the G-torsor $V = W \times G^0$, for some k-irreducible H-torsor W. We let F denote the fixed field $E^{G^0} = k(W)$ and write $F = k(w)$ for some F-point $w \in W$ and $E = F(g) = k(g, w)$ for some E-point $g \in G^0$. For $(\sigma, \tau) \in G(\mathcal{C}) = H(\mathcal{C}) \ltimes G^0(\mathcal{C})$ we have

$$^{(\sigma,\tau)}(w, g) = (w \cdot \sigma, \sigma^{-1} g \sigma \tau),$$

and in particular for $\sigma \in H(\mathcal{C})$

$$^{\sigma}g = \sigma^{-1} g \sigma. \tag{1}$$

Regarding G as a subgroup of some GL_n and its Lie algebra $\mathcal{G}$ as being a subalgebra of the Lie algebra gl_n of all $n \times n$ matrices one has that $A = g' g^{-1} \in \mathrm{gl}_n(F)$, since the entries of A are invariant under the action of the constant group G^0. By (1) we have

$$\begin{aligned} ^{\sigma}A &= {}^{\sigma}(g' g^{-1}) = (\sigma^{-1} g \sigma)' \sigma^{-1} g^{-1} \sigma & (2) \\ &= \sigma^{-1} A \sigma. & (3) \end{aligned}$$

Definition. [Mitschi-Singer and Hartmann] *Let K be a Galois extension of k with Galois group H. Let V be a right H-module over k. We consider $K \otimes_k V$ as a left H-module via the action $\sigma \cdot a \otimes v = \sigma(a) \otimes v$ and as a right H-module via the action $a \otimes (v \cdot \sigma)$ for any $\sigma \in H$. We say an element $u \in K \otimes_k V$ is H-equivariant if $\sigma \cdot u = u \cdot \sigma$ for all $\sigma \in H$.*

With notation as above, consider $V = \mathcal{G}$ as a right H-module via $v \to h^{-1}vh$ for all $h \in H$ and $v \in \mathcal{G}$.

Assume that k is a differential field with field of constants $\mathcal{C}$ and E is the function field of a k-irreducible G-torsor of the form $W \times G^0$, for some k-irreducible H-torsor W.

Proposition 1. [MSH] *Let E be a Picard-Vessiot extension of k with Galois group $G = H \ltimes G^0$. Then*

1. *$K = E^{G^0}$ is the function field of a k-irreducible H-torsor (and so K is a Galois extension of k with Galois group H),*

2. *E is a Picard-Vessiot extension of K for an equation of the form $X' = AX$ where A is an H-equivariant element of $\mathcal{G}(K)$. Furthermore, the Galois group of E over K is G^0.*

Proposition 2. [MSH] *Le k be a differential field of characteristic zero with algebraically closed field of constants $\mathcal{C}$. Let $G = H \ltimes G^0 \subset$ GL_n be an algebraic group over $\mathcal{C}$, with H finite and G^0 connected with Lie algebra $\mathcal{G}$. Let W be a k-irreducible H-torsor and let $K = k(W)$.*

Let $A \in \mathcal{G}(K)$ and assume that

1. *A is H-equivariant.*

2. *The Picard-Vessiot extension E of K corresponding to the equation $X' = AX$ has Galois group G^0.*

Then E is the function field of the k-irreducible G-torsor $W \times G^0$ and a Picard-Vessiot extension of k with Galois group G. Furthermore the action of the Galois group corresponds to the action of G on E induced by the action of G on $W \times G^0$.

Criterion for H-equivariance

As before, let $G = H \ltimes G^0$, where H is finite, G^0 is connected with Lie algebra $\mathcal{G}$, and the adjoint H-action on $\mathcal{G}$ is faithful. Moreover, assume that H acts on G^0 via the right conjugation $h^{-1}gh$, $h \in H$, $g \in G^0$, so that the adjoint action is also a right action. The adjoint H-action is in fact a $\mathcal{C}$-linear action on $\mathcal{G}$ and it induces a faithful representation $\rho : H \to \mathsf{GL}_n(\mathcal{C})$, where $n = \dim(G^0)$, with respect to a basis $\{D_1, \ldots, D_n\}$ of $\mathcal{G}$. Identify H with its isomorphic image $\rho(H) \leq \mathsf{GL}_n(\mathcal{C})$. Then for $h \in H$, the H-action on the D_i reads:

$$D_i \cdot h_i = \sum_{j=1}^{n} \rho(h)_{ij} D_j \qquad (4)$$

where the $\rho(h)_{ij}$ are the entries of the matrix $\rho(h)$.

Let $M = \sum_{i=1}^{n} \mathcal{C}\, m_i$ be the $\mathcal{C}$-span of n elements m_i in some field extension of $\mathcal{C}$.

Assume that H and the m_i (not necessarily linearly independent over $\mathcal{C}$) are such that there is a left H-action on M.

Example. take $m_1 = x \notin \mathcal{C}$, $m_2 = -x$ and $H = C_2 = \{h_1, h_2\}$. Let $\rho : H \to \mathrm{GL}_2$ be the representation given by $\rho(h_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\rho(h_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} m_2 \\ m_1 \end{pmatrix}$.

So, there is a well defined action on $\sum_{i=1}^{2} \mathcal{C}\, m_i$ by letting $h_1 \cdot m_i = m_i$, $h_2 \cdot m_1 = m_2$ and $h_2 \cdot m_2 = m_1$.

The extended Lie algebra $M \otimes_{\mathcal{C}} \mathcal{G}$ can then be seen as a left H-module via the action $h \cdot m \otimes D$ and as a right H-module via the action $m \otimes D \cdot h$.

Proposition 3. *Let $\mathcal{D} = \sum_{i=1}^{n} m_i \otimes D_i \in M \otimes_{\mathcal{C}} \mathcal{G}$. Then, for $h \in H$, $h \cdot \mathcal{D} = \mathcal{D} \cdot h$ if and only if the H-action on the m_i is given by*

$$h \cdot m_i = \sum_{j=1}^{n} \rho(h)_{ij}^{T} m_j. \quad (5)$$

where $\rho(h)^T$ denotes the transpose of the matrix $\rho(h)$.

Proof. The condition $h \cdot \mathcal{D} = \mathcal{D} \cdot h$ reads $\sum_{i=1}^{n}(h \cdot m_i) \otimes D_i = \sum_{i=1}^{n} m_i \otimes (D_i \cdot h)$. Substituting the expression in (4) for $D_i \cdot h$, this last equation implies

$$\begin{aligned}\sum_{i=1}^{n}(h \cdot m_i) \otimes D_i &= \sum_{i=1}^{n} m_i \otimes \{\sum_{j=1}^{n} \rho(h)_{ij} D_j\} \\ &= \sum_{j=1}^{n} \{\sum_{i=1}^{n} \rho(h)_{ij} m_i\} \otimes D_j \\ &= \sum_{k=1}^{n} \{\sum_{\ell=1}^{n} \rho(h)^T_{k\ell} m_\ell\} \otimes D_k,\end{aligned}$$

that is,

$$\sum_{i=1}^{n} \{(h \cdot m_i) - \sum_{j=1}^{n} \rho(h)^T_{ij} m_j\} \otimes D_i = 0. \qquad (6)$$

Let $\{b_1, \ldots, b_s\}$ be a $\mathcal{C}$-basis of M so that the set $B = \{b_i \otimes D_j \,|\, i = 1, \ldots, s,\ j = 1, \ldots, n\}$, is a $\mathcal{C}$-basis of $M \otimes_{\mathcal{C}} \mathcal{G}$. For each $i = 1, \ldots, n$, we have for the coefficient of D_i in (6):

$$h \cdot m_i - \sum_{j=1}^{n} \rho(h)_{ij}^{T} m_j = \sum_{\ell=1}^{s} c_{i\ell}\, b_\ell \qquad (7)$$

with $c_{i\ell} \in \mathcal{C}$.

Now, (6) and (7) imply that $\sum_{i=1}^{n} \sum_{\ell=1}^{s} c_{i\ell}\, b_\ell \otimes D_i = 0$. This, in turn, implies that each $c_{i\ell} = 0$, since the set B is a $\mathcal{C}$-basis. This shows that $h \cdot m_i = \sum_{j=1}^{n} \rho(h)_{ij}^{T} m_j$.

□

The following is an immediate consequence:

Corollary 1. *Let $m_1, \ldots, m_n$, $D_1, \ldots, D_n$ be as above and suppose that the m_i are algebraically (respectively differentially) independent over $\mathcal{C}$. A left H-action may then be defined in the field $\mathcal{C}(m_1, \ldots, m_n)$ (respectively differential field $\mathcal{C}\langle m_1, \ldots, m_n \rangle$) via Eqn. (5), such that the element $\mathcal{D} = \sum_{i=1}^{n} m_i \otimes D_i$ satisfies $h \cdot \mathcal{D} = \mathcal{D} \cdot h$.*

Remarks. Suppose we start with a left H-action on G and make it a right action in the standard way, namely, via $h^{-1} \cdot g$, $h \in H$, $g \in G$. In this case, if $\rho : H \to \mathsf{GL}_n$ denotes the representation induced by the adjoint left action, with respect to the same basis, then the dual representation to ρ given by $\rho^\vee(h) = (\rho(h)^{-1})^T$, $h \in H$, produces the result.

The H-equivariance of $\mathcal{D}$ is related to the fact that for a k-vector space V with basis $\{v_i\}$ and dual basis $\{v_i^\vee\}$ under the isomporphism $E^\vee \otimes E \cong \mathsf{End}_k(E)$ the Casimir element $\sum_{i=1}^n v_i^\vee \otimes v_i$ is sent to the identity.

Let $G = H \ltimes G^0$ as before.

$\mathcal{F} = \mathcal{C}\langle Y_1, \ldots, Y_n \rangle$, $n = \dim(G^0)$.

By Corollary 1 we can define a left H-action on $\mathcal{F}$ via

$$h \cdot Y_i = \sum_{j=1}^{n} \rho(h)_{ij}^T Y_j$$

where the homomorphism ρ represents H in GL_n with respect to a basis $\{D_1, \cdots, D_n\}$ of $\mathcal{G} = \mathrm{Lie}(G^0)$. This action is such that the element $\mathcal{D} = \sum_{i=1}^{n} Y_i \otimes D_i$ satisfies $h \cdot \mathcal{D} = \mathcal{D} \cdot h$ in the left and right H-module $\mathcal{F} \otimes_{\mathcal{C}} \mathcal{G}$.

Since the adjoint H-action on $\mathcal{G}$ is by hypothesis faithful, so is the above action on $\mathcal{F}$. Therefore, the extension $\mathcal{F} \supset \mathcal{F}^H$ is Galois with group H. The latter implies that $\mathcal{F} = \mathcal{F}^H(\mathcal{W})$ for some $\mathcal{F}^H$-irreducible H-torsor $\mathcal{W}$. Then, as an element of the left and right H-module

$\mathcal{F} \otimes_{\mathcal{F}^H} (\mathcal{F}^H \otimes_{\mathcal{C}} \mathcal{G})$ we have that $\sum_{i=1}^n Y_i \otimes D_i$ is H-equivariant in the sense of Definition . Now, if we identify G with a subgroup of GL_m for some m, the D_i can be identified with linearly independent matrices $A_i \in \mathsf{gl}_m$ and $\mathcal{D} = \sum_{i=1}^n Y_i \otimes D_i$ corresponds to the H-equivariant matrix

$$A(Y) = \sum_{i=1}^{n} Y_i A_i \in \mathcal{G}(\mathcal{F}).$$

By Theorem 4, the field $\mathcal{E} = \mathcal{F}(G^0)$ is a Picard-Vessiot extension of $\mathcal{F}$ with Galois group G^0 for the equation $X' = A(Y)X$. By virtue of Proposition 2 we have then shown:

Proposition 4. *$\mathcal{E}$ is the function field of the $\mathcal{F}^H$-irreducible G-torsor $\mathcal{W} \times G^0$ and a Picard-Vessiot extension of $\mathcal{F}^H$ with Galois group G.*

Next we show that the extension $\mathcal{E} \supset \mathcal{F}$ is H-equivariantly generic for G^0.

Let $E \supset k$ be a Picard-Vessiot equation with Galois group $G = H \ltimes G^0$ and assume that E is also the function field of a k-irreducible G-torsor $W \times G^0$, where W is a k-irreducible H-torsor.

Write $F = E^{G^0}$, by proposition 1, there is an H-equivariant matrix $A \in \mathcal{G}(F)$ such that $E \supset F$ is a Picard-Vessiot extension with group G^0 for the equation $X' = AX$ and $E = F(G^0)$. Since $A \in \mathcal{G}(F)$ there is a specialization $Y_i \to f_i$, with $f_i \in F$, such that $A = \sum_{i=1}^n f_i A_i = A(f_i)$. Since A is H-equivariant, by Proposition 3, the H-action on the f_i is given by (5). The latter implies that the specialization $Y_i \to f_i$ is H-equivariant, that is, $h \cdot Y_i = \sum_{j=1}^n \rho(h)_{ij}^T Y_j \to \sum_{j=1}^n \rho(h)_{ij}^T f_j = h \cdot f_i$.

We summarize the preceding discussion in the following:

Theorem 6. *The Picard-Vessiot extension $\mathcal{E} \supset \mathcal{F}^H$ is generic for $G = H \ltimes G^0$ in the following sense: let $E \supset k$ be a Picard-Vessiot extension with differential Galois group G, and assume that E is the function field of a k-irreducible G-torsor $W \times G^0$, where W is a k-irreducible H-torsor. Then there is an H-equivariant specialization $Y_1, \ldots, Y_n \to f_1, \ldots, f_n$, with the $f_i \in F = k(W)$, such that the derivation $\mathcal{D}$ of $\mathcal{E}$ specializes to an H-equivariant derivation D on E, which is equivalent to D_E.*

Remarks on descent genericity.

First, we point out that in the context of polynomial Galois theory, in principle, descent is not a required property in the definition of generic polynomial (see Definition above). Nontheless, it is known (Kemper) that generic polynomials are descent generic. This means that condition (ii) in Definition can be replaced with the stronger one:

(ii') For any subgroup $H \leq G$, every H-extension of a field L containing K is the splitting field (over L) of the polynomial $P(\mathbf{a}, X)$ for some $\mathbf{a} = (a_1, \ldots, a_n) \in L^n$.

We don't know at present whether this result can be generalized to the case when G is an infinite linear algebraic group. However, we can show a weaker condition.

Let $G = H \ltimes G^0$ and let $\mathcal{E} \supset \mathcal{F}^H$ be the generic G-extension in Theorem 6. Write $G' = H \ltimes J^0 \leq G$ where J^0 denotes an H-stable connected subgroup of G^0. We have,

Corollary 2. *The extension $\mathcal{E} \supset \mathcal{F}^H$ has the following descent property: for every subgroup $G' \leq G$ as above and every Picard-Vessiot G'-extension $E \supset k$, with E the function field of a k-irreducible G'-torsor of the form $W \times J^0$, where W is a k-irreducible H-torsor, there is an H-equivariant specialization $Y_i \to f_i$ with $f_i \in F = k(W)$, such that the derivation $\mathcal{D}$ of $\mathcal{E}$ specializes to an H-equivariant derivation D on E, which is equivalent to D_E.*

Proof. This is an immediate consequence of Theorem 5.(ii).

□

Example

Let $G^0 = \mathsf{SL}_2(\mathcal{C})$ and $H = C_2$, where the action of C_2 on $\mathsf{SL}_2(\mathcal{C})$ is given by inverse transposition, that is, conjugation by the element $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Applying the adjoint action to the basis $A_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $A_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ of $\mathsf{Lie}(\mathsf{SL}_2)$, we see that this action is diagonalizable with eigenvectors A_1, A_2, A_3. The corresponding eigenvalues are 1, -1, -1.

Let σ denote the nontrivial elment of C_2 and $\rho : C_2 \to \mathsf{GL}_3(\mathcal{C})$ be the representation given by the above action on the basis A_1, A_2, A_3. Then, $\rho(\sigma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

Observe that in this case $\rho^\vee(\sigma) = \rho(\sigma)$. Following the method above, we let C_2 act on the differentially independent elements Y_i, $i = 1, \ldots, 3$, by $\sigma{\cdot}Y_1 = Y_1$, $\sigma{\cdot}Y_2 = -Y_2$, $\sigma{\cdot}Y_3 = -Y_3$. Then $\mathcal{C}\langle Y_1, Y_2, Y_3\rangle(\mathrm{SL}_2) \supset \mathcal{C}\langle Y_1, Y_2, Y_3\rangle^{C_2}$ is a generic Picard-Vessiot extension for $C_2 \ltimes \mathrm{SL}_2$. The equation corresponding to the standard representation of SL_2 in GL_2 is $X' = A(Y)X$, where $A(Y) = \begin{bmatrix} Y_3 & -Y_1 + Y_2 \\ Y_1 + Y_2 & -Y_3 \end{bmatrix}$.

In fact, the matrix $\begin{bmatrix} -\sqrt{x} & -1+\sqrt{x} \\ 1+\sqrt{x} & \sqrt{x} \end{bmatrix}$ is a specialization of $A(Y)$ with group $C_2 \ltimes \mathsf{SL}_2$. The following Maple code gives a proof to this fact:

```
with(DEtools):_Envdiffopdomain:=[Dx,x];
A:=matrix(2,2,[-sqrt(x),-1+sqrt(x), 1+sqrt(x),
sqrt(x)]);
B,P:=cyclic(A,[1,I]);
L:=Dx^2-B[2,2]*Dx-B[2,1];
eq:=diffop2de(L,y(x)):
kovacicsols(eq,y(x));
```

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