Introduction to Rota-Baxter Algebras

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- **1. Rota-Baxter algebras:**
  Fix λ in the base ring $\mathbf{k}$. A Rota-Baxter operator or a Baxter operator of weight λ on a $\mathbf{k}$-algebra R is a linear map $P: R \to R$ such that
  $$P(x)P(y) = P(xP(y)) + P(P(x)y) + \lambda P(xy), \ \forall x, y \in R.$$

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- **Examples: Integration:** $R = \mathbf{Cont}(\mathbb{R})$ (ring of continuous functions on $\mathbb{R}$).
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  Then P is a weight 0 Rota-Baxter operator:
- $$F(x) := P[f](x) = \int_0^x f(t)\,dt, \quad G(x) := P[g](x) = \int_0^x g(t)\,dt.$$
  Then the integration by parts formula states
  $$\int_0^x F(t)G'(t)dt = F(x)G(x) - \int_0^x F'(t)G(t)dt$$
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- that is,

$$P[P[f]g](x) = P[f](x)P[g](x) - P[fP[g]](x).$$

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$$P[f](x)P[g](x) = \Big(\sum_{n\geq 1} f(x+n)\Big)\Big(\sum_{m\geq 1} g(x+m)\Big)$$

$$= \sum_{n\geq 1, m\geq 1} f(x+n)g(x+m)$$

$$= \Big(\sum_{n>m\geq 1} + \sum_{m>n\geq 1} + \sum_{m=n\geq 1}\Big) f(x+n)g(x+m)$$

$$= \sum_{m\geq 1} \Big(\sum_{k\geq 1} f(x+\underbrace{k+m}_{=n})\Big)g(x+m) + \sum_{n\geq 1}\Big(\sum_{k\geq 1} g(x+\underbrace{k+n}_{=m})\Big)f(x+n)$$

$$+ \sum_{n\geq 1} f(x+n)g(x+n)$$

$$= P(P(f)g)(x) + P(fP(g))(x) + P(fg)(x).$$

- **Partial sum:** Let R be the set of sequences $\{a_n\}$ with values in **k**. Then R is a **k**-algebra with termwise addition, multiplication and scalar product. Define

$$P : R \to R, P(a_1, a_2, \cdots) = (a_1, a_1 + a_2, \cdots).$$

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- **Matrices:** On the algebra of upper triangular $n \times n$ matrices $M_n^u(\mathbf{k})$, define

$$P((c_{k\ell}))_{ij} = \delta_{ij} \sum_{k \geq i} c_{ik}.$$

  Then P is a Rota-Baxter operator of weight -1.

- **Scalar product:** Let R be a **k**-algebra. For a given $\lambda \in \mathbf{k}$, define

$$P_\lambda : R \to R, x \mapsto -\lambda x, \forall x \in R.$$

Then (R, P_λ) is a Rota-Baxter algebra of weight λ. In particular, id is a Rota-Baxter operator of weight -1 and any **k**-algebra is a Rota-Baxter algebra.

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- **QFT dimensional regularization:** Let $R = \mathbb{C}[t^{-1}, t]]$ be the ring of Laurent series $\sum_{n=-k}^{\infty} a_n t^n,\ k \geq 0$. Define

$$P(\sum_{n=-k}^{\infty} a_n t^n) = \sum_{n=-k}^{-1} a_n t^n.$$

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- **Classical Yang-Baxter equation:** Let $\mathfrak{g}$ be a Lie algebra with a self-duality $\mathfrak{g}^* := \mathrm{Hom}(\mathfrak{g}, \mathbf{k}) \cong \mathfrak{g}$. Then $\mathfrak{g}^{\otimes 2} \cong \mathfrak{g} \otimes \mathfrak{g}^* \cong \mathrm{End}(\mathfrak{g})$. Let $r_{12} \in \mathfrak{g}^{\otimes 2}$ be anti-symmetric. Then r_{12} is a solution (r-matrix) of the classical Yang-Baxter equation (CYB)

$$CYB(r) := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$$

if and only if the corresponding $P \in \mathrm{End}(\mathfrak{g})$ is a (Lie algebra) Rota-Baxter operator of weight 0:

$$[P(x), P(y)] = P[P(x), y] + P[x, P(y)].$$

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- **Associative Yang-Baxter equations** (Aguiar) Let A be an associative algebra and let $r := \sum_i u_i \otimes v_i \in R \otimes R$ be a solution of the associative Yang-Baxter equation

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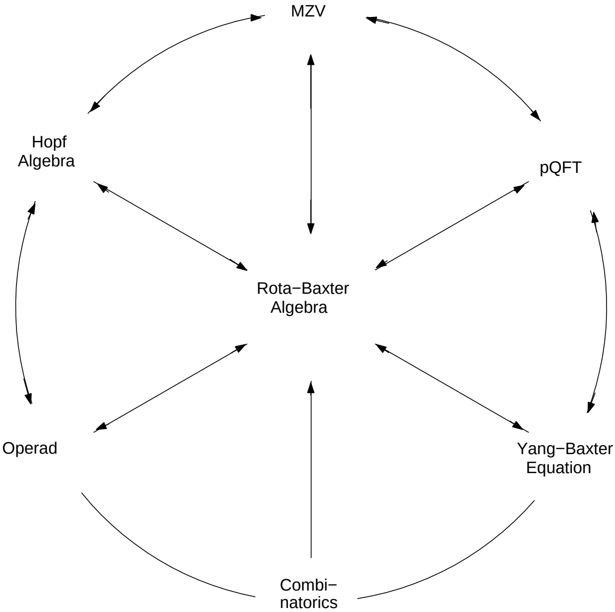
- **Others** Divided powers, Hochschild homology ring, dendriform algebras, rooted trees, quasi-shuffles, Chen integral symbols, ...


MZV
Hopf Algebra
pQFT
Rota−Baxter Algebra
Operad
Yang−Baxter Equation
Combi−natorics

- **2. Free commutative Rota-Baxter algebras**
  Let A be a commutative $\mathbf{k}$-algebra. Let $\text{Ш}^+(A) = \bigoplus_{n\geq 0} A^{\otimes n} (= T(A))$. Consider the following products on $\text{Ш}^+(A)$. Define $\mathbf{1_k} \in \mathbf{k}$ to be the unit. Let $\mathfrak{a} = a_1 \otimes \cdots \otimes a_m \in A^{\otimes m}$ and $\mathfrak{b} = b_1 \otimes \cdots \otimes b_n \in A^{\otimes n}$.

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- **Quasi-shuffle product:** Hoffman (2000) on multiple zeta values. Write $\mathfrak{a} = a_1 \otimes \mathfrak{a}'$, $\mathfrak{b} = b_1 \otimes \mathfrak{b}'$. Recursively define

$$(a_1\otimes\mathfrak{a}')*(b_1\otimes\mathfrak{b}') = a_1\otimes(\mathfrak{a}'*(b_1\otimes\mathfrak{b}'))+b_1\otimes((a_1\otimes\mathfrak{a}')*\mathfrak{b}')+a_1b_1\otimes(\mathfrak{a}'*\mathfrak{b}'),$$

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- **Example.**
  $a_1 * (b_1 \otimes b_2) = a_1 \otimes (\mathfrak{a}' * (b_1 \otimes b_2)) + b_1 \otimes (a_1 * b_2) + (a_1 b_1) \otimes (\mathfrak{a}' * b_2)$
  $= a_1 \otimes (b_1 \otimes b_2) + b_1 \otimes (a_1 * b_2) + (a_1 b_1) \otimes b_2.$
  $= a_1 \otimes b_1 \otimes b_2 + b_1 \otimes a_1 \otimes b_2 + b_1 \otimes b_2 \otimes a_1 + b_1 \otimes a_1 b_2 + a_1 b_1 \otimes b_2.$

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- A mixable shuffle is a shuffle in which some pairs $a_i \otimes b_j$ are merged into $a_i b_j$.
  Define $(a_1 \otimes \ldots \otimes a_m) \diamond (b_1 \otimes \ldots \otimes b_n)$ to be the sum of mixable shuffles of $a_1 \otimes \ldots \otimes a_m$ and $b_1 \otimes \ldots \otimes b_n$.

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- **Example:**

$$\begin{aligned} & a_1 \diamond (b_1 \otimes b_2) \\ & = a_1 \otimes b_1 \otimes b_2 + b_1 \otimes a_1 \otimes b_2 + b_1 \otimes b_2 \otimes a_1 \quad \text{(shuffles)} \\ & + a_1 b_1 \otimes b_2 + b_1 \otimes a_1 b_2 \quad \text{(merged shuffles)}. \end{aligned}$$

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- Let $\overline{S}_c(m,n)$ be the set of triples (k,α,β) where $\max\{m,n\} \leq k \leq m+n$ and $\alpha : [m] \to [k]$, $\beta : [n] \to [k]$ are order preserving injections with $\mathrm{im}(\alpha) \cup \mathrm{im}(\beta) = [k]$.

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- For a given $(k,\alpha,\beta) \in \overline{S}_c(m,n)$ and $1 \leq i \leq k$, $\alpha^{-1}(i)$ is either a singleton $\{j\}$ or $\emptyset$. Then accordingly define $a_{\alpha^{-1}(i)} = a_j$ or 1. Similarly define $b_{\beta^{-1}(i)}$. Call $a_{\alpha^{-1}(1)}b_{\beta^{-1}(1)} \otimes \cdots \otimes a_{\alpha^{-1}(k)}b_{\beta^{-1}(k)}$ a stuffle.

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- Define a product

$$\mathfrak{a} \diamond \mathfrak{b} = \sum_{(k,\alpha,\beta)\in\overline{S}_c(m,n)} a_{\alpha^{-1}(1)}b_{\beta^{-1}(1)} \otimes \cdots \otimes a_{\alpha^{-1}(k)}b_{\beta^{-1}(k)}$$

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- **Example:** $\mathfrak{a} = a_1, m = 1$, $\mathfrak{b} = b_1 \otimes b_2$, $n = 2$.

k	α	β	stuffles
3	$\alpha(1) = 1$	$\beta(1) = 2, \beta(2) = 3$	$a_1 \otimes b_1 \otimes b_2$
3	$\alpha(1) = 2$	$\beta(1) = 1, \beta(2) = 3$	$b_1 \otimes a_1 \otimes b_2$
3	$\alpha(1) = 3$	$\beta(1) = 1, \beta(2) = 2$	$b_1 \otimes b_2 \otimes a_1$
2	$\alpha(1) = 1$	$\beta(1) = 1, \beta(2) = 2$	$a_1b_1 \otimes b_2$
2	$\alpha(1) = 2$	$\beta(1) = 1, \beta(2) = 2$	$b_1 \otimes a_1b_2$

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- Let $D(m,n)$ be the set of lattice paths from $(0,0)$ to (m,n) consisting steps either to the right, to the above, or to the above-right. For $d \in D(m,n)$ define $d(\mathfrak{a},\mathfrak{b})$ to be the path d with $\mathfrak{a}=(a_1,\cdots,a_m)$ (resp. $\mathfrak{b}=(b_1,\cdots,b_n)$) sequentially labeling the horizontal (resp. vertical) and diagonal segments of d. Define

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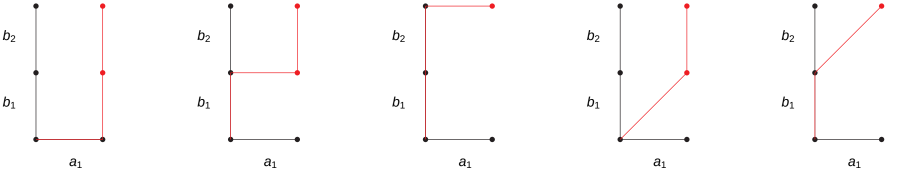
- **Example:** $\mathfrak{a} = a_1, \mathfrak{b} = b_1 \otimes b_2$.



$a_1 \otimes b_1 \otimes b_2$ $\quad b_1 \otimes a_1 \otimes b_2$ $\quad b_1 \otimes b_2 \otimes a_1$ $\quad a_1 b_1 \otimes b_2$ $\quad b_1 \otimes a_1 b_2$

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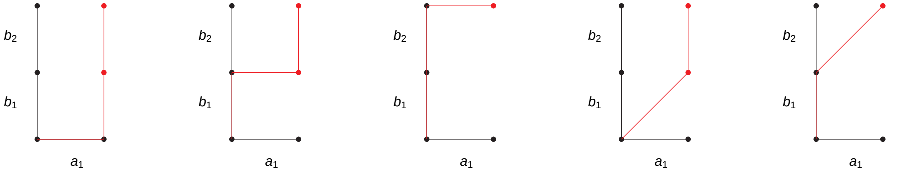
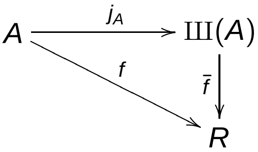


$a_1 \otimes b_1 \otimes b_2 \quad b_1 \otimes a_1 \otimes b_2 \quad b_1 \otimes b_2 \otimes a_1 \quad a_1 b_1 \otimes b_2 \quad b_1 \otimes a_1 b_2$

- **Theorem** All the above products define the same algebra on $Ш^+(A)$ (of weight $\lambda = 1$).

- A **free commutative Rota-Baxter algebra over another commutative algebra** A is a commutative Rota-Baxter algebra $Ш(A)$ with an algebra homomorphism $j_A : A \to Ш(A)$ such that for any commutative Rota-Baxter algebra R and algebra homomorphism $f : A \to R$, there is a unique Rota-Baxter algebra homomorphism making the diagram commute



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$$\begin{array}{ccc} A & \xrightarrow{j_A} & Ш(A) \\ & \searrow^{f} & \downarrow{\bar{f}} \\ & & R \end{array}$$

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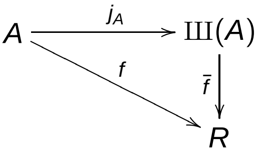

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- Recall $(\text{Ш}^+(A), \diamond)$ is a commutative algebra. Then the tensor product algebra (scalar extension) $\text{Ш}(A) := A \otimes \text{Ш}^+(A)$ is a commutative A-algebra.
  Theorem (Guo-Keigher) $\text{Ш}(A)$ with the shift operator $P(\mathfrak{a}) := 1 \otimes \mathfrak{a}$ is the free commutative RBA over A.

- **Free (noncommutative) Rota-Baxter algebras of rooted forests.** Let $Ш^{\mathrm{NC}}(X)$ be the free Rota-Baxter algebra on a set X. Let $\mathcal{F}(X)$ be the set of rooted forests with leafs decorated by X,

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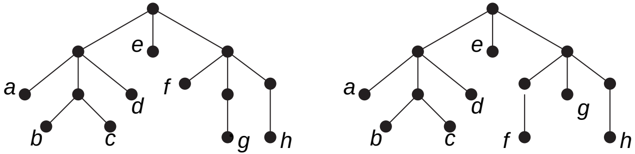
- 

  For example, the left tree is not leaf-spaced since the two right most branches are not separated by a leaf branch. But the right tree is leaf-spaced.

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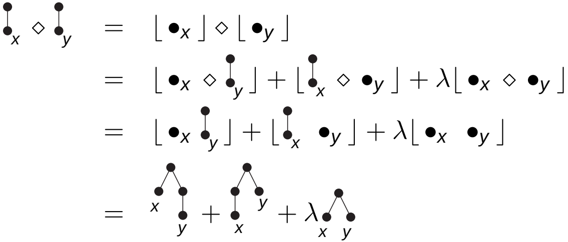
  2. If $F = F_1 \cdots F_b$ and $F' = F'_1 \cdots F'_{b'}$ are in $\mathcal{F}_\ell(X)$ with their corresponding decompositions into leaf decorated trees. Then

$$F \diamond F' = F_1 \cdots (F_b \diamond F'_1) \cdots F_{b'}.$$



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- Theorem. The quadruple $(\mathbf{k}\mathcal{F}_\ell(X), \diamond, \lfloor\ \rfloor, j_X)$ is the free nonunitary (noncommutative) Rota-Baxter algebra on X.

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